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On Peak Sets for Certain Function Spaces

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Introduction.

Let A be a function space on a compact Hausdorff space X. In this paper, we show that some theorems on function algebras can be generalized to the case of function spaces A having certain conditions. E. Briem [2] proved the following: Let A be a function algebra. If any peak set for the real part $\mathrm{Re}\,A$ of A is a peak set for A, then $A=C(X)$, where $C(X)$ denotes the Banach algebra of complex-valued continuous functions on X with the supremum norm. In association with the theorem of Briem, we consider the class of function spaces having the condition (A) (see §1). It is a wider class containing the class of function algebras. We here discuss whether theorems on function algebras can be generalized to the case of the class.

In §1, the Bishop antisymmetric decomposition theorem for function spaces is given. This is a generalization of Bishop's theorem [1] on function algebras. In §2 we give some examples of function spaces having (A). In §3 we consider the class $\mathscr{A}$ of function spaces having (A) and give characterizations to assert that $A=C(X)$ for $A\in\mathscr{A}$. These results are generalizations of theorems on function algebras.

§1. Bishop antisymmetric decomposition for function spaces.

Throughout this parer, X will denote a compact Hausdorff space. A is said to be a *function space* (resp. *function algebra*) on X if A is a closed subspace (resp. subalgebra) in $C(X)$ containing constant functions and separating points in X.

Let A be a function space on X. For a subset E in X, we denote

$$A(E)=\{f\in C(E): fg\in A|_E \text{ for any } g\in A|_E\},$$
$$A_R(E)=\{f\in C_R(E): fg\in A|_E \text{ for any } g\in A|_E\}$$

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where $A|_E$ denotes the restriction of A to E and $C_R(E)$ is the set of all real-valued continuous functions on E.

We here see that $1 \in A_R(E) \subset A(E) \subset A|_E$, $A_R(E) = A(E) \cap C_R(E)$ and that $A(E)$ and $A_R(E)$ are both algebras.

Let E be a subset in X. Then we call E an *antisymmetric set* for A if any function in $A_R(E)$ is constant. We easily see that (i) if $\cap_\lambda E_\lambda \neq \emptyset$ for a family $\{E_\lambda\}_{\lambda \in \Lambda}$ of antisymmetric sets for A then $\cup_\lambda E_\lambda$ is an antisymmetric set for A and (ii) the closure $\bar{E}$ of an antisymmetric set E for A is an antisymmetric set for A. Hence the sum of all antisymmetric sets for A containing a point x in X is a maximal antisymmetric set for A which is closed in X. Thus X is decomposed by a family of maximal antisymmetric sets for A. We write $\mathscr{K}(A)$ the family of maximal antisymmetric sets for A.

Let A be a uniformly closed subspace in $C(X)$ or $C_R(X)$. Then a closed subset F in X is called a *peak set* for A if $f(x) = 1$ $(x \in F)$ and $|f(x)| < 1$ $(x \in X \setminus F)$ for an $f \in A$. A *p-set* for A is an intersection of peak sets for A. A closed subset F is called a *BEP-set* for A if for any $f \in A|_F$ and for any closed subset G in X with $G \cap F = \emptyset$ and any $\varepsilon > 0$, there is a $g \in A$ such that $g = f$ on F, $|g(x)| < \varepsilon$ on G and $\|g\| = \|f\|_F$, where $\|g\| = \sup_{x \in X} |g(x)|$ and $\|f\|_F = \sup_{x \in F} |f(x)|$. For a uniformly closed subspace A in $C(X)$, F is a BEP-set for A if and only if $\mu_F \in A^\perp$ for any $\mu \in A^\perp$, where $A^\perp$ denotes the set of measures μ on X such that $\int f\, d\mu = 0$ for any $f \in A$ (cf. [7]).

Now we here consider the Bishop antisymmetric decomposition theorem for function spaces. This is a generalization of Bishop's theorem on function algebras ([1], [3], [9]).

THEOREM 1.1. *Let A be a function space on a compact Hausdorff space X. Then X is decomposed by the family $\mathscr{K}(A)$ of maximal antisymmetric sets for A and the following is satisfied.*

(i) *Any $K \in \mathscr{K}(A)$ is a BEP-set for A:*

(ii) *If $f \in C(X)$ and if $f|_K \in A|_K$ for any $K \in \mathscr{K}(A)$, then $f \in A$.*

PROOF. (i) For $K \in \mathscr{K}(A)$, let F be the intersection of all BEP-sets for A containing K. Then F is the smallest BEP-set for A containing K. We here need only to show that $F = K$. If $K \subsetneqq F$, then there is an $f \in A_R(F)$ which is not constant since K is a maximal antisymmetric set for A. As $f|_K \in A_R(K)$, f is a constant c on K. Hence if we put $E = \{x \in F : f(x) = c\}$, then $K \subset E \subsetneqq F$.

For $\varepsilon > 0$, we put

$$g = -\varepsilon(f - c)^2 + 1 .$$

Since $A_R(F)$ is a real algebra containing 1, $g \in A_R(F)$. For a sufficiently small $\varepsilon > 0$, we see that $g|_E = 1$ and $|g| < 1$ on $F \setminus E$. So E is a peak set for $A|_F$ and $g^n \in A_R(F)$ $(n=1, 2, 3, \cdots)$. Since F is a BEP-set, $\mu_F \in A^\perp$ for any $\mu \in A^\perp$. Here for any $h \in A$, $g^n h|_F \in A|_F$ and $0 = \mu_F(g^n h) \to \mu_F(\chi_E h) = \mu_E(h)$, where χ_E is the characteristic function for E. This shows that $\mu_E \in A^\perp$ for any $\mu \in A^\perp$, that is, E is a BEP-set for A. This is a contradiction since F is the smallest BEP-set for A containing K.

(ii) It is proved by a similar method to the proof due to Glicksberg [9]. We first show that the support F of μ is an antisymmetric set for A for $\mu \in \text{ext}(\text{ball } A^\perp)$, where $\text{ext}(\text{ball } A^\perp)$ is the set of all extreme points of closed unit ball in $A^\perp$. To do this we prove that f is constant for any $f \in A_R(F)$. We here can assume that $0 < f < 1$ on F and f is considered as a function in A by extending on X for convenience. From the definition of $A_R(F)$, $f\mu/\|f\mu\|$, $(1-f)\mu/\|(1-f)\mu\| \in \text{ball } A^\perp$.

Moreover we have

$$\|f\mu\| + \|(1-f)\mu\| = \int |f| d|\mu| + \int |1-f| d|\mu|$$
$$= \int_F f\, d|\mu| + \int_F (1-f) d|\mu| = \int_F d|\mu| = \|\mu\| = 1 ,$$

and

$$\mu = \|f\mu\| \cdot \frac{f\mu}{\|f\mu\|} + \|(1-f)\mu\| \cdot \frac{(1-f)\mu}{\|(1-f)\mu\|} .$$

Since μ is an extreme point of ball $A^\perp$, we have $\mu = f\mu/\|f\mu\|$ and $f = \|f\mu\|$ (a.e. $|\mu|$). So $U = \{x \in X : f(x) \neq \|f\mu\|\}$ is open in X and $|\mu|(U) = 0$. It implies that $\varnothing = U \cap F$ and $f(x) = \|f\mu\|$ on F. This shows that F is an antisymmetric set for A. Hence there is a $K_0 \in \mathscr{K}(A)$ such that $F \subset K_0$.

Now let $f \in C(X)$ and $f|_K \in A|_K$ for any $K \in \mathscr{K}(A)$. Then there is a $g \in A$ with $g|_{K_0} = f|_{K_0}$, and for any $\mu \in \text{ext}(\text{ball } A^\perp)$

$$\int f\, d\mu = \int_{K_0} f\, d\mu = \int_{K_0} g\, d\mu = \int g\, d\mu = 0 .$$

By the Krein-Milman theorem, $\int f\, d\mu = 0$ for any $\mu \in A^\perp$. It follows that $f \in A$.

REMARK. In [6], Ellis discussed the Bishop antisymmetric decomposition for a function space on its Shilov boundary.

We here describe $A(X)$ for some function spaces A on X. Let Γ, D and $\bar{D}$ be $\{z \in \boldsymbol{C} : |z| = 1\}$, $\{z \in \boldsymbol{C} : |z| < 1\}$ and $\{z \in \boldsymbol{C} : |z| \leqq 1\}$ respectively.

EXAMPLES. (1) Let A be the function space on $\bar{D}$ consisting of continuous functions on $\bar{D}$ which are complex harmonic on D. Then $A(\bar{D})$ is the set of constant functions.

(2) Let B be the disc algebra on Γ and $\varphi \in B$, $\varphi \neq 0$ on Γ. Then $A = \varphi^{-1}B$ is a function space on Γ and $A(\Gamma) = B$.

(3) Let B be the disc algebra on Γ. We put $A = \{f \in B : f = \lambda + g, \lambda \in \boldsymbol{C}, g(0) = 0$ and $g(1/2) + g(-1/2) = 0\}$, where $g(\lambda)$ is the value at λ of g which is considered as a function in the disc algebra on $\bar{D}$. Then A is a function space on Γ and $A(\Gamma) = \{f \in B : f = \lambda + g, \lambda \in \boldsymbol{C}$ and $g(0) = g(1/2) = g(-1/2) = 0\}$.

Next we consider the following three conditions for a function space A on X.

(A) Any peak set for A is a peak set for $A(X)$.

(B) For each $K \in \mathscr{K}(A)$, any peak set for $A|_K$ is a peak set for $A(K)$.

(C) For each $K \in \mathscr{K}(A)$, any peak set for $A|_K$ is a peak set for $A_R(K)$.

Here that (C)$\rightarrow$(B) is clear and furthermore we have

THEOREM 1.2. *Let A be a function space on X. Then the following are satisfied*:

(i) *If A has* (A), *then it has* (B).

(ii) *If A satisfies* (C), *then* $A = C(X)$.

(iii) *If A has* (B) *and it is self-adjoint, then* $A = C(X)$.

PROOF. (i) Let $K \in \mathscr{K}(A)$ and F be a peak set for $A|_K$. Then F is a p-set for A since K is a BEP-set for A. So F is a p-set for $A(K)$ by the hypothesis. But since F is a peak set for $A|_K$, it is a G_δ-set in K. It implies that F is a peak set for $A(K)$.

(ii) By Theorem 1.1 it suffices to show that any $K \in \mathscr{K}(A)$ is a singleton. If some $K_0 \in \mathscr{K}(A)$ has at least two points, there is a subset F in K_0 such that F is a peak set for $A|_{K_0}$ and $F \subsetneqq K_0$. By the hypothesis, F is a peak set for $A_R(K_0)$. This is a contradiction since K_0 is an antisymmetric set for A.

(iii) Since A is self-adjoint, for any $K \in \mathscr{K}(A)$, $A|_K$ is self-adjoint and so is $A(K)$. It implies $A_R(K) = \mathrm{Re}\, A(K)$. By the hypothesis, any peak set for $A|_K$ is a peak set for $A(K)$ and it is a peak set for $A_R(K)$. By (ii), the proof is complete.

Theorem 1.2 (ii) implies a theorem of Ellis [6] as follows.

COROLLARY 1.3 (Ellis). *Let A be a function space on X. If any peak set for A is a peak set for $A_R(X)$, then* $A = C(X)$.

PROOF. For any $K \in \mathscr{K}(A)$, $A_R(X)|_K \subset A_R(K)$. Any peak set F for $A|_K$ is a p-set for A and it will follow by the hypothesis that F is a p-set for $A_R(X)$. Hence F is a p-set for $A_R(K)$, and F is a peak set for $A_R(K)$ since it is a G_δ-set in K. From Theorem 1.2 (ii), we have $A=C(X)$.

§ 2. Condition (A).

We here consider function spaces A which satisfy the condition (A), i.e., if any peak set for A is a peak set for $A(X)$.

We first give examples of such function spaces.

EXAMPLES. (1) Any function algebra has (A).

(2) In § 1, Examples (2), A has (A). For, if F is a peak set for A, then there is an $f \in B$ such that $\varphi^{-1}f=1$ on F and $|\varphi^{-1}f|<1$ on $\Gamma \setminus F$. So $f=\varphi$ on F and $f \neq \varphi$ on $\Gamma \setminus F$. Hence $F=\{x \in \Gamma : f-\varphi=0\}$ is a zero-set for B. We easily see that F is a peak set for $B=A(\Gamma)$.

(3) In general, if B is a function algebra on X satisfying that any zero-set for B is a peak set for B and if $\varphi \in B$, $\varphi \neq 0$ on X, then $A=\varphi^{-1}B$ has (A). Examples of B which satisfy the condition above are the disc algebra on Γ and the algebra of generalized analytic functions ([5]).

(4) Let B be the disc algebra on Γ. Then $A=(z-a)^{-1}B+(z-b)^{-1}B$ ($|a|<1$, $|b|<1$) is a function space and satisfies (A) (a special case of (2)).

(5) In § 1, Examples (3), A satisfies (A). For, let $\lambda+f_0 \in A$ be a peaking function of a peak set F for A. Then $F=\{z \in \Gamma : \lambda+f_0(z)-1=0\}$. Hence F is Γ or a set of zero Lebesgue measure on Γ. It is not hard to see that F is a peak set for $A(\Gamma)$.

(6) If $\{A_\lambda\}$ is a family of function spaces having (A), then the direct sum $\bigoplus A_\lambda$ has (A). It is proved in the following proposition.

PROPOSITION 2.1. *Let A_λ be a function space on X_λ having* (A) ($\lambda \in \Lambda$). *Then the direct sum $\bigoplus A_\lambda$ of $\{A_\lambda\}_{\lambda \in \Lambda}$ has* (A).

PROOF. The direct sum $A=\bigoplus A_\lambda$ is regarded as a function space on the one-point compactification $X=\cup X_\lambda \cup \{p\}$ of $\cup X_\lambda$. Let E be a peak set for A. Then there is an $f \in A$ such that $f(x)=1$ on E and $|f(x)|<1$ on $X \setminus E$. If $E \cap X_\lambda \neq \varnothing$, $E \cap X_\lambda$ is a peak set for A_λ. Since A_λ has (A), it is also a peak set for $A_\lambda(X_\lambda)$. So there is an $f_\lambda \in A_\lambda(X_\lambda)$ such that $f_\lambda(x)=1$ on $E \cap X_\lambda$ and $|f_\lambda(x)|<1$ on $X_\lambda \setminus E$. We here put $f_\lambda \equiv 0$ when $E \cap X_\lambda = \varnothing$. If $p \notin E$, there is a finite subset $\Lambda_0 \subset \Lambda$ with $E \subset \cup_{\lambda \in \Lambda_0} X_\lambda$. So by putting $g(x)=f_\lambda(x)$ ($x \in X_\lambda$, $\lambda \in \Lambda_0$), $g(x)=0$ ($x \in X_\lambda$, $\lambda \notin \Lambda_0$) and $g(p)=0$, $E=\{x \in X : g(x)=1\}$ becomes a peak set for $A(X)$. If $p \in E$, we put $g_{\lambda'}(x)=f_{\lambda'}(x)$ ($x \in X_{\lambda'}$), $g_{\lambda'}(x)=1$ ($x \in X_\lambda$, $\lambda \neq \lambda'$) and $g_{\lambda'}(p)=1$ for any fixed $\lambda' \in \Lambda$. Then

$E_{\lambda'}=\{x\in X: g_{\lambda'}(x)=1\}$ is a peak set for $A(X)$. Hence $E=\cap\{E_{\lambda'}: \lambda'$ is any element in $\Lambda\}$ is a p-set for $A(X)$. Since E is a G_δ-set in X, it is a peak set for $A(X)$.

§ 3. Characterizations assert that $A=C(X)$.

Let A be a function space on X satisfying (A). We consider conditions under which A is identical with $C(X)$.

The following is the Stone-Weierstrass theorem for function spaces satisfying (A).

THEOREM 3.1. *Let A have* (A). *If A is self-adjoint, then* $A=C(X)$.

PROOF. By Theorem 1.2 (i), if A has (A), then it has (B). By Theorem 1.2 (iii) we have $A=C(X)$.

A Briem's theorem [2] is generalized as follows:

THEOREM 3.2. *Let A satisfy* (A). *If any peak set for* Re A *is a peak set for A, then* $A=C(X)$.

PROOF. Let F be any peak set for Re A. Then if it can be proved that F is a BEP-set for A, we have $A=C(X)$ by ([12], Theorem 2.2) and this proves the theorem. To do this suppose that F is a peak set for Re A. By the hypothesis F is a peak set for A. Since A has (A), it is a peak set for $A(X)$. Let g be a peaking function in $A(X)$ for F. Then for any $f\in A$ and for any $\mu\in A^\perp$, $0=\mu(g^n f)\to\mu(\chi_F f)=\mu_F(f)\ (n\to\infty)$ and so $\mu_F(f)=0$. This shows that F is a BEP-set for A.

The following is a generalization of a Glicksberg's theorem ([10], [13]) on function algebras.

THEOREM 3.3. *Let A have* (A). *If $A|_F$ is closed in $C(F)$ for any closed subset F in X, then* $A=C(X)$.

We need the following lemma in order to prove the theorem.

LEMMA 3.4. *Let A satisfy the hypothesis in Theorem* 3.3. *Then any peak set for* Re $A(X)$ *is a peak set for* $A(X)$.

PROOF. We use a similar argument to Briem ([2], Prop. 2). Let F be a peak set for Re $A(X)$. Then there is an $a\in A(X)$ such that $a=u+iv$, u and v are real functions, $u=0$ on F and $u<0$ on $X\setminus F$. Since $A(X)$ is a closed subalgebra containing 1, $a_1=\exp a\in A(X)\subset A$. Here $|a_1|=\exp u=1$ on F and $|a_1|=\exp u<1$ on $X\setminus F$. By the hypothesis, $A|_F$

is closed in $C(F)$. So the open mapping theorem implies the existence of a constant c_F such that each $f \in A|_F$ has an extension $g \in A$ with $\|g\| \leqq c_F\|f\|_F$. Since $\exp(-na) \in A(X) \subset A$, there is a $b_n \in A$ such that $b_n = \exp(-na)$ on F and $\|b_n\| \leqq c_F \|\exp(-na)\|_F = c_F$. From that $a_1^n \in A(X)$, it follows $a_1^n b_n \in A$, $\|a_1^n b_n\| \leqq c_F$ and $a_1^n b_n = 1$ on F. Moreover for any compact set G in X with $G \cap F = \emptyset$, we have $a_1^n b_n \Rightarrow 0$ on G. So given $\varepsilon > 0$, by taking a sufficiently large n, $f = a_1^n b_n \in A$ satisfies that $\|f\| \leqq c_F$, $f|_F = 1$ and $|f| \leqq \varepsilon$ on G. Since F is a G_δ-set, there is an $f \in A$ such that $\|f\| = 1$, $f|_F = 1$ and $|f(x)| < 1$ $(x \notin F)$ ([4], Lemma 13), that is, F is a peak set for A. By the hypothesis, F is a peak set for $A(X)$.

Proof of Theorem 3.3. We introduce a relation $\sim$ in X as follows:

$$x \sim y \iff f(x) = f(y) \text{ for any } f \in A(X) .$$

Then $\sim$ is an equivalence relation in X.

We put $\tilde{x} = \{y \in X : y \sim x\}$ for $x \in X$ and $\tilde{X} = \{\tilde{x} : x \in X\}$. By defining the topology in $\tilde{X}$ such that the mapping $x \to \tilde{x}$ from X to $\tilde{X}$ is continuous, $\tilde{X}$ becomes a compact Hausdorff space. By putting $\tilde{f}(\tilde{x}) = f(x)$ for $f \in A(X)$, $A(X)^\sim = \{\tilde{f} : f \in A(X)\}$ becomes a function algebra on $\tilde{X}$. Now if $\tilde{F}$ is a peak set for $\mathrm{Re}\, A(X)^\sim$ then there is an $\tilde{f} \in A(X)^\sim$ such that $\mathrm{Re}\, \tilde{f}(\tilde{x}) = 1$ $(\tilde{x} \in \tilde{F})$ and $|\mathrm{Re}\, \tilde{f}(\tilde{x})| < 1$ $(\tilde{x} \notin \tilde{F})$. If we put $F = \{x \in X : \tilde{x} \in \tilde{F}\}$, F is a closed set in X, $\mathrm{Re}\, f(x) = 1$ $(x \in F)$ and $|\mathrm{Re}\, f(x)| < 1$ $(x \notin F)$. From this F is a peak set for $\mathrm{Re}\, A(X)$, and Lemma 3.4 implies that F is a peak set for $A(X)$. So $\tilde{F}$ is a peak set for $A(X)^\sim$. By a theorem of Briem [2] we have that $A(X)^\sim = C(\tilde{X})$. Hence $A(X)$ is self-adjoint. By the hypothesis any peak set F for A is a peak set for $A(X)$ and a peak set for $\mathrm{Re}\, A(X)$. Since $A(X)$ is self-adjoint, F is a peak set for $A_R(X) = \mathrm{Re}\, A(X)$ and so $A = C(X)$ from Corollary 1.3.

We next generalize a wellknown theorem on function algebras to the case of function spaces having (A) (cf. [8], [11], [3]).

Let A be a function space. Then a closed subset F in X is called an *interpolation set* for A if $A|_F = C(F)$.

Theorem 3.5. *Let A be a function space on X having* (A). *If X is the sum of a sequence $\{F_n\}_{n=1}^\infty$ of interpolation sets for A, then $A = C(X)$.*

We need some lemmas to prove the theorem.

Let A be a function space and $K \in \mathscr{K}(A)$. Then we denote by $\partial(A|_K)$ the Shilov boundary for $A|_K$. Since $A(K) \subset A|_K$, $\partial(A|_K)$ is a boundary for $A(K)$. So $A(K)|_{\partial(A|_K)}$ is a uniformly closed subalgebra in $C(\partial(A|_K))$. A function space A is called to be *essential* if for any proper closed subset

F in X there is an $f \in C(X)$ such that $f(x)=0$ ($x \in F$) and $f \notin A$. We begin with the following

LEMMA 3.6. *If A has* (A), *for any $K \in \mathscr{K}(A)$ a peak set for $A|_{\partial(A|_K)}$ is a peak set for $A(K)|_{\partial(A|_K)}$. And $A(K)|_{\partial(A|_K)}$ is an essential algebra and so $A|_{\partial(A|_K)}$ is essential.*

PROOF. Let F be a peak set for $A|_{\partial(A|_K)}$. Then there is an $f \in A|_{\partial(A|_K)}$ such that $f=1$ on F and $|f|<1$ on $\partial(A|_K) \setminus F$. That $\partial(A|_K)$ is a boundary for $A|_K$ implies the existence of $g \in A|_K$ such that $g|_F=f|_F=1$, $|g|<1$ on $\partial(A|_K) \setminus F$ and $\|g\|_K=1$. So $E=\{x \in K: g(x)=\|g\|=1\}$ is a peak set for $A|_K$ and $F=\partial(A|_K) \cap E$. By (A) E is a peak set for $A(K)$ and so F is a peak set for $A(K)|_{\partial(A|_K)}$. Since any $f \in A_R(K)$ is constant, if $g \in A(K)|_{\partial(A|_K)}$ is real on $\partial(A|_K)$, then it is constant. It follows that $A(K)|_{\partial(A|_K)}$ is an essential algebra. We here show that $A|_{\partial(A|_K)}$ is essential. Suppose otherwise. Then there is a closed subset $E \subsetneqq \partial(A|_K)$ such that if $f \in C(\partial(A|_K))$, $f|_E=0$ then $f \in A|_{\partial(A|_K)}$. For a fixed closed subset F in $\partial(A|_K)$ with $F \cap E=\varnothing$, there is an $f \in C_R(\partial(A|_K))$ such that $f|_F=1$, $f|_E=0$ and $0 \leqq f \leqq 1$. So $f \in A|_{\partial(A|_K)}$. From this F is a p-set for $A|_{\partial(A|_K)}$ and a p-set for $A(K)|_{\partial(A|_K)}$. Since $A(K)|_{\partial(A|_K)}$ is a uniformly closed algebra, F is a BEP-set for $A(K)|_{\partial(A|_K)}$. Similarly, any closed subset which is contained in F is a BEP-set for $A(K)|_{\partial(A|_K)}$. It implies $A(K)|_F=C(F)$ since $1 \in A(K)|_F$. Now for any $\mu \in (A(K)|_{\partial(A|_K)})^{\perp}$ and any $\varepsilon>0$, there is an open subset U in $\partial(A|_K)$ such that $U \supset E$ and $|\mu|(U \setminus E)<\varepsilon$. If $F=\partial(A|_K) \setminus U$, it is a closed subset in $\partial(A|_K)$ and $F \cap E=\varnothing$. By the fact stated above, $A(K)|_F=C(F)$. Suppose that $g \in C(\partial(A|_K))$ and $g|_E=0$. Then $g|_F \in A(K)|_F$ and there is an $h \in A(K)|_{\partial(A|_K)}$ such that $h|_F=g|_F$, $\|h\|=\|g\|_F$ and $|h|<\varepsilon$ on E. Hence

$$
\begin{aligned}
|\mu(g)| &= |\mu(g)-\mu(h)| = |\mu(g-h)| \\
&\leqq \left|\int_E (g-h)d\mu\right| + \left|\int_{U\setminus E}(g-h)d\mu\right| + \left|\int_{\partial(A|_K)\setminus U}(g-h)d\mu\right| \\
&\leqq \int_E |h|d|\mu| + \int_{U\setminus E}|g-h|d|\mu| \\
&\leqq \varepsilon\|\mu\| + \|g-h\|\,|\mu|(U\setminus E) \leqq \varepsilon(\|\mu\|+2\|g\|)\,.
\end{aligned}
$$

Since ε is arbitrary, $\mu(g)=0$. So $g \in A(K)|_{\partial(A|_K)}$. It follows that $A(K)|_{\partial(A|_K)}$ is not essential. This contradiction proves that $A|_{\partial(A|_K)}$ is essential.

LEMMA 3.7. *For each $K \in \mathscr{K}(A)$, any p-set for $A(K)$ is a BEP-set for $A|_K$.*

PROOF. Let F be a p-set for $A(K)$. For any $\mu \in (A|_K)^{\perp}$ and any

$\varepsilon>0$, $|\mu|(U\setminus F)<\varepsilon$ for some open subset U in K containing F. Here take a peak set E for $A(K)$ with $F\subset E\subset U$, and choose $f\in A(K)$ such that $f|_E=1$ and $|f|<1$ on $K\setminus E$. Then since $f^n\in A(K)$ for any n, $f^n g\in A|_K$ for $g\in A|_K$. Hence

$$0=\mu(f^n g)\to\mu(\chi_E\cdot g)=\mu_E(g)\qquad(n\to\infty)$$

and so $\mu_E(g)=0$. Furthermore we have

$$\begin{aligned}|\mu_F(g)|&=|\mu_F(g)-\mu_E(g)|\leqq\|g\|\,|\mu|(E\setminus F)\\&\leqq\|g\|\,|\mu|(U\setminus F)\leqq\|g\|\cdot\varepsilon\,.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, $\mu_F(g)=0$. This implies $\mu_F\in(A|_K)^\perp$. This shows that F is a BEP-set for $A|_K$.

LEMMA 3.8. *Let A has* (A). *Then for any $K\in\mathscr{K}(A)$, $\partial(A|_K)$ is equal to the closure $\bar{P}_{A(K)}$ of $P_{A(K)}$, where $P_{A(K)}=\{x\in K\colon\{x\}$ is a BEP-set for $A(K)\}$.*

PROOF. A BEP-set for $A|_K$ is a p-set for $A|_K$. By (A) it is a p-set for $A(K)$. Since $A(K)$ is an algebra, it is a BEP-set for $A(K)$. Conversely, a BEP-set for $A(K)$ is a p-set for $A(K)$ and it is a BEP-set for $A|_K$ from Lemma 3.7. Hence

$$P_{A(K)}=\{x\in K\colon\{x\}\text{ is a BEP-set for }A|_K\}\,.$$

We first show that if F is a peak set for $A|_K$, then $F\cap P_{A(K)}\neq\emptyset$. If F is a peak set for $A|_K$, then it is a peak set for $A(K)$ by (A). From Lemma 3.7 it is a BEP-set for $A|_K$. Now we put $\mathscr{F}=\{E\colon E$ is a BEP-set for $A|_K$ and $E\subset F\}$. Then $\mathscr{F}$ becomes a partially ordered set by the inclusion. For any chain $\{F_\alpha\}$ in $\mathscr{F}$, $\bigcap_\alpha F_\alpha$ is a BEP-set for $A|_K$. By Zorn's lemma, there is a minimal element F_0 in $\mathscr{F}$. It suffices to show that F_0 is a singleton to prove that $F\cap P_{A(K)}\neq\emptyset$. If F_0 has at least two points there is an $E\subsetneqq F_0$ such that E is a peak set for $A|_{F_0}$. Since F_0 is a BEP-set for $A|_K$, E is a p-set for $A|_K$. By (A) E is a p-set for $A(K)$. By Lemma 3.7, it is a BEP-set for $A|_K$. This is a contradiction since F_0 is a minimal element in $\mathscr{F}$. We next show that $P_{A(K)}$ is a boundary for $A|_K$. Put $F=\{x\in K\colon|f(x)|=\|f\|_K\}$ for $f\in A|_K$. Then $E=\{x\in K\colon\alpha f(x)=\|\alpha f\|_K=\|f\|_K\}$ (some $\alpha\in\boldsymbol{C}$, $|\alpha|=1$) is a non-void peak set for $A|_K$. Take an $x_0\in P_{A(K)}\cap E$. Then

$$|f(x_0)|=\alpha f(x_0)=\|\alpha f\|_K=\|f\|_K\,,$$

and

$$P_{A(K)}\cap F\neq\emptyset\,.$$

This shows that $P_{A(K)}$ is a boundary for $A|_K$. We denote by $\mathrm{Ch}(A|_K)$ the Choquet boundary for $A|_K$. Then we show that $P_{A(K)}\subset \mathrm{Ch}(A|_K)$. Indeed, for each $x\in P_{A(K)}$, for any open set U in K containing x and for any $\varepsilon>0$, there is an $f\in A|_K$ such that $f(x)=\|f\|_K=1$ and $|f|<\varepsilon$ on $K\setminus U$, since $\{x\}$ is a BEP-set for $A|_K$. If μ is a representing measure for x,

$$1=|f(x)|=\left|\int_U f\,d\mu+\int_{K\setminus U} f\,d\mu\right|$$
$$\leqq \mu(U)+\varepsilon .$$

So $\mu(U)\geqq 1-\varepsilon$. Since $\varepsilon>0$ is arbitrary, the support of $\mu=\{x\}$. Hence $x\in \mathrm{Ch}(A|_K)$ and so $P_{A(K)}\subset \mathrm{Ch}(A|_K)$. $\bar{P}_{A(K)}\subset\overline{\mathrm{Ch}(A|_K)}=\partial(A|_K)$. So $\bar{P}_{A(K)}=\partial(A|_K)$.

LEMMA 3.9. *Let A has* (A). *Then if, for any $K\in\mathscr{K}(A)$, $A|_{\bar V}=C(\bar V)$ for some non-void open subset V in $\partial(A|_K)$, then $A=C(X)$.*

PROOF. For any $K\in\mathscr{K}(A)$, $\bar{P}_{A(K)}=\partial(A|_K)$ from Lemma 3.8. By the hypothesis, $V\cap P_{A(K)}\neq\varnothing$. If $x_0\in V\cap P_{A(K)}$, we show that $\partial(A|_K)=\{x_0\}$. If otherwise, there is a $y\in\partial(A|_K)$, $y\neq x_0$. Then $x_0\in U\subset V$, $y\notin U$ for some open subset U in $\partial(A|_K)$. From that $\{x_0\}$ is a BEP-set for $A(K)$, there is a $g\in A(K)$ such that $g(x_0)=\|g\|_K=1$ and $|g|<1/3$ on $\partial(A|_K)\setminus U$. If we put $U_0=\{x\in\partial(A|_K)\colon |1-g(x)|<1/3\}$, then $x_0\in U_0\subset U\subset V$ and $y\notin U_0$.

Put $B=\{z\in \boldsymbol{C}\colon |z|<1/3\}$ and $D=\{z\in\boldsymbol{C}\colon |1-z|<1/3\}$. Then there is a sequence $\{p_n\}$ of polynomials of z such that p_n converges to χ_D uniformly on $B\cup D$ by Runge's theorem. We here show that if $f\in C(\partial(A|_K))$ and $f|_{\partial(A|_K)\setminus U_0}=0$ then $f\in A|_{\partial(A|_K)}$. If it should be proved, $A|_{\partial(A|_K)}$ would be not essential. This is a contradiction by Lemma 3.6 and this shows that $\partial(A|_K)=\{x_0\}$. This means that for any $K\in\mathscr{K}(A)$ K is a singleton. Thus the lemma is proved. Hence to prove the lemma, it suffices to show that if $f\in C(\partial(A|_K))$, $f|_{\partial(A|_K)\setminus U_0}=0$, then $f\in A|_{\partial(A|_K)}$. If $f\in C(\partial(A|_K))$, $f|_{\partial(A|_K)\setminus U_0}=0$, then there is an $h\in A|_K$ with $h=f$ on $\bar V$ since $\bar V\subset K$ and $A|_{\bar V}=C(\bar V)$. Hence $h|_{(\bar V\setminus U_0)}=0$. Since $g(U_0)\subset D$, $g(\partial(A|_K)\setminus V)\subset B$, $p_n\circ g$ converges uniformly to χ_{U_0} on $U_0\cup(\partial(A|_K)\setminus V)$. Of course, $h\cdot(p_n\circ g)\Rightarrow h\cdot\chi_{U_0}=f$ on $\partial(A|_K)$. Since $A(K)$ is an algebra containing 1 and $h\in A|_K$, $h\cdot(p_n\circ g)\in A|_K$. So $f\in A|_{\partial(A|_K)}$.

PROOF OF THEOREM 3.5. Let $X=\bigcup_{n=1}^{\infty}F_n$ for a sequence $\{F_n\}_{n=1}^{\infty}$ of interpolation sets for A. For any $K\in\mathscr{K}(A)$,

$$\partial(A|_K)=\bigcup_{n=1}^{\infty}(\partial(A|_K)\cap F_n) .$$

By Baire's theorem, for some n_0, $\bar V\subset\partial(A|_K)\cap F_{n_0}$, where V is a non-void open subset in $\partial(A|_K)$. Since $A|_{F_{n_0}}=C(F_{n_0})$, $A|_{\bar{V}}=C(\bar{V})$. By Lemma 3.9, $A=C(X)$.

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