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On the Algebraicity of the Ratio of Special Theta Constants

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In this paper we treat the special value of the Riemann θ function with characteristic ${}^t[a, b]$ $(a, b \in \boldsymbol{Q})$:

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z, \tau)=\sum_{n \in \boldsymbol{Z}} \exp\{\pi i(n+a)^2\tau+2\pi i(n+a)(z+b)\} ,$$

where $z, \tau \in \boldsymbol{C}$ and $\operatorname{Im}\tau>0$. We show the following proposition:

PROPOSITION RA. *Suppose τ is an imaginary quadratic number with* $\operatorname{Im}\tau>0$, *then*

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(0, m\tau)\Big/\theta\begin{bmatrix} a' \\ b' \end{bmatrix}(0, \tau)$$

is an algebraic number for any a, b, a', b' of $\boldsymbol{Q}$ and any positive integer m, provided the denominator does not vanish.

This result plays a role of key stone in the forthcoming work concerning the modular form relative to the Picard modular group (it acts on 2-dimensional hyperball B^2).

PROOF. We divide the assertion in two parts:

(i) $\theta\begin{bmatrix} a \\ b \end{bmatrix}(0, \tau)\Big/\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0, \tau)$ is algebraic,

(ii) $\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0, m\tau)\Big/\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0, \tau)$ is algebraic.

At first we show (i). Let us recall the transformation formula:

$$\text{(R-1)} \qquad \theta\begin{bmatrix} \tilde{\varepsilon}' \\ \tilde{\varepsilon}'' \end{bmatrix}(\tilde{z}, \tilde{\tau})=K(M, \varepsilon)\sqrt{c\tau+d}\cdot\exp\left(\frac{\pi i c z^2}{c\tau+d}\right)\theta\begin{bmatrix} \varepsilon' \\ \varepsilon'' \end{bmatrix}(z, \tau) ,$$

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where $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \boldsymbol{Z})$, $\tilde{z}=\dfrac{z}{c\tau+d}$, $\tilde{\tau}=\dfrac{a\tau+b}{c\tau+d}$, $\tilde{\varepsilon}'=d\varepsilon'-c\varepsilon''+cd/2$, $\tilde{\varepsilon}''=-b\varepsilon'+a\varepsilon''+ab/2$ and $K(M, \varepsilon)$ is a certain root of 1 (cf. [R-F]).

If we apply (R-1) to $M=\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we obtain

(R-2)
$$\theta\begin{bmatrix} \varepsilon' \\ -n\varepsilon'+\varepsilon''+\dfrac{n}{2} \end{bmatrix}(0, \tau+n)=\exp \pi i(-n\varepsilon'^2+n\varepsilon')\cdot\theta\begin{bmatrix} \varepsilon' \\ \varepsilon'' \end{bmatrix}(0, \tau) ,$$
$$\theta\begin{bmatrix} -\varepsilon'' \\ \varepsilon' \end{bmatrix}\left(0, -\frac{1}{\tau}\right)=\exp\left(-\frac{1}{4}\pi i-2\pi i\varepsilon'\varepsilon''\right)\sqrt{\tau}\cdot\theta\begin{bmatrix} \varepsilon' \\ \varepsilon'' \end{bmatrix}(0, \tau) ,$$

where we suppose $\mathrm{Re}\sqrt{\tau}>0$.

Now we set

(R-3)
$$\begin{aligned} \Psi(t) &= \prod_{k_2=0}^{m-1}\prod_{k_1=0}^{m-1}\left\{\theta^{4m^2}\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0, \tau)t-\theta^{4m^2}\begin{bmatrix} k_1/m \\ k_2/m \end{bmatrix}(0, \tau)\right\} \\ &= \theta^{4m^4}\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0, \tau)t^{m^2}+\cdots+(-1)^d\theta^{4m^2(m^2-d)}\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0, \tau)\cdot A_d(\tau)t^{m^2-d} \\ &\quad +\cdots+(-1)^{m^2}\prod_{k_1=0}^{m-1}\prod_{k_2=0}^{m-1}\theta^{4m^2}\begin{bmatrix} k_1/m \\ k_2/m \end{bmatrix}(0, \tau) , \end{aligned}$$

where $A_d(\tau)$ is the elementary symmetric polynomial of degree d relative to m^2 theta constants θ^{4m^2} with characteristic ${}^t[k_1/m, k_2/m]$.

From (R-2) we have

(R-4)
$$\theta^{4m^2}\begin{bmatrix} k_1/m \\ k_2/m-2k_1/m \end{bmatrix}(0, \tau+2)=\theta^{4m^2}\begin{bmatrix} k_1/m \\ k_2/m \end{bmatrix}(0, \tau) ,$$
$$\theta^{4m^2}\begin{bmatrix} -k_2/m \\ k_1/m \end{bmatrix}\left(0, -\frac{1}{\tau}\right)=(-1)^{m^2}\tau^{2m^2}\theta^{4m^2}\begin{bmatrix} k_1/m \\ k_2/m \end{bmatrix}(0, \tau) ,$$

namely, $A_d(\tau)$ satisfies

(R-5)
$$A_d(\tau+2)=A_d(\tau) ,$$
$$A_d\left(-\frac{1}{\tau}\right)=(-1)^{dm^2}\tau^{2dm^2}A_d(\tau) .$$

On the other hand it is well known that $\theta^{4dm^2}\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0, \tau)$ satisfies the same periodic relation. Hence every coefficient $C_d(\tau)$ of $\Psi(t)\Big/\theta^{4m^4}\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0, \tau)$ (as a polynomial in t) is a holomorphic modular function w.r.t. $\Gamma_{1,2}$, the subgroup of $SL(2, \boldsymbol{Z})$ generated by $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ and $\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$, especially relative to $\Gamma(2)$, where $\Gamma(2)$ is the principal congruence subgroup of level 2 of the modular group Γ. By the definition of $A_d(\tau)$ the above coefficient has a Fourier expansion with coefficients in $\boldsymbol{Q}$. By the classical q development principle $C_d(\tau)$ belongs to $\boldsymbol{Q}(\lambda(\tau))$, i.e., the field obtained by adjoining $\lambda(\tau)$ to $\boldsymbol{Q}$, where $\lambda(\tau)$ is the elliptic modular function that is the generator of the field of meromorphic modular functions w.r.t. $\Gamma(2)$. By the complex multiplication theory $\lambda(\tau)$ is an algebraic number. Hence every $C_d(\tau)$ is also algebraic. Therefore the root $\theta^{4m^2}\begin{bmatrix}k_1/m\\ k_2/m\end{bmatrix}(0, \tau)\Big/\theta^{4m^2}\begin{bmatrix}0\\ 0\end{bmatrix}(0, \tau)$ of the equation $\Psi(t)=0$ is algebraic.

Next we investigate (ii). We show it by induction on m. We have the following addition formula ([M] Chap. II, Prop. 6.4):

$$\text{(R-6)}\qquad \theta\begin{bmatrix}a/n_1\\ 0\end{bmatrix}(n_1z, n_1\tau)\cdot\theta\begin{bmatrix}b/n_2\\ 0\end{bmatrix}(n_2z, n_2\tau)$$
$$=\sum_{d\in \boldsymbol{Z}/(n_1+n_2)\boldsymbol{Z}}\left\{\theta\begin{bmatrix}\dfrac{n_1n_2d+n_2a-n_1b}{n_1n_2(n_1+n_2)}\\ 0\end{bmatrix}(0, n_1n_2(n_1+n_2)\tau)\right.$$
$$\left.\times\theta\begin{bmatrix}\dfrac{n_1d+a+b}{n_1+n_2}\\ 0\end{bmatrix}((n_1+n_2)z, (n_1+n_2)\tau)\right\},$$

where $a, b\in\boldsymbol{Z}$, and n_1 and n_2 are positive integers.

For $m=1$ the assertion is trivial. Then we suppose (ii) is true for $m\leqq k$. Putting $a=b=0$, $z=0$, $n_1=1$, $n_2=k$ in (R-6), we have

$$\theta\begin{bmatrix}0\\ 0\end{bmatrix}(0, \tau)\cdot\theta\begin{bmatrix}0\\ 0\end{bmatrix}(0, k\tau)$$
$$=\sum_{d\in\boldsymbol{Z}/(k+1)\boldsymbol{Z}}\left\{\theta\begin{bmatrix}\dfrac{kd}{k(k+1)}\\ 0\end{bmatrix}(0, k(k+1)\tau)\cdot\theta\begin{bmatrix}\dfrac{d}{k+1}\\ 0\end{bmatrix}(0, (k+1)\tau)\right\}.$$

Dividing both sides by $\theta^2\begin{bmatrix}0\\ 0\end{bmatrix}(0, (k+1)\tau)$, the right-hand side becomes algebraic because of (i) and the assumption of induction. On the other hand, $\theta\begin{bmatrix}0\\ 0\end{bmatrix}(0, k\tau)\Big/\theta\begin{bmatrix}0\\ 0\end{bmatrix}(0, \tau)$ is algebraic by the assumption. Therefore $\theta^2\begin{bmatrix}0\\ 0\end{bmatrix}(0, \tau)\Big/\theta^2\begin{bmatrix}0\\ 0\end{bmatrix}(0, (k+1)\tau)$ is algebraic. q.e.d.

References

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