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On a Full Spectrum Condition for 2-Dimensional Linear Quasi-Periodic Systems

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§1. In this paper, we consider the problem concerning linear quasi-periodic systems. Before starting our discussions, we state the following definitions.

DEFINITION 1 ([2, 3]). A real number λ is called a characteristic exponent of the system

$$\dot{x}=C(t)x\ , \qquad t\in \boldsymbol{R}\ , \quad \dot{}=\frac{d}{dt}\ , \tag{1.1}$$

if there exists a solution $x(t)$ of (1.1) which satisfies

$$\limsup_{t\to\infty} t^{-1}\log|x(t)|=\lambda\ .$$

DEFINITION 2 ([4]). If the number of characteristic exponents of (1.1) is equal to the dimension of (1.1), then we say that (1.1) has full spectrum.

Now, let there be given a linear almost periodic system

$$\dot{x}=C(t)x\ , \qquad x\in \boldsymbol{R}^n\ . \tag{1.2}$$

If (1.2) has full spectrum, then it is shown in [1] that (1.2) has a fundamental matrix $X(t)$ of the form

$$X(t)=F(t)\,\mathrm{diag}\Big(\exp\Big(\int_0^t d_1(s)ds\Big),\ \cdots,\ \exp\Big(\int_0^t d_n(s)ds\Big)\Big)$$

where $F(t)$ is an almost periodic matrix function and $d_i(t)$ $(i=1, \cdots, n)$ are almost periodic functions. Also it is shown in [4] that, if (1.2) is especially a linear quasi-periodic system whose coefficient matrix satisfies a nonresonance condition and a smoothness condition and if (1.2) has

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full spectrum, then (1.2) has a fundamental matrix with a Floquet-type representation.

However, it cannot be judged directly from the coefficient matrix whether (1.2) has full spectrum or not. Thus, it is required to give a condition for judging this to the coefficient matrix.

The aim of this paper is to obtain such a sufficient condition which is easy to examine its validity, if the given system is

$$\dot{x}=(\Lambda+\varepsilon A(t))x \tag{1.3}$$

where

$$x\in \boldsymbol{C}^2\,,\qquad \Lambda=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$$

and ε is a real parameter sufficiently close to 0. Except for the case when $A(t)$ is supposed to be an almost periodic matrix function as in §2, we suppose that $A(t)$ is a quasi-periodic matrix function whose mean is zero. If $A(t)$ is quasi-periodic, then $A(t)$ can be represented by a Fourier series of the form

$$A(t)\sim\sum_m \hat{A}(m)\exp(m\cdot\omega\sqrt{-1}\,t)$$

where

$$m=(m_1,\,\cdots,\,m_K)\in\boldsymbol{Z}^K\,,\qquad \omega=(\omega_1,\,\cdots,\,\omega_K)\in\boldsymbol{R}^K\,.$$

Whenever $A(t)$ is supposed to be quasi-periodic, we suppose that there exist positive constants c and σ such that

$$|m\cdot\omega|\geqq c|m|^{-\sigma} \tag{1.4}$$

where, for vectors, $|\cdot|$ denotes the Euclidean norm.

Also, we put

$$A(t)=\begin{bmatrix}a_{11}(t) & a_{12}(t)\\ a_{21}(t) & a_{22}(t)\end{bmatrix}.$$

For stating a main result, we introduce the following operators.

DEFINITION 3. For an arbitrary almost periodic function $h(t)$, $\mu(h(t))$ denotes the mean value of $h(t)$, and $\iota(h(t))$ denotes the unique almost periodic solution of

$$\dot{x}=h(t)-\mu(h(t))$$

whose mean value is zero if it exists. Also we put

$$\iota_{ij}h(t)=\iota(a_{ij}(t)h(t)) ,$$
$$\iota_{k,ij}h(t)=\iota^k(a_{ij}(t)h(t)) ,$$
$$\mu_{ij}h(t)=\mu(a_{ij}(t)h(t)) ,$$

for $(i, j)=(1, 2)$, $(2, 1)$ and $k=2, 3$, and

$$\iota_d h(t)=\iota((a_{11}(t)-a_{22}(t))h(t)) ,$$
$$\iota_{2,d}h(t)=\iota^2((a_{11}(t)-a_{22}(t))h(t)) ,$$
$$\mu_d h(t)=\mu((a_{11}(t)-a_{22}(t))h(t)) .$$

Our main result is

THEOREM 1. *Suppose that*
(i) *$A(t)$ is a quasi-periodic matrix function whose mean is zero,*
(ii) *Nonresonance condition* (1.4) *holds,*
(iii) *$A(t)\in C^N(\boldsymbol{R})$, where $N\geqq 8\sigma+4(K+1)$,*
(iv) $\mathrm{Re}\sqrt{\mu_{21}(\iota_{2,21}+\iota_d)1}\neq 0$, *where $\sqrt{}$ denotes a square root whose branch is taken arbitrarily.*
Then, if $|\varepsilon|$ is sufficiently small, (1.3) *has full spectrum.*

There are some results concerning existence of an exponential dichotomy in [2]. We can conclude from them that the linear system has full spectrum, if it can be transformed into the system which satisfies the hypotheses of them. Also, as will be discussed in §2, we can give a sufficient condition for (1.3) having full spectrum to the mean of $A(t)$ in (1.3) by following the proof of Proposition 5.3 of [2], even if $A(t)$ is an almost periodic matrix function. However, the coefficient matrix of the linear system considered in these results of [2] is not given by the form of (1.3). Therefore, for example, it is not easy to know from the results mentioned above whether a linear quasi-periodic system

$$\dot{x}=\left\{\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}+\varepsilon\begin{bmatrix}0 & a(t)\\ \sin t & 0\end{bmatrix}\right\}x \tag{1.5}$$

has full spectrum or not, if $a(t)$ is a quasi-periodic function with mean value zero. While, our theorem states that (1.5) does, if $a(t)$ satisfies (i)-(iii) of our theorem. Namely, concerning more systems having the form of (1.3), we can conclude from our theorem that they have full spectrum.

The proof of Theorem 1 will be done in §§3-8. We shall show that (1.3) can be transformed into a system whose coefficient matrix is sufficiently close to that of a system which has an exponential dichotomy and shall finish the proof by applying Proposition 5.1 of [2] which states the roughness of exponential dichotomies.

§ **2.** We show the following

THEOREM 2. *If, in* (1.3), $A(t)$ *is an almost periodic matrix function, and if*

$$\mathrm{Re}\sqrt{\mu_{21}1}\neq 0\,, \tag{2.1}$$

then (1.3) *has full spectrum, if* $\varepsilon>0$ *is sufficiently small.*

PROOF. Let

$$\mu(A(t))=\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22}\end{bmatrix}$$

be the mean of $A(t)$, and put

$$D=4\varepsilon m_{21}+\varepsilon^2\{(m_{11}-m_{22})^2+4m_{12}m_{21}\}\,,$$

$$P(\varepsilon)=\begin{bmatrix} 2(1+\varepsilon m_{12}) & 2(1+\varepsilon m_{12}) \\ -\varepsilon(m_{11}-m_{22})+\sqrt{D} & -\varepsilon(m_{11}-m_{22})-\sqrt{D}\end{bmatrix}. \tag{2.2}$$

Then, if $\eta=\sqrt{\varepsilon}$, we have

$$\sqrt{D}=2\sqrt{m_{21}}\,\eta(1+o(1)) \tag{2.3}$$

as $\eta\to 0$. From (2.1), we have

$$\mathrm{Re}\sqrt{D}\neq 0 \tag{2.4}$$

if η is sufficiently close to 0. Hence, we have

$$P^{-1}(\varepsilon)=\begin{bmatrix} \dfrac{\varepsilon(m_{11}-m_{22})+\sqrt{D}}{4(1+\varepsilon m_{12})\sqrt{D}} & \dfrac{1}{2\sqrt{D}} \\ -\dfrac{\varepsilon(m_{11}-m_{22})-\sqrt{D}}{4(1+\varepsilon m_{12})\sqrt{D}} & -\dfrac{1}{2\sqrt{D}}\end{bmatrix} \tag{2.5}$$

and

$$P^{-1}(\varepsilon)\{\Lambda+\varepsilon\mu(A(t))\}P(\varepsilon)=\mathrm{diag}\left(\frac{\varepsilon(m_{11}+m_{22})+\sqrt{D}}{2},\ \frac{\varepsilon(m_{11}+m_{22})-\sqrt{D}}{2}\right).$$

Hence, by

$$x=e^{\varepsilon(m_{11}+m_{22})t/2}P(\varepsilon)y\,,$$

we have

$$\dot{y}=\left\{\operatorname{diag}\left(\frac{\sqrt{D}}{2},\ \frac{-\sqrt{D}}{2}\right)+\varepsilon C(t,\varepsilon)\right\}y \tag{2.6}$$

where

$$C(t,\varepsilon)=P^{-1}(\varepsilon)\{A(t)-\mu(A(t))\}P(\varepsilon)\,.$$

On the other hand, from (2.2), (2.3) and (2.5), we have

$$P(\varepsilon)=\begin{bmatrix} 2(1+o(1)) & 2(1+o(1)) \\ o(1) & o(1) \end{bmatrix}$$

$$P^{-1}(\varepsilon)=\begin{bmatrix} O(1) & (4\sqrt{m_{21}}\eta)^{-1}(1+o(1)) \\ O(1) & -(4\sqrt{m_{21}}\eta)^{-1}(1+o(1)) \end{bmatrix}$$

as $\eta\to 0$. Hence, if

$$A(t)-\mu(A(t))=\begin{bmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{bmatrix},$$

then

$$\varepsilon C(t,\varepsilon)=\frac{b_{21}(t)}{2\sqrt{m_{21}}}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\eta+o(\eta)$$

as $\eta\to 0$.

Since $\sqrt{D}$ and $\varepsilon C(t,\varepsilon)$ are holomorphic in the neighborhood of $\eta=0$, we can put

$$\Delta(\eta)=\sum_{k=0}^{\infty}\Delta_k\eta^k=\eta^{-1}\operatorname{diag}\left(\frac{\sqrt{D}}{2},\ \frac{-\sqrt{D}}{2}\right),$$

$$G(t,\eta)=\eta C(t,\varepsilon)=\sum_{k=0}^{\infty}G_k(t)\eta^k\,,$$

where

$$\Delta_0=\operatorname{diag}(\sqrt{m_{21}},\ -\sqrt{m_{21}})\,,$$

$$G_0(t)=\frac{b_{21}(t)}{2\sqrt{m_{21}}}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}.$$

Hence, from (2.6), we have

$$\dot{y}=\{\Delta(\eta)\eta+G(t,\eta)\eta\}y\,.$$

Put

$$\tau=\eta t\ ,$$

then we have

$$\frac{dy}{d\tau}=\{\Delta(\eta)+G(\eta^{-1}\tau,\ \eta)\}y\ . \tag{2.7}$$

Here we can suppose that $\Delta(\eta)$ and $G(t, \eta)$ are holomorphic in the disk $|\eta|<\eta_0$.

Now, put

$$\Delta(\eta)=\mathrm{diag}(\lambda(\eta),\ -\lambda(\eta))\ ,$$

then we can write

$$\lambda(\eta)=\sqrt{m_{21}}+\tilde{\lambda}(\eta)\ , \qquad \tilde{\lambda}(\eta)=o(1)$$

as $\eta\to 0$. Hence, there exists a positive number η_1 such that $|\eta|<\eta_1$ implies

$$|\mathrm{Re}\tilde{\lambda}(\eta)|\leqq\left|\frac{\mathrm{Re}\sqrt{m_{21}}}{2}\right|\ .$$

Hence, if

$$Z(t,\ \eta)=\mathrm{diag}(\exp(\lambda(\eta)t),\ \exp(-\lambda(\eta)t))\ ,$$

$$P=\begin{cases}\mathrm{diag}(1,\ 0) & \text{if}\quad \mathrm{Re}\sqrt{m_{21}}<0\\ \mathrm{diag}(0,\ 1) & \text{if}\quad \mathrm{Re}\sqrt{m_{21}}>0\ ,\end{cases}$$

and if $|\eta|<\eta_1$, then

$$\frac{dz}{d\tau}=\Delta(\eta)z$$

has an exponential dichotomy

$$|Z(t,\ \eta)PZ^{-1}(s,\ \eta)|\leqq e^{-\nu(t-s)}\ , \qquad t\geqq s\ ,$$
$$|Z(t,\ \eta)(I-P)Z^{-1}(s,\ \eta)|\leqq e^{-\nu(s-t)}\ , \qquad s\geqq t\ ,$$

where $\nu=|\mathrm{Re}\sqrt{m_{21}}|/2$.

Since $G(t, \eta)$ is almost periodic in t uniformly for η, $G(t, \eta)$ is uniformly bounded. Also, since the mean of $G(t, \eta)$ is zero, it follows from Theorem 3.1 of [3] that

$$\lim_{T\to\infty}\frac{1}{T}\int_s^{s+T}G(t,\ \eta)dt$$

converges uniformly for s and η. Thus, by the same line as in the proof of Proposition 5.3 of [2], we conclude that there exists a positive number η_2 such that $|\eta|<\eta_2$ implies that (2.7) has an exponential dichotomy with P. Hence, if η is fixed so that $|\eta|<\min(\eta_0, \eta_1, \eta_2)$, then (1.3) has the same property. Since the characteristic exponents of (1.3) are given by positive and negative numbers in this case, (1.3) has full spectrum. Hence the proof is completed.

§ **3.** Now we start the proof of Theorem 1. Hence, $A(t)$ in (1.3) is a quasi-periodic function with mean zero.

Since we use the scheme introduced in § 3 of [6], we must consider the matrix differential equation

$$\dot{\Phi}=\{\Lambda+\varepsilon A(t)\}\Phi-\Phi\{\Lambda+\varepsilon A(t)\} . \tag{3.1}$$

Put

$$\Phi=\Phi(t, \varepsilon)=\sum_{k=0}^{\infty} \Phi_k(t)\varepsilon^k$$

formally, then, from the formal calculation, we have

$$\dot{\Phi}_0(t)=\Lambda\Phi_0(t)-\Phi_0(t)\Lambda , \tag{3.2}$$

$$\dot{\Phi}_k(t)=\Lambda\Phi_k(t)-\Phi_k(t)\Lambda+F_k(t) , \tag{3.3}$$

$$F_k(t)=A(t)\Phi_{k-1}(t)-\Phi_{k-1}(t)A(t) , \tag{3.4}$$
$$(k=1, 2, \cdots) .$$

By (3.2), we can take

$$\Phi_0(t)=C_0=\begin{bmatrix} c_1 & c_2 \\ 0 & c_1 \end{bmatrix}$$

where c_1 and c_2 are constants.

Put

$$\Phi_k(t)=[\phi_{ij}^{(k)}(t)] , \qquad F_k(t)=[f_{ij}^{(k)}(t)] ,$$

then, from (3.3), we have

$$\dot{\phi}_{11}^{(k)}(t)=\phi_{21}^{(k)}(t)+f_{11}^{(k)}(t) , \tag{3.5}$$

$$\dot{\phi}_{12}^{(k)}(t)=\phi_{22}^{(k)}(t)-\phi_{11}^{(k)}(t)+f_{12}^{(k)}(t) , \tag{3.6}$$

$$\dot{\phi}_{21}^{(k)}(t)=f_{21}^{(k)}(t) , \tag{3.7}$$

$$\dot{\phi}_{22}^{(k)}(t)=-\phi_{21}^{(k)}(t)+f_{22}^{(k)}(t) . \tag{3.8}$$

Now, suppose that $\Phi_r(t)$ $(r=1, 2, \cdots, k-1)$ can be calculated to be quasi-periodic and $\Phi_k(t)$ is known to be quasi-periodic. Then, if

$$\mu(f_{21}^{(k)}(t))=0 \tag{3.9}$$

can be shown, it follows from (3.7) that

$$\phi_{21}^{(k)}(t)=\iota(f_{21}^{(k)}(t))+c_{21}^{(k)} \tag{3.10}$$

where $c_{21}^{(k)}$ is a constant. Substituting (3.10) into (3.5), we have

$$\dot{\phi}_{11}^{(k)}(t)=\iota(f_{21}^{(k)}(t))+c_{21}^{(k)}+f_{11}^{(k)}(t) . \tag{3.11}$$

Hence, if we take

$$c_{21}^{(k)}=-\mu(f_{11}^{(k)}(t)) , \tag{3.12}$$

then

$$\phi_{11}^{(k)}(t)=\iota^2(f_{21}^{(k)}(t))+\iota(f_{11}^{(k)}(t))+c_{11}^{(k)} . \tag{3.13}$$

However, we have from (3.4)

$$\operatorname{trace} F_k(t)=0 .$$

Hence

$$\mu(f_{11}^{(k)}(t))=-\mu(f_{22}^{(k)}(t)) . \tag{3.14}$$

It follows from (3.8), (3.10) and (3.14) that

$$\dot{\phi}_{22}^{(k)}(t)=-\iota(f_{21}^{(k)}(t))-\mu(f_{22}^{(k)}(t))+f_{22}^{(k)}(t) .$$

Hence

$$\phi_{22}^{(k)}(t)=-\iota^2(f_{21}^{(k)}(t))+\iota(f_{22}^{(k)}(t))+c_{22}^{(k)} , \tag{3.15}$$

where $c_{22}^{(k)}$ is a constant. Substituting (3.13) and (3.15) into (3.6), we have

$$\dot{\phi}_{12}^{(k)}(t)=-2\iota^2(f_{21}^{(k)}(t))+\iota(f_{22}^{(k)}(t)-f_{11}^{(k)}(t))+c_{22}^{(k)}-c_{11}^{(k)}+f_{12}^{(k)}(t) .$$

Here we take

$$c_{11}^{(k)}=0 , \qquad c_{22}^{(k)}=-\mu(f_{12}^{(k)}(t)) . \tag{3.16}$$

Then we can take

$$\phi_{12}^{(k)}(t)=-2\iota^3(f_{21}^{(k)}(t))-\iota^2(f_{11}^{(k)}(t)-f_{22}^{(k)}(t))+\iota(f_{12}^{(k)}(t)) . \tag{3.17}$$

Here, for an almost periodic function $h(t)$, we regard $\iota(h(t))$ as a product of ι and $h(t)$. Also we regard $\mu(h(t))$ as a product of μ and $h(t)$. Then, if we notice (3.12) and (3.16), we can rewrite (3.10), (3.13), (3.15) and (3.17) into a linear transformation such as

$$\begin{bmatrix}\phi_{11}^{(k)}(t)\\ \phi_{12}^{(k)}(t)\\ \phi_{21}^{(k)}(t)\\ \phi_{22}^{(k)}(t)\end{bmatrix}=\begin{bmatrix}\iota & 0 & \iota^2 & 0\\ -\iota^2 & \iota & -2\iota^3 & \iota^2\\ -\mu & 0 & \iota & 0\\ 0 & -\mu & -\iota^2 & \iota\end{bmatrix}\begin{bmatrix}f_{11}^{(k)}(t)\\ f_{12}^{(k)}(t)\\ f_{21}^{(k)}(t)\\ f_{22}^{(k)}(t)\end{bmatrix}. \tag{3.18}$$

Here the multiplication of the matrix with operators as its entries and the vector with functions as its entries follows the usual rule of the multiplication of a matrix and a vector.

On the other hand, we have from (3.4)

$$\begin{bmatrix}f_{11}^{(k)}(t)\\ f_{12}^{(k)}(t)\\ f_{21}^{(k)}(t)\\ f_{22}^{(k)}(t)\end{bmatrix}=\begin{bmatrix}0 & -a_{21}(t) & a_{12}(t) & 0\\ -a_{12}(t) & d(t) & 0 & a_{12}(t)\\ a_{21}(t) & 0 & -d(t) & -a_{21}(t)\\ 0 & a_{21}(t) & -a_{12}(t) & 0\end{bmatrix}\begin{bmatrix}\phi_{11}^{(k-1)}(t)\\ \phi_{12}^{(k-1)}(t)\\ \phi_{21}^{(k-1)}(t)\\ \phi_{22}^{(k-1)}(t)\end{bmatrix} \tag{3.19}$$

where

$$d(t)=a_{11}(t)-a_{22}(t)\ .$$

Substitute (3.19) into (3.18) and make a multiplication of matrices, then we have

$$\begin{bmatrix}\phi_{11}^{(k)}(t)\\ \phi_{12}^{(k)}(t)\\ \phi_{21}^{(k)}(t)\\ \phi_{22}^{(k)}(t)\end{bmatrix}=\begin{bmatrix}\iota_{2,21} & -\iota_{21} & \iota_{12}-\iota_{2,d} & -\iota_{2,21}\\ -\iota_{12}-2\iota_{3,21} & 2\iota_{2,21}+\iota_d & -2\iota_{2,12}+2\iota_{3,d} & \iota_{12}+2\iota_{3,21}\\ \iota_{21} & \mu_{21} & -\mu_{12}-\iota_d & -\iota_{21}\\ \mu_{12}-\iota_{2,21} & -\mu_d+\iota_{21} & \iota_{2,d}-\iota_{12} & -\mu_{12}+\iota_{2,21}\end{bmatrix}\begin{bmatrix}\phi_{11}^{(k-1)}(t)\\ \phi_{12}^{(k-1)}(t)\\ \phi_{21}^{(k-1)}(t)\\ \phi_{22}^{(k-1)}(t)\end{bmatrix}. \tag{3.20}$$

Put

$$p^{(r)}(t)=\phi_{11}^{(r)}(t)-\phi_{22}^{(r)}(t)\ ,\qquad r=1,2,\cdots .$$

Then, from (3.20), we have

$$p^{(k)}(t)=(2\iota_{2,21}-\mu_{12})p^{(k-1)}(t)-(2\iota_{21}-\mu_d)\phi_{12}^{(k-1)}(t)+2(\iota_{12}-\iota_{2,d})\phi_{21}^{(k-1)}(t)\ , \tag{3.21}$$

$$\begin{cases}\phi_{11}^{(k)}(t)=\iota_{2,21}p^{(k-1)}(t)-\iota_{21}\phi_{12}^{(k-1)}(t)+(\iota_{12}-\iota_{2,d})\phi_{21}^{(k-1)}(t)\ ,\\ \phi_{22}^{(k)}(t)=(\mu_{12}-\iota_{2,21})p^{(k-1)}(t)+(-\mu_d+\iota_{21})\phi_{12}^{(k-1)}(t)+(\iota_{2,d}-\iota_{12})\phi_{21}^{(k-1)}(t)\ .\end{cases} \tag{3.22}$$

From (3.20) and (3.21), we have

$$(3.23)\qquad \begin{bmatrix} p^{(k)}(t) \\ \phi_{12}^{(k)}(t) \\ \phi_{21}^{(k)}(t) \end{bmatrix} = \begin{bmatrix} 2\iota_{2,21}-\mu_{12} & -2\iota_{21}+\mu_d & 2\iota_{12}-2\iota_{2,d} \\ -\iota_{12}-2\iota_{3,21} & 2\iota_{2,21}+\iota_d & -2\iota_{2,12}+2\iota_{3,d} \\ \iota_{21} & \mu_{21} & -\mu_{12}-\iota_d \end{bmatrix} \begin{bmatrix} p^{(k-1)}(t) \\ \phi_{12}^{(k-1)}(t) \\ \phi_{21}^{(k-1)}(t) \end{bmatrix} .$$

Thus we have

LEMMA 1. *Suppose that the following statements are valid:*
(i) $\Phi_r(t)$ $(r=1, 2, \cdots, k-1)$ *can be calculated to be quasi-periodic,*
(ii) $\Phi_k(t)$ *is known to be quasi-periodic,*
(iii) (3.9) *holds.*
Then $\Phi_k(t)$ *can be calculated by* (3.22) *and* (3.23).

Furthermore, we have

$$(3.24)\qquad \mu(f_{21}^{(k)}(t))=\mu_{21}p^{(k-1)}(t)-\mu_d\phi_{21}^{(k-1)}(t) ,$$

which will be used for showing (3.9).

§4. For discussions of the later sections, we here prove two lemmas.

LEMMA 2. *If* $h(t)$ *is a quasi-periodic function satisfying*
(i) $h(t)$ *can be represented by a Fourier expansion*

$$h(t) \sim \sum_{m\in Z^K} \hat{h}(m)\exp(m\cdot\omega\sqrt{-1}\,t) , \qquad \omega\in R^K ,$$

(ii) *Nonresonance condition* (1.4) *holds,*
(iii) $h(t)\in C^M(R)$, *where* $M\geqq k\sigma+K+1$ *for some* $k\in N$,
then, for $r=1, 2, \cdots, k$, *there exists* $\iota^r(h(t))$ *such that*

$$\mathrm{Mod}(\iota^r(h(t)))=\mathrm{Mod}(h(t)) ,$$
$$\iota^r(h(t))\in C^{M-r\sigma-K-1}(R) .$$

Here, "Mod" *denotes the module of an almost periodic function.*

For the proof, we need the following

LEMMA 3 ([4], Lemma 2). *Put*

$$D_M=\{h\colon T^K\to C\ ;\ h(\theta)\sim \sum_{m\in Z^K}\hat{h}(m)\exp(m\cdot\theta\sqrt{-1})\ (\theta\in T^K) ,\ \sup_{m\in Z^K}|m|^M|\hat{h}(m)|<\infty\}$$

where T^K *denotes a K-dimensional torus and M is a positive integer. Then*

$$D_{M+K+1}\subset C^M(T^K)\subset D_M .$$

PROOF OF LEMMA 2. It follows from (1.4) and Lemma 3 that, if $r=1, \cdots, k$, we have

$$\left|\frac{\hat{h}(m)}{(m\cdot\omega\sqrt{-1})^r}\right| \leqq \frac{|m|^{r\sigma}|\hat{h}(m)|}{c^r} \leqq \frac{L}{|m|^{M-r\sigma}}, \tag{4.1}$$

where L is a constant. On the other hand, we have

$$M-k\sigma \geqq K+1 .$$

Hence

$$\sum_{m\in Z^K-\{0\}} \frac{\hat{h}(m)}{(m\cdot\omega\sqrt{-1})^r} \exp(m\cdot\omega\sqrt{-1}\, t) \tag{4.2}$$

converges uniformly and therefore (4.2) is equal to $\iota^r(h(t))$. This shows the existence of $\iota^r(h(t))$ together with $\mathrm{Mod}(\iota^r(h(t)))=\mathrm{Mod}(h(t))$.

Furthermore, we have from (4.1)

$$\iota^r(h(t)) \in D_{M-r\sigma} .$$

Hence it follows from Lemma 3 that

$$\iota^r(h(t)) \in C^{M-r\sigma-K-1}(\boldsymbol{R}) .$$

LEMMA 4. *If $f(t)$ and $g(t)$ are almost periodic functions such that $\iota(f(t))$ and $\iota(g(t))$ exist, then we have*

$$\mu(f(t)\iota(g(t))) = -\mu(\iota(f(t))g(t)) , \tag{4.3}$$

$$\mu(f(t)\iota(f(t)))=0 . \tag{4.4}$$

PROOF. Since

$$\frac{d}{dt}\{\iota(f(t))\iota(g(t))\} = \{f(t)-\mu(f(t))\}\iota(g(t)) + \iota(f(t))\{g(t)-\mu(g(t))\} ,$$

we have

$$\mu\left(\frac{d}{dt}\{\iota(f(t))\iota(g(t))\}\right) = \mu(f(t)\iota(g(t))) + \mu(\iota(f(t))g(t)) .$$

Since the right-hand side of this equality is equal to zero, we have (4.3).

Take $f(t)=g(t)$ especially, then we have (4.4) from (4.3).

§5. In this section, we show that, for $k=0, 1, 2, 3, 4$, $\Phi_k(t)$ can be determined to be quasi-periodic.

First, take

$$\begin{bmatrix} p^{(0)}(t) \\ \phi_{12}^{(0)}(t) \\ \phi_{21}^{(0)}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} .$$

Then, from (3.24), we have

$$\mu(f_{21}^{(1)}(t)) = \mu_{21}(p^{(0)}(t)) - \mu_d(\phi_{21}^{(0)}(t)) = 0 .$$

Hence, from (3.23) and Lemma 2, we have

$$\begin{bmatrix} p^{(1)}(t) \\ \phi_{12}^{(1)}(t) \\ \phi_{21}^{(1)}(t) \end{bmatrix} = \begin{bmatrix} -2\iota_{21}1 \\ (2\iota_{2,21} + \iota_d)1 \\ 0 \end{bmatrix} ,$$

and

$$p^{(1)}(t) \in C^{N-\sigma-K-1}(\boldsymbol{R}) , \quad \phi_{12}^{(1)}(t) \in C^{N-2\sigma-K-1}(\boldsymbol{R}) , \quad \phi_{21}^{(1)}(t) \in C^{\infty}(\boldsymbol{R}) .$$

Hence, from (3.24) and Lemma 4, we have

$$\mu(f_{21}^{(2)}(t)) = -2\mu_{21}\iota_{21}1 = 0 .$$

Hence, we have from (3.23) and Lemma 2,

$$\begin{aligned} p^{(2)}(t) &= -4\iota_{2,21}\iota_{21}1 + 2\mu_{12}\iota_{21}1 - 4\iota_{21}\iota_{2,21}1 - 2\iota_{21}\iota_d 1 + 2\mu_d\iota_{2,21}1 , \\ \phi_{12}^{(2)}(t) &= 2\iota_{12}\iota_{21}1 + 4\iota_{3,21}\iota_{21}1 + 4\iota_{2,21}\iota_{2,21}1 + 2\iota_{2,21}\iota_d 1 + 2\iota_d\iota_{2,21}1 + \iota_d\iota_d 1 , \\ \phi_{21}^{(2)}(t) &= -2\iota_{21}\iota_{21}1 + 2\mu_{21}\iota_{2,21}1 + \mu_{21}\iota_d 1 , \end{aligned}$$

$$p^{(2)}(t) \in C^{N-3\sigma-2(K+1)}(\boldsymbol{R}) , \quad \phi_{12}^{(2)}(t) \in C^{N-4\sigma-2(K+1)}(\boldsymbol{R}) , \quad \phi_{21}^{(2)}(t) \in C^{N-2\sigma-2(K+1)}(\boldsymbol{R}) .$$

Hence, we have from (3.24)

$$\mu(f_{21}^{(3)}(t)) = -4\mu_{21}\iota_{2,21}\iota_{21}1 - 4\mu_{21}\iota_{21}\iota_{2,21}1 - 2\mu_{21}\iota_{21}\iota_d 1 + 2\mu_d\iota_{21}\iota_{21}1 .$$

On the other hand, it follows from Lemma 4 that

$$\begin{aligned} \mu_d\iota_{21}\iota_{21}1 &= \mu(d(t)\iota(a_{21}(t)\iota(a_{21}(t)))) \\ &= -\mu(\iota(d(t))a_{21}(t)\iota(a_{21}(t))) \\ &= \mu(a_{21}(t)\iota(a_{21}(t)\iota(d(t)))) \\ &= \mu_{21}\iota_{21}\iota_d 1 . \end{aligned}$$

In the similar manner, we have

$$\mu_{21}\iota_{2,21}\iota_{21}1 = -\mu_{21}\iota_{21}\iota_{2,21}1 .$$

Hence we have

$$\mu(f_{21}^{(3)}(t))=0 \ .$$

Also it follows from (3.23) and Lemma 2 that

$$\begin{aligned}
p^{(3)}(t)= & -8\iota_{2,21}\iota_{2,21}\iota_{21}1+4\iota_{2,21}1\times\mu_{12}\iota_{21}1-8\iota_{2,21}\iota_{21}\iota_{2,21}1 \\
& -4\iota_{2,21}\iota_{21}\iota_{d}1+4\iota_{2,21}1\times\mu_{d}\iota_{2,21}1 \\
& +4\mu_{12}\iota_{2,21}\iota_{21}1+4\mu_{12}\iota_{21}\iota_{2,21}1+2\mu_{12}\iota_{21}\iota_{d}1 \\
& -4\iota_{21}\iota_{12}\iota_{21}1-8\iota_{21}\iota_{3,21}\iota_{21}1-8\iota_{21}\iota_{2,21}\iota_{2,21}1 \\
& -4\iota_{21}\iota_{2,21}\iota_{d}1-4\iota_{21}\iota_{d}\iota_{2,21}1-2\iota_{21}\iota_{d}\iota_{d}1 \\
& +2\mu_{d}\iota_{12}\iota_{21}1+4\mu_{d}\iota_{3,21}\iota_{21}1+4\mu_{d}\iota_{2,21}\iota_{2,21}1 \\
& +2\mu_{d}\iota_{2,21}\iota_{d}1+2\mu_{d}\iota_{d}\iota_{2,21}1+\mu_{d}\iota_{d}\iota_{d}1 \\
& -4\iota_{12}\iota_{21}\iota_{21}1+4\iota_{12}1\times\mu_{21}\iota_{2,21}1+2\iota_{12}1\times\mu_{21}\iota_{d}1 \\
& +4\iota_{2,d}\iota_{21}\iota_{21}1-4\iota_{2,d}1\times\mu_{21}\iota_{2,21}1-2\iota_{2,d}1\times\mu_{21}\iota_{d}1 \ , \\
\phi_{21}^{(3)}(t)= & -4\iota_{21}\iota_{2,21}\iota_{21}1+2\iota_{21}1\times\mu_{12}\iota_{21}1-4\iota_{21}\iota_{21}\iota_{2,21}1 \\
& -2\iota_{21}\iota_{21}\iota_{d}1+2\iota_{21}1\times\mu_{d}\iota_{2,21}1+2\mu_{21}\iota_{12}\iota_{21}1 \\
& +4\mu_{21}\iota_{3,21}\iota_{21}1+4\mu_{21}\iota_{2,21}\iota_{2,21}1+2\mu_{21}\iota_{2,21}\iota_{d}1 \\
& +2\mu_{21}\iota_{d}\iota_{2,21}1+\mu_{21}\iota_{d}\iota_{d}1+2\mu_{12}\iota_{21}\iota_{21}1 \\
& +2\iota_{d}\iota_{21}\iota_{21}1-2\iota_{d}1\times\mu_{21}\iota_{2,21}1-\iota_{d}1\times\mu_{21}\iota_{d}1 \ .
\end{aligned}$$

Hence

$$\begin{aligned}
\mu_{21}p^{(3)}(t)= & -8\mu_{21}\iota_{2,21}\iota_{2,21}\iota_{21}1+4\mu_{21}\iota_{2,21}1\times\mu_{12}\iota_{21}1-8\mu_{21}\iota_{2,21}\iota_{21}\iota_{2,21}1 \\
& -4\mu_{21}\iota_{2,21}\iota_{21}\iota_{d}1+4\mu_{21}\iota_{2,21}1\times\mu_{d}\iota_{2,21}1 \\
& -4\mu_{21}\iota_{21}\iota_{12}\iota_{21}1-8\mu_{21}\iota_{21}\iota_{3,21}\iota_{21}1-8\mu_{21}\iota_{21}\iota_{2,21}\iota_{2,21}1 \\
& -4\mu_{21}\iota_{21}\iota_{2,21}\iota_{d}1-4\mu_{21}\iota_{21}\iota_{d}\iota_{2,21}1-2\mu_{21}\iota_{21}\iota_{d}\iota_{d}1 \\
& -4\mu_{21}\iota_{12}\iota_{21}\iota_{21}1+4\mu_{21}\iota_{12}1\times\mu_{21}\iota_{2,21}1+2\mu_{21}\iota_{12}1\times\mu_{21}\iota_{d}1 \\
& +4\mu_{21}\iota_{2,d}\iota_{21}\iota_{21}1-4\mu_{21}\iota_{2,d}1\times\mu_{21}\iota_{2,21}1-2\mu_{21}\iota_{2,d}1\times\mu_{21}\iota_{d}1 \ , \\
\mu_{d}\phi_{21}^{(3)}(t)= & -4\mu_{d}\iota_{21}\iota_{2,21}\iota_{21}1+2\mu_{d}\iota_{21}1\times\mu_{12}\iota_{21}1-4\mu_{d}\iota_{21}\iota_{21}\iota_{2,21}1 \\
& -2\mu_{d}\iota_{21}\iota_{21}\iota_{d}1+2\mu_{d}\iota_{21}1\times\mu_{d}\iota_{2,21}1+2\mu_{d}\iota_{d}\iota_{21}\iota_{21}1 \ .
\end{aligned}$$

On the other hand, it follows from Lemma 4 that

$$\begin{aligned}
& \mu_{21}\iota_{2,21}1\times\mu_{12}\iota_{21}1=-\mu_{21}\iota_{2,21}1\times\mu_{21}\iota_{12}1 \ , \\
& \mu_{21}\iota_{2,21}1\times\mu_{d}\iota_{2,21}1=\mu_{21}\iota_{2,21}1\times\mu_{21}\iota_{2,d}1 \ , \\
& \mu_{21}\iota_{12}1\times\mu_{21}\iota_{d}1=\mu_{12}\iota_{21}1\times\mu_{d}\iota_{21}1 \ , \\
& \mu_{21}\iota_{2,d}1\times\mu_{21}\iota_{d}1=-\mu_{d}\iota_{2,21}1\times\mu_{d}\iota_{21}1 \ .
\end{aligned}$$

Also we have

$$\begin{aligned}\mu_{21}\iota_{2,21}\iota_{2,21}\iota_{21}1 &= \mu(a_{21}(t)\iota^2(a_{21}(t)\iota^2(a_{21}(t)\iota(a_{21}(t)))))\\ &= \mu(\iota^2(a_{21}(t))a_{21}(t)\iota^2(a_{21}(t)\iota(a_{21}(t))))\\ &= \mu(\iota^2(a_{21}(t)\iota^2(a_{21}(t)))a_{21}(t)\iota(a_{21}(t)))\\ &= -\mu(a_{21}(t)\iota(a_{21}(t)\iota^2(a_{21}(t)\iota^2(a_{21}(t)))))\\ &= -\mu_{21}\iota_{21}\iota_{2,21}\iota_{2,21}1.\end{aligned}$$

In the similar manner, we have

$$\mu_{21}\iota_{2,21}\iota_{21}\iota_{2,21}1 = \mu_{21}\iota_{21}\iota_{3,21}\iota_{21}1 = \mu_d\iota_{21}\iota_{21}\iota_d 1 = 0 ,$$

$$\begin{aligned}\mu_{21}\iota_{2,21}\iota_{21}\iota_d 1 &= \mu_d\iota_{21}\iota_{21}\iota_{2,21}1 ,\\ \mu_{21}\iota_{21}\iota_{12}\iota_{21}1 &= -\mu_{21}\iota_{12}\iota_{21}\iota_{21}1 ,\\ \mu_{21}\iota_{21}\iota_{2,21}\iota_d 1 &= \mu_d\iota_{21}\iota_{2,21}\iota_{21}1 ,\\ \mu_{21}\iota_{21}\iota_d\iota_{2,21}1 &= \mu_{21}\iota_{2,d}\iota_{21}\iota_{21}1 ,\\ \mu_{21}\iota_{21}\iota_d\iota_d 1 &= -\mu_d\iota_d\iota_{21}\iota_{21}1 .\end{aligned}$$

Hence, from (3.24), we have

$$\mu(f_{21}^{(4)}(t)) = \mu_{21}p^{(3)}(t) - \mu_d\phi_{21}^{(3)}(t) = 0 .$$

Hence, from (3.23) and Lemma 2, we have

$$p^{(4)}(t) \in C^{N-7\sigma-4(K+1)}(\boldsymbol{R}) , \quad \phi_{12}^{(4)}(t) \in C^{N-8\sigma-4(K+1)}(\boldsymbol{R}) , \quad \phi_{21}^{(4)}(t) \in C^{N-6\sigma-4(K+1)}(\boldsymbol{R}) ,$$

and $p^{(4)}(t)$, $\phi_{12}^{(4)}(t)$, and $\phi_{21}^{(4)}(t)$ can be determined to be quasi-periodic.

Thus we conclude that $\Phi_k(t)$ $(k=1, 2, 3, 4)$ can be determined to be quasi-periodic.

§6. According to the scheme introduced in §3 of [6], we here represent a fundamental matrix of

(6.1) $$\dot{x} = C(t)x$$

where

$$C(t) = \begin{bmatrix} \alpha_{11}(t) & \alpha_{12}(t) \\ \alpha_{21}(t) & \alpha_{22}(t) \end{bmatrix} ,$$

by using a solution of the matrix differential equation

(6.2) $$\dot{\Phi} = C(t)\Phi - \Phi C(t) .$$

Suppose that we have already obtained a solution

$$\Phi(t)=\begin{bmatrix}\phi_{11}(t) & \phi_{12}(t)\\ \phi_{21}(t) & \phi_{22}(t)\end{bmatrix}$$

of (6.2) which is not a scalar matrix. Also, put

$$p(t)=\phi_{11}(t)-\phi_{22}(t)\ ,$$
$$R=p(t)^2+4\phi_{12}(t)\phi_{21}(t)\ .$$

Then we have

Lemma 5. *If $R\neq 0$ and $\phi_{12}(t)\neq 0$ for all $t\in\boldsymbol{R}$, then there exists a fundamental matrix $X(t)$ of* (6.1) *which can be represented by*

$$X(t)=\begin{bmatrix}2\phi_{12}(t) & 2\phi_{12}(t)\\ -p(t)+\sqrt{R} & -p(t)-\sqrt{R}\end{bmatrix}\operatorname{diag}\Bigl(\exp\Bigl(\int_0^t\Bigl(\alpha_{22}(s)+\frac{p(s)+\sqrt{R}}{2\phi_{12}(s)}\alpha_{12}(s)\Bigr)ds\Bigr),\ \exp\Bigl(\int_0^t\Bigl(\alpha_{22}(s)+\frac{p(s)-\sqrt{R}}{2\phi_{12}(s)}\alpha_{12}(s)\Bigr)ds\Bigr)\Bigr)\ . \tag{6.3}$$

Proof. The general solution of (6.2) is equal to

$$X(t)CX^{-1}(t)$$

where C is a constant matrix. Hence the eigenvalues of $\Phi(t)$ are constants and the discriminant R of the characteristic equation of $\Phi(t)$ is a constant. The eigenvalues of $\Phi(t)$ are given by

$$\frac{\phi_{11}(t)+\phi_{22}(t)\pm\sqrt{R}}{2}\ .$$

Since $R\neq 0$, $\Phi(t)$ can be changed into a diagonal matrix by

$$V(t)=\begin{bmatrix}2\phi_{12}(t) & 2\phi_{12}(t)\\ -p(t)+\sqrt{R} & -p(t)-\sqrt{R}\end{bmatrix}.$$

Also we have

$$V^{-1}(t)=\begin{bmatrix}\dfrac{p(t)+\sqrt{R}}{4\phi_{12}(t)\sqrt{R}} & \dfrac{1}{2\sqrt{R}}\\ -\dfrac{p(t)-\sqrt{R}}{4\phi_{12}(t)\sqrt{R}} & -\dfrac{1}{2\sqrt{R}}\end{bmatrix}.$$

It follows from (6.2) that

$$\begin{bmatrix} \dot{\phi}_{11}(t) & \dot{\phi}_{12}(t) \\ \dot{\phi}_{21}(t) & \dot{\phi}_{22}(t) \end{bmatrix} = \begin{bmatrix} \alpha_{12}(t)\phi_{21}(t)-\phi_{12}(t)\alpha_{21}(t) & (\alpha_{11}(t)-\alpha_{22}(t))\phi_{12}(t)-\alpha_{12}(t)p(t) \\ \alpha_{21}(t)p(t)-(\alpha_{11}(t)-\alpha_{22}(t))\phi_{21}(t) & \alpha_{21}(t)\phi_{12}(t)-\phi_{21}(t)\alpha_{12}(t) \end{bmatrix}.$$

Hence we have

$$\dot{V}(t) = \begin{bmatrix} 2(\alpha_{11}(t)-\alpha_{22}(t))\phi_{12}(t)-2\alpha_{12}(t)p(t) & 2(\alpha_{11}(t)-\alpha_{22}(t))\phi_{12}(t)-2\alpha_{12}(t)p(t) \\ -2(\alpha_{12}(t)\phi_{21}(t)-\phi_{12}(t)\alpha_{21}(t)) & -2(\alpha_{12}(t)\phi_{21}(t)-\phi_{12}(t)\alpha_{21}(t)) \end{bmatrix}.$$

Hence, it follows from Theorem 1 of [6] that (6.1) can be transformed into

$$\dot{y} = \operatorname{diag}\left(\alpha_{22}(t)+\frac{p(t)+\sqrt{R}}{2\phi_{12}(t)}\alpha_{12}(t),\ \alpha_{22}(t)+\frac{p(t)-\sqrt{R}}{2\phi_{12}(t)}\alpha_{12}(t)\right)y$$

by $x=V(t)y$ and therefore we can obtain (6.3).

§ 7. Now, put

$$\Phi(t,\ \varepsilon)=\sum_{k=0}^{\infty}\Phi_k(t)\varepsilon^k$$

where $\Phi_k(t)$ $(k=0,\ 1,\ 2,\ 3,\ 4)$ are quasi-periodic solutions of (3.3) whose existence is shown in § 3, and $\Phi_k(t)$ $(k=5,\ 6,\ \cdots)$ are arbitrary solutions of (3.3) such that $\sum_{k=5}^{\infty}\Phi_k(t)\varepsilon^k$ converges uniformly for t in a compact set in the neighborhood of $\varepsilon=0$. Then $\Phi(t,\ \varepsilon)$ is a solution of (3.1). The existence of such $\Phi_k(t)$ $(k=5,\ 6,\ \cdots)$ can be shown as follows.

The general solution of (3.1) is given by

$$X(t,\ \varepsilon)C(\varepsilon)X^{-1}(t,\ \varepsilon) \tag{7.1}$$

where $C(\varepsilon)$ is a matrix independent of t and $X(t,\ \varepsilon)$ is a fundamental matrix of (1.3). If we take $X(t,\ \varepsilon)$ so that $X(0,\ \varepsilon)=I$, then $X(t,\ \varepsilon)$ is holomorphic in the neighborhood of $\varepsilon=0$. Also, we take $C(\varepsilon)$ to be holomorphic in the neighborhood of $\varepsilon=0$. Then $\Phi(t,\ \varepsilon)$ is the same. Hence, we can put

$$X(t,\ \varepsilon)=\sum_{k=0}^{\infty}X_k(t)\varepsilon^k\ , \qquad X^{-1}(t,\ \varepsilon)=\sum_{k=0}^{\infty}\tilde{X}_k(t)\varepsilon^k\ ,$$

$$C(\varepsilon)=\sum_{k=0}^{\infty}C_k\varepsilon^k\ ,$$

and therefore (7.1) is equal to

$$\sum_{k=0}^{\infty}\sum_{q_1+q_2+q_3=k}X_{q_1}(t)C_{q_2}\tilde{X}_{q_3}(t)\varepsilon^k\ .$$

Since $X_k(t), \tilde{X}_k(t)$ $(k=0, 1, 2, \cdots)$ and $\Phi_k(t)$ $(k=0, 1, 2, 3, 4)$ have been already determined, only C_k $(k=5, 6, \cdots)$ can be taken arbitrarily. However, if we take C_k $(k=5, 6, \cdots)$ so that

$$\sum_{k=5}^{\infty} C_k \varepsilon^k$$

converges, and if we put

$$\Phi_k(t) = \sum_{q_1+q_2+q_3=k} X_{q_1}(t) C_{q_2} \tilde{X}_{q_3}(t) ,$$

then the above $\Phi_k(t)$ $(k=5, 6, \cdots)$ can be obtained.

Let $\phi_{ij}(t, \varepsilon)$ denote the (i, j)-entries of $\Phi(t, \varepsilon)$ and put

$$p(t, \varepsilon) = \phi_{11}(t, \varepsilon) - \phi_{22}(t, \varepsilon) ,$$
$$R(\varepsilon) = p(t, \varepsilon)^2 + 4\phi_{12}(t, \varepsilon)\phi_{21}(t, \varepsilon) .$$

Then we have

LEMMA 6. *If $R(\varepsilon) = \xi \varepsilon^2 (1+o(1))$ as $\varepsilon \to 0$, and if $\operatorname{Re}\sqrt{\xi} \neq 0$, then there exists a matrix function $W(t, \varepsilon)$ which is holomorphic in the neighborhood of $\varepsilon = 0$ and is quasi-periodic in t uniformly for ε such that, if $|\varepsilon|$ is sufficiently small, $x = W(t, \varepsilon)y$ transforms (1.3) into*

$$\dot{y} = (\Delta(t)\varepsilon + G(t, \varepsilon))y \tag{7.2}$$

where $\Delta(t)$ is a diagonal matrix whose entries are quasi-periodic functions and $G(t, \varepsilon)$ is a matrix whose entries are holomorphic in the neighborhood of $\varepsilon = 0$ and is quasi-periodic in t uniformly for ε such that

$$G(t, \varepsilon) = O(\varepsilon^2)$$

as $\varepsilon \to 0$.

PROOF. Put

$$V(t, \varepsilon) = \begin{bmatrix} 2\phi_{12}(t, \varepsilon) & 2\phi_{12}(t, \varepsilon) \\ -p(t, \varepsilon) + \sqrt{R(\varepsilon)} & -p(t, \varepsilon) - \sqrt{R(\varepsilon)} \end{bmatrix} .$$

Then, since $\phi_{12}(t, \varepsilon) = 1 + o(1)$ as $\varepsilon \to 0$, we have

$$V^{-1}(t, \varepsilon) = \begin{bmatrix} \dfrac{p(t, \varepsilon) + \sqrt{R(\varepsilon)}}{4\phi_{12}(t, \varepsilon)\sqrt{R(\varepsilon)}} & \dfrac{1}{2\sqrt{R(\varepsilon)}} \\ -\dfrac{p(t, \varepsilon) - \sqrt{R(\varepsilon)}}{4\phi_{12}(t, \varepsilon)\sqrt{R(\varepsilon)}} & -\dfrac{1}{2\sqrt{R(\varepsilon)}} \end{bmatrix}$$

$$=\begin{bmatrix} O(1) & \dfrac{1}{2\sqrt{\xi}\,\varepsilon}(1+o(1)) \\ O(1) & -\dfrac{1}{2\sqrt{\xi}\,\varepsilon}(1+o(1)) \end{bmatrix}$$

as $\varepsilon \to 0$. Hence we can put

$$V(t, \varepsilon)=\sum_{k=0}^{\infty} V_k(t)\varepsilon^k , \qquad V^{-1}(t, \varepsilon)=\varepsilon^{-1}\sum_{k=0}^{\infty} \tilde{V}_k(t)\varepsilon^k ,$$

$$\dot{V}(t, \varepsilon)=\sum_{k=0}^{\infty} \dot{V}_k(t)\varepsilon^k .$$

Hence, if

$$D(t, \varepsilon)=V^{-1}(t, \varepsilon)\{(\Lambda+\varepsilon A(t))V(t, \varepsilon)-\dot{V}(t, \varepsilon)\} ,$$

then it follows from Lemma 5 that $D(t, \varepsilon)$ is a diagonal matrix and

$$D(t, \varepsilon)=\Delta(t)\varepsilon+\cdots ,$$

where $\cdots$ denotes a power series of ε starting from the term whose degree is greater than 1 and

$$\Delta(t)=\sum_{q_1+q_2=2} \tilde{V}_{q_1}(t)(\Lambda V_{q_2}(t)-\dot{V}_{q_2}(t))+\sum_{q_1+q_2=1} \tilde{V}_{q_1}(t)A(t)V_{q_2}(t) . \tag{7.3}$$

In fact, if $d_i(t, \varepsilon)$ $(i=1, 2)$ denote diagonal entries of $D(t, \varepsilon)$, then we have from Lemma 5

$$\begin{aligned} d_1(t, \varepsilon)&=a_{22}(t)\varepsilon+\frac{p(t, \varepsilon)+\sqrt{R(\varepsilon)}}{2\phi_{12}(t, \varepsilon)}(1+a_{12}(t)\varepsilon) , \\ d_2(t, \varepsilon)&=a_{22}(t)\varepsilon+\frac{p(t, \varepsilon)-\sqrt{R(\varepsilon)}}{2\phi_{12}(t, \varepsilon)}(1+a_{12}(t)\varepsilon) . \end{aligned} \tag{7.4}$$

Since $\phi_{12}(t, \varepsilon)=1+o(1)$ as $\varepsilon \to 0$, the power series of ε representing $d_i(t, \varepsilon)$ starts from the term whose degree is greater than 1.

Now, put

$$\begin{aligned} &\Psi(t, \varepsilon)=\sum_{k=0}^{4} \Phi_k(t)\varepsilon^k=[\psi_{ij}(t, \varepsilon)] , \\ &p_*(t, \varepsilon)=\psi_{11}(t, \varepsilon)-\psi_{22}(t, \varepsilon) , \\ &R_*(t, \varepsilon)=(p_*(t, \varepsilon))^2+4\psi_{12}(t, \varepsilon)\psi_{21}(t, \varepsilon) , \end{aligned}$$

and

$$W(t, \varepsilon)=\left[\frac{2\psi_{12}(t, \varepsilon)}{-p_*(t, \varepsilon)+\sqrt{R_*(t, \varepsilon)}} \quad \frac{2\psi_{12}(t, \varepsilon)}{-p_*(t, \varepsilon)-\sqrt{R_*(t, \varepsilon)}}\right].$$

Then, if

$$\phi_{ij}(t, \varepsilon)=\sum_{k=0}^{\infty} \phi_{ij}^{(k)}(t)\varepsilon^k , \qquad \psi_{ij}(t, \varepsilon)=\sum_{k=0}^{\infty} \psi_{ij}^{(k)}(t)\varepsilon^k ,$$

we have

(7.5) $$\phi_{ij}^{(k)}(t)=\psi_{ij}^{(k)}(t)$$

for $k=0, 1, 2, 3, 4$. Hence, if

$$p(t, \varepsilon)=\sum_{k=0}^{\infty} p^{(k)}(t)\varepsilon^k , \qquad p_*(t, \varepsilon)=\sum_{k=0}^{\infty} p_*^{(k)}(t)\varepsilon^k ,$$

then we have

(7.6) $$p^{(k)}(t)=p_*^{(k)}(t)$$

for $k=0, 1, 2, 3, 4$.

Furthermore, it follows from our hypotheses that we can put

$$R(\varepsilon)=\xi\varepsilon^2+\sum_{k=3}^{\infty} R_k\varepsilon^k ,$$

where

(7.7) $$\xi=(p^{(1)}(t))^2+4\phi_{21}^{(2)}(t)+4\phi_{12}^{(1)}(t)\phi_{21}^{(1)}(t) ,$$

(7.8) $$R_k=\sum_{q=0}^{k} (p^{(k-q)}(t)p^{(q)}(t)+4\phi_{12}^{(k-q)}(t)\phi_{21}^{(q)}(t)) .$$

Hence

$$\sqrt{R(\varepsilon)}=\pm\sqrt{\xi}\,\varepsilon\left\{1+\frac{1}{2}\xi^{-1}R_3\varepsilon+\left(\frac{1}{2}\xi^{-1}R_4-\frac{1}{8}(\xi^{-1}R_3)^2\right)\varepsilon^2+\cdots\right\} .$$

Hence, if

$$\sqrt{R(\varepsilon)}=\sum_{k=0}^{\infty} \beta^{(k)}\varepsilon^k , \qquad \sqrt{R_*(t, \varepsilon)}=\sum_{k=0}^{\infty} \beta_*^{(k)}(t)\varepsilon^k ,$$

then it follows from (7.5), (7.6) and the calculation similar to that for obtaining (7.7) and (7.8) that

(7.9) $$\beta^{(k)}=\beta_*^{(k)}(t)$$

for $k=0, 1, 2$. Hence, if

$$W(t, \varepsilon)=\sum_{k=0}^{\infty} W_k(t)\varepsilon^k ,$$

then, from (7.5), (7.6) and (7.9), we have

$$W_k(t)=V_k(t) , \qquad \dot{W}_k(t)=\dot{V}_k(t) \tag{7.10}$$

for $k=0, 1, 2$.

Moreover, we have

$$W^{-1}(t, \varepsilon) = \begin{bmatrix} \dfrac{p_*(t, \varepsilon)+\sqrt{R_*(t, \varepsilon)}}{4\psi_{12}(t, \varepsilon)\sqrt{R_*(t, \varepsilon)}} & \dfrac{1}{2\sqrt{R_*(t, \varepsilon)}} \\ -\dfrac{p_*(t, \varepsilon)-\sqrt{R_*(t, \varepsilon)}}{4\psi_{12}(t, \varepsilon)\sqrt{R_*(t, \varepsilon)}} & -\dfrac{1}{2\sqrt{R_*(t, \varepsilon)}} \end{bmatrix},$$

which is quasi-periodic in t. On the other hand,

$$\frac{1}{\sqrt{R_*(t, \varepsilon)}}=\pm\frac{1}{\sqrt{\xi}\,\varepsilon}\left\{1-\frac{1}{2}\xi^{-1}R_3\varepsilon+\left(\frac{3}{8}(\xi^{-1}R_3)^2-\frac{1}{2}\xi^{-1}R_4\right)\varepsilon^2+\cdots\right\} .$$

Hence, if

$$W^{-1}(t, \varepsilon)=\varepsilon^{-1}\sum_{k=0}^{\infty} \tilde{W}_k(t)\varepsilon^k ,$$

then

$$\tilde{W}_k(t)=\tilde{V}_k(t) \tag{7.11}$$

for $k=0, 1, 2$.

Hence, it follows from (7.3), (7.10) and (7.11) that $x=W(t)y$ transforms (1.3) into (7.2).

§8. It follows from (7.7) and the calculations of §5 that

$$\xi=4(\iota_{21}1)^2-8\iota_{21}\iota_{21}1+8\mu_{21}\iota_{2,21}1+4\mu_{21}\iota_d 1 .$$

However, since $R(\varepsilon)$ is independent of t, ξ is a constant. Hence we have

$$\xi=\mu(\xi)=4\mu(\iota_{21}1)^2+8\mu_{21}\iota_{2,21}1+4\mu_{21}\iota_d 1 .$$

On the other hand, it follows from Lemma 4 that

$$\mu(\iota_{21}1)^2=-\mu_{21}\iota_{2,21}1 .$$

Hence we have

$$\xi = 4\mu_{21}(\iota_{2,21}+\iota_d)1 .$$

Hence, the hypotheses of Theorem 1 implies the hypotheses of Lemma 6 and therefore we can transform (1.3) into (7.2). Thus, for the proof of Theorem 1, it suffices to show that (7.2) has full spectrum.

From (7.4) we have

$$\Delta(t) = \mathrm{diag}\Big(a_{22}(t)+\frac{1}{2}(p^{(1)}(t)+\sqrt{\xi}),\ a_{22}(t)+\frac{1}{2}(p^{(1)}(t)-\sqrt{\xi})\Big) .$$

Hence, by

$$y = \exp\Big(\int_0^t\Big(a_{22}(\tau)+\frac{1}{2}p^{(1)}(\tau)\Big)d\tau\cdot\varepsilon\Big)z ,$$

we have

$$\dot{z} = (\Delta_0\varepsilon + G(t,\varepsilon))z \tag{8.1}$$

where

$$\Delta_0 = \mathrm{diag}\Big(\frac{\sqrt{\xi}}{2},\ -\frac{\sqrt{\xi}}{2}\Big) .$$

Here the following lemma is required.

LEMMA 7 ([2], Proposition 5.1). *If the system*

$$\dot{x} = C(t)x$$

has an exponential dichotomy

$$|X(t)PX^{-1}(s)| \leqq Ke^{-\alpha(t-s)} , \qquad t \geqq s ,$$
$$|X(t)(I-P)X^{-1}(s)| \leqq Ke^{-\alpha(s-t)} , \qquad s \geqq t ,$$

where $\alpha>0$, $K\geqq 1$ and P is an orthogonal projection, and if

$$\delta = \sup_{t\in R}|\Gamma(t)| \leqq \frac{\alpha}{36K^5} ,$$

then the perturbed system

$$\dot{y} = (C(t)+\Gamma(t))y$$

has an exponential dichotomy

$$|Y(t)PY^{-1}(s)| \leqq 12K^3 e^{-(\alpha-6K^3\delta)(t-s)} , \qquad t \geqq s ,$$
$$|Y(t)(I-P)Y^{-1}(s)| \leqq 12K^3 e^{-(\alpha-6K^3\delta)(s-t)} , \qquad s \geqq t .$$

The system

$$\dot{z}=\varDelta_0\varepsilon z \tag{8.2}$$

has an exponential dichotomy

$$|Z(t)PZ^{-1}(s)|\leqq e^{-(1/2)|\varepsilon\mathrm{Re}\sqrt{\xi}|(t-s)}, \qquad t\geqq s,$$
$$|Z(t)(I-P)Z^{-1}(s)|\leqq e^{-(1/2)|\varepsilon\mathrm{Re}\sqrt{\xi}|(s-t)}, \qquad s\geqq t,$$

where $Z(t)$ is a fundamental matrix of (8.2) of the form

$$Z(t)=\mathrm{diag}\left(\exp\left(\frac{\sqrt{\xi}\,\varepsilon t}{2}\right),\ \exp\left(\frac{-\sqrt{\xi}\,\varepsilon t}{2}\right)\right)$$

and

$$P=\begin{cases}\mathrm{diag}(0,1) & \text{if}\quad \varepsilon\,\mathrm{Re}\sqrt{\varepsilon}>0\,,\\ \mathrm{diag}(1,0) & \text{if}\quad \varepsilon\,\mathrm{Re}\sqrt{\varepsilon}<0\,.\end{cases}$$

Also, since $G(t,\varepsilon)$ is quasi-periodic in t uniformly for ε in a sufficiently small neighborhood of $\varepsilon=0$, there exist constants ε_0 and L such that

$$|G(t,\varepsilon)|\leqq L$$

on $\boldsymbol{R}\times\{\varepsilon\in\boldsymbol{R};\ |\varepsilon|\leqq\varepsilon_0\}$. On the other hand, $G(t,\varepsilon)$ is a holomorphic function with

$$G(t,\varepsilon)=O(\varepsilon^2)\,.$$

Hence we have

$$|G(t,\varepsilon)|\leqq L\left(\frac{|\varepsilon|}{|\varepsilon_0|}\right)^2\Big/\left(1-\frac{|\varepsilon|}{|\varepsilon_0|}\right).$$

Now, take ε such that

$$L\left(\frac{|\varepsilon|}{|\varepsilon_0|}\right)^2\Big/\left(1-\frac{|\varepsilon|}{|\varepsilon_0|}\right)\leqq\frac{|\varepsilon\,\mathrm{Re}\sqrt{\xi}\,|}{72},$$

or equivalently

$$|\varepsilon|\leqq\frac{|\mathrm{Re}\sqrt{\xi}\,|\,|\varepsilon_0|^2}{72L+|\mathrm{Re}\sqrt{\xi}\,|\,|\varepsilon_0|},$$

then it follows from Lemma 7 that (8.1) has an exponential dichotomy with the projection P. Hence, (8.1) has full spectrum. That is, (1.3) also has full spectrum. Now the proof is completed.

Since the mean of $a_{22}(t)+p^{(1)}(t)/2$ is equal to zero, we have

COROLLARY. *Suppose that* (i)-(iv) *of Theorem* 1 *hold, then, if* $|\varepsilon|$ *is sufficiently small,* (1.3) *has an exponential dichotomy.*

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