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Logarithmic Fano Manifolds Are Simply Connected

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§1. Introduction.

The purpose of this short note is to show that the branched covering method in [6] is also effective in the case of positive first Chern class. Hence there is nothing new from the technical point of view. But it will be worthwhile to point out that even in differential geometry it is useful to study smooth differentiable manifolds through orbifold structures.

Let X be a smooth projective algebraic variety over $\boldsymbol{C}$. X is said to be a Fano manifold if the canonical bundle K_X is negative. By the solution of Calabi's conjecture ([7]), X is Fano if and only if X admits a Kähler metric of positive Ricci curvature. Then by using Myer's theorem ([4]) and the Kodaira vanishing theorem, one can easily see that every Fano manifold is simply connected (cf. [1]).

Let X be a smooth projective algebraic variety over $\boldsymbol{C}$ and let D be a divisor on X with simple normal crossings. The pair (X, D) is said to be a logarithmic Fano manifold if the logarithmic canonical bundle K_X+D is negative. This is a natural generalization of the notion of Fano manifolds.

In this paper, we prove the following theorem.

THEOREM 1. *Let (X, D) be a logarithmic Fano manifold. Then X is simply connected.*

The essential point of the proof is to see that X has an orbifold structure with a Kähler metric of positive Ricci tensor. The rest of the proof is a minor modification of the argument in [1].

REMARK 1. It is plausible that if a projective manifold X over $\boldsymbol{C}$ satisfies that $\kappa(-K_X)=\dim X$, then X is simply connected. Theorem 1 is a partial answer to this problem.

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§ 2. Proof of Theorem 1.

In this section, we shall prove Theorem 1.

Let (X, D) be as in Theorem 1. Let m be a sufficiently large natural number such that $K_X+((m-1)/m)D$ is negative. Let us fix a smooth hermitian metric on the line bundle $\mathcal{O}_X(D)$ and let $\|\ \|$ denote the hermitian norm. Let σ be a holomorphic section of $\mathcal{O}_X(D)$ with divisor D. Then there exists a smooth volume form Ψ on X such that

$$\operatorname{Ric}\|\sigma\|^{-2(m-1)/m}\Psi=-\sqrt{-1}\partial\bar{\partial}\log(\|\sigma\|^{-2(m-1)/m}\Psi)$$

is a smooth Kähler form on X. We set

$$\Omega=\frac{\|\sigma\|^{-2(m-1)/m}\Psi}{(1+\|\sigma\|^{2/m})^2} \tag{1}$$

and

$$\omega=\operatorname{Ric}\Omega\ . \tag{2}$$

Let us cover X by a finite collection of polydisks $\{\Delta_\alpha\}$,

$$\Delta_\alpha=\{(z_\alpha^1, \cdots, z_\alpha^n)\in \boldsymbol{C}^n \mid |z_\alpha^i|<1,\ 1\leqq i\leqq n\} \qquad (n=\dim M)$$

such that if $\Delta_\alpha\cap D\neq\varnothing$

$$\Delta_\alpha\cap D=\{z_\alpha^1\cdots z_\alpha^k=0\}$$

for some k depending on α. Let

$$\tilde{\Delta}_\alpha=\{(t_\alpha^1, \cdots, t_\alpha^n)\in \boldsymbol{C}^n \mid |t_\alpha^i|<1,\ 1\leqq i\leqq n\}$$

be a copy of Δ_α and let $p_\alpha\colon \tilde{\Delta}_\alpha\to\Delta_\alpha$ be a morphism defined by

$$p_\alpha(t_\alpha^1, \cdots, t_\alpha^n)=((t_\alpha^1)^m, \cdots, (t_\alpha^k)^m, t_\alpha^{k+1}, \cdots, t_\alpha^n)\ . \tag{3}$$

One can easily see that if we multiply a sufficiently small positive number to σ we may assume that $p_\alpha^*\omega$ is a smooth Kähler form and $p_\alpha^*\Omega$ is a smooth nondegenerate volume form for every α.

Let us consider the Monge-Ampère equation:

$$\frac{(\omega+\sqrt{-1}\partial\bar{\partial}u)^n}{\Omega}=\int_M\omega^n\Big/\int_M\Omega\ . \tag{4}$$

DEFINITION 1. A function f on X is said to be B-C^∞, if $p_\alpha^* f$ is C^∞ for every α.

Now it is easy to see that the equation (4) has a B-C^∞ solution u by the same strategy in [7] (cf. [3, pp. 63-66]). We set

$$\tilde{\omega}=\omega+\sqrt{-1}\partial\bar{\partial}u .$$

Then by (4) we have

$$\mathrm{Ric}\,\tilde{\omega}=\omega . \tag{5}$$

Hence $\tilde{\omega}$ is a B-C^∞-metric of positive Ricci curvature. In other words, X has a structure of an orbifold with positive Ricci curvature. We note that the real codimension of D in X is two. Hence general two distinct points on X can be jointed by a minimizing geodesic which does not intersect D.

The following lemma can be proven by using Jacobi fields along a minimizing geodesic which joints two general distinct points on X (cf. [2, p. 74, Theorem 3.4]).

LEMMA 1 (An orbifold version of [4]). *X has a finite fundamental group.*

On the other hand, since $-(K_X+D)$ is ample, by the vanishing theorem in [5], we have:

LEMMA 2. *$H^p(X, \mathcal{O}_X)=0$ for $1\leqq p\leqq n$. In particular $\chi(\mathcal{O}_X)=1$.*

Now we proceed as in [1]. Let $\pi\colon \tilde{X}\to X$ be the universal covering of X. By Lemma 1, $\tilde{X}$ is projective. By Lemma 2, we see that $\chi(\mathcal{O}_{\tilde{X}})=\chi(\mathcal{O}_X)=1$. Since the Riemann-Roch theorem implies that $\chi(\mathcal{O}_{\tilde{X}})=(\deg\pi)\cdot\chi(\mathcal{O}_X)$, we conclude that X is simply connected.

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