TOKYO J. MATH.
VOL. 11, NO. 2, 1988

Complete Space-Like Surfaces with Constant Mean Curvature in the Minkowski 3-Space

Kotaro YAMADA

Keio University

(Communicated by M. Obata)

Introduction.

Let L^3 be the Minkowski 3-space, that is, $\boldsymbol{R}^3$ with the indefinite metric $\langle\ ,\ \rangle=(dx^1)^2+(dx^2)^2-(dx^3)^2$. A surface in L^3 is called *space-like* if the induced metric on the surface is positive definite. On a space-like surface, the notions of the first fundamental form, the second fundamental form, and the mean curvature are defined in the same way as on a surface in the euclidean space.

In particular, we shall consider complete space-like surfaces with constant mean curvature H. For example, in [2] and [4], Calabi and Cheng-Yau established the Bernstein-type theorem when $H\equiv 0$, *maximal* space-like surface. In other words, the uniqueness theorem holds for maximal surfaces.

In this paper, we investigate complete space-like surfaces with *non-zero* constant mean curvature H. In this case, uniqueness does not hold and there are several examples. The most well-known example of such a surface is the *pseudosphere*:

$$S(H)=\left\{(x^1,\ x^2,\ x^3)\in L^3\ ;\ (x^1)^2+(x^2)^2-(x^3)^2=-\frac{1}{H^2},\ x^3>0\right\}\ , \tag{0.1}$$

which is the only complete, totally umbilical space-like surface with constant mean curvature H. Note that $S(H)$ is isometric to the Poincaré disc with constant Gaussian curvature $-H^2$.

Among non-umbilical space-like surfaces, the following *hyperbolic cylinder* is the simplest one:

$$C(H)=\left\{(x^1,\ x^2,\ x^3)\in L^3\ ;\ (x^1)^2-(x^3)^2=-\frac{1}{4H^2},\ x^3>0\right\}\ . \tag{0.2}$$

This is the only complete, flat space-like surface with non-zero constant mean curvature H.

Received September 28, 1987

Although many other constant mean curvature surfaces are constructed by Treibergs [9] as entire graphs on the x^1x^2-plane which solve his asymptotic Dirichlet problem, $S(H)$ and $C(H)$ are distinctive among such surfaces. For example, Choquet-Bruhat [3] characterized $S(H)$ as the only constant mean curvature slices in L^3 with some assumptions, and Goddard [5] showed that any perturbation of $S(H)$ with constant mean curvature must be a translation of L^3.

In this paper, we shall give a new proof of the following theorem characterizing the hyperbolic cylinder $C(H)$ among the complete space-like surfaces with non-zero constant mean curvature H which are "uniformly" non-umbilical.

THEOREM. *The hyperbolic cylinder $C(H)$ is the only complete space-like surface in L^3 with non-zero constant mean curvature H whose principal curvatures k_1 and k_2 satisfy*

$$(k_1-k_2)^2 \geqq \varepsilon^2 \tag{0.3}$$

for some positive number ε.

This theorem was firstly proved by T. K. Milnor [7]. In her proof, the theorem is the consequence of the fact that Gaussian curvature of the surface must be non-positive [4], and of Liouville's theorem. On the other hand, we use a maximum principle for a non-linear elliptic equation on $\boldsymbol{R}^2$ to prove the theorem. More precisely, outline of our proof is the following.

In § 1, the fundamental equations for a space-like surface are reviewed. Using these equations, we show in § 2 that the second fundamental form of a space-like surface satisfying the assumption of the theorem is determined when the surface is conformal to $\boldsymbol{R}^2$. In this case, the Gauss equation shows that there exists an entire solution of the equation $\Delta\rho=\lambda\sinh\rho$ on $\boldsymbol{R}^2$, where λ is a positive constant. As a consequence of the maximum principle, we prove in § 3 that the only entire solution of this equation is the trivial one, which gives $C(H)$. The proof of the theorem follows immediately from this fact.

Note that the assumption (0.3) is necessary. In fact, we can construct complete non-umbilical space-like surfaces with constant mean curvature H on which $(k_1-k_2)^2$ tends to 0 at infinity (see § 4 Remark 1).

The author would like to thank Dr. Masahiko Kanai and the referee for their helpful remarks.

§ 1. Space-like surfaces with constant mean curvature.

Let Σ be a space-like surface in L^3 with constant mean curvature

H. Then the first fundamental form, i.e., the induced metric $g=\langle\ ,\ \rangle|_\Sigma$ gives a riemannian metric on Σ. So we can take isothermal parameters (u, v) as local coordinates of Σ in which g is written as

$$g=e^{\sigma}(du^2+dv^2) \tag{1.1}$$

with some smooth function $\sigma(u, v)$. Using a complex parameter $z=u+\sqrt{-1}v$, we can also write

$$g=e^{\sigma}dzd\bar{z}\ .$$

Take the unit normal vector field of Σ, i.e., a vector field ν along Σ which satisfies $\langle\nu, \nu\rangle=-1$. So, the second fundamental form h of Σ is defined as a symmetric 2-tensor on Σ by

$$h(X, Y)=-\langle\bar{\nabla}_X\nu, Y\rangle \qquad \text{for} \quad X, Y\in T_p\Sigma$$

at each point p on Σ, where $\bar{\nabla}$ is the canonical connection of L^3. Since the mean curvature $H=(1/2)\mathrm{trace}_g h$, h is written as

$$h=Ldu^2+2Mdudv+(2e^{\sigma}H-L)dv^2$$

in the present isothermal coordinates.

Let k_1 and k_2 be principal curvatures of Σ, i.e., the eigenvalues of h with respect to the metric g. So, the Gaussian curvature K and the mean curvature H are written as

$$K=-k_1k_2=e^{-2\sigma}\{M^2-L(2e^{\sigma}H-L)\}\ ,$$

$$H=\frac{1}{2}(k_1+k_2)\ ,$$

and

$$(k_1-k_2)^2=4(H^2+K)=4e^{-2\sigma}\{(L-e^{\sigma}H)^2+M^2\} \tag{1.2}$$

holds.

Define a function Φ on Σ locally as

$$\Phi(z)=(L-e^{\sigma}H)-\sqrt{-1}M\ . \tag{1.3}$$

So,

$$(k_1-k_2)^2=4|\Phi|^2e^{-2\sigma}\ . \tag{1.4}$$

Note that a point p of Σ with a complex coordinate z is an umbilical point if and only if $\Phi(z)=0$.

In the present coordinates, the fundamental equations of Σ imply the following:

Lemma 1.1. *Let Σ be a space-like surface in L^3 with constant mean curvature H, and (u, v) its isothermal coordinates in which the first fundamental form g is written as* (1.1). *Then,*

(1) (Codazzi equation) *The locally defined function $\Phi(z)$ in* (1.3) *is holomorphic.*

(2) (Gauss equation) *The Gaussian curvature K of Σ is the intrinsic sectional curvature of (Σ, g), i.e.,*

$$K=-\frac{1}{2}e^{-\sigma}\Delta\sigma=-e^{-\sigma}(H^2e^{\sigma}-|\Phi|^2e^{-\sigma}), \qquad where \quad \Delta=\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}.$$

For example, let $\Sigma=C(H)$, the hyperbolic cylinder defined in (0.2). Putting $u=(2H)^{-1}\tanh^{-1}(x^1/x^3)$ and $v=x^2$, we have the global isothermal coordinates (u, v) of Σ in which g, h and Φ are written as:

$$\begin{cases} g=du^2+dv^2 \\ h=2Hdu^2 \\ \Phi=H=\text{constant}. \end{cases} \tag{1.5}$$

In particular, $C(H)$ is isometric to the euclidean plane $\boldsymbol{R}^2$.

Conversely, a flat, complete space-like surface with non-zero constant mean curvature H is congruent to $C(H)$.

§ 2. Complete space-like surface conformal to $\boldsymbol{R}^2$.

Let Σ be a complete space-like surface with constant mean curvature H. In this section, Σ is assumed to be conformal to the euclidean plane $\boldsymbol{R}^2$. So, we can take the standard coordinates (u, v) of $\boldsymbol{R}^2$ as the *global* isothermal coordinates of Σ in which the first fundamental form g has the form

$$g=e^{\sigma}(du^2+dv^2)=e^{\sigma}dzd\bar{z} \tag{2.1}$$

with some smooth function σ on $\boldsymbol{R}^2$. Then the complex-valued function $\Phi(z)$ is defined on the whole plane $\boldsymbol{C}=\boldsymbol{R}^2$, and holomorphic because of Lemma 1.1 (1). That is Φ is an entire holomorphic function on $\boldsymbol{R}^2$. Though there are many entire functions on $\boldsymbol{C}$, Φ must be constant under the assumptions of our theorem. Namely we have

Lemma 2.1. *Let Σ be a complete surface as above whose principal curvatures k_1 and k_2 satisfy*

(2.2) $$(k_1-k_2)^2 \geqq \varepsilon^2 > 0$$

for some positive ε. *Then the function* $\Phi(z)$ *in* (1.3) *must be constant.*

PROOF. Substituting (1.4) into (2.2), we have

(2.3) $$2\varepsilon^{-1}|\Phi| \geqq e^{\sigma} .$$

Consider a riemannian metric

$$\hat{g} = 2\varepsilon^{-1}|\Phi|(du^2+dv^2) = 2\varepsilon^{-1}|\Phi| dz d\bar{z}$$

on $R^2 = C$. Then, (2.3) shows $\hat{g} \geqq g$ as quadratic forms on TR^2. So, by the completeness of g, $\hat{g}$ is also a complete metric on R^2.

On the other hand, the Gaussian curvature of $\hat{g}$ is

$$K_{\hat{g}} = -\frac{\varepsilon}{4}|\Phi|^{-1}\Delta \log|\Phi| = 0 ,$$

since Φ is holomorphic.

Hence $\hat{g}$ is the flat complete metric on R^2. Then there exists an isometry

$$\mu: (C, \hat{g}) \longrightarrow (C, g_0) ,$$

where g_0 is the standard metric of C. The isometry μ can be considered as an entire holomorphic function which maps C onto C injectively, since it is conformal. Moreover, the injectivity of μ shows that μ must have a pole of order 1 at ∞. Thus μ is linear, i.e.,

$$\mu(z) = az + b$$

for some constants $a \neq 0$ and b.

Hence

$$2\varepsilon^{-1}|\Phi| dz d\bar{z} = \hat{g} = \mu^* g_0 = |a|^{-2} dz d\bar{z} ,$$

and then, Φ must be constant. □

Substituting this into the Gauss equation, Lemma 1.1 (2), and putting $\lambda = 4|H\Phi|$, we have the following equation.

COROLLARY 2.2. *Let* Σ *be as in Lemma* 2.1 *and* $\rho = \sigma + \log|H/\Phi|$. *Then* ρ *satisfies the equation*

(2.4) $$\Delta\rho = \lambda \sinh \rho \quad \text{on} \quad R^2 ,$$

where $\Delta=\partial^2/\partial u^2+\partial^2/\partial v^2$, *and* $\lambda=4|H\Phi|$, *a positive constant.*

The trivial solution $\rho\equiv 0$ gives the flat metric on Σ, and hence, it corresponds to the hyperbolic cylinder $C(H)$.

§ 3. Non existence of non-trivial solutions of (2.4).

In this section, we shall prove the following proposition, the maximum principle for the equation (2.4).

PROPOSITION 3.1. *Let* λ *be a positive number. Then the equation*

$$\Delta\rho=\lambda\sinh\rho \quad \text{on} \quad \boldsymbol{R}^2 \tag{3.1}$$

has no entire solutions except $\rho\equiv 0$.

To prove this, we look at radially symmetric solution of (3.1).
Consider the ordinary differential equation

$$\varphi''(r)+\frac{1}{r}\varphi'(r)=\lambda\sinh\varphi(r) \quad \text{for} \quad r\geqq 0\ , \tag{3.2 (a)}$$

$$\varphi(0)=a>0\ , \qquad \varphi'(0)=0\ , \tag{3.2 (b)}$$

where $'$ is the derivation with respect to r. So, the solution of (3.2) is a radially symmetric solution of (3.1) with $r=\sqrt{u^2+v^2}$. First, we claim the local existence of a solution of (3.2).

LEMMA 3.2. *There exists a local solution of* (3.2) (a) *and* (3.2) (b).

PROOF. Write (3.2) as

$$\varphi(r)=a+\int_0^r\frac{ds}{s}\int_0^s t\lambda\sinh\varphi(t)dt\ ,$$

and use a usual iteration argument. □

Nevertheless, there exist no global solutions of (3.1) except the trivial solution $\rho\equiv 0$.

LEMMA 3.3. *There exists no entire, radially symmetric solution* $\varphi(r)$ *of* (3.1) *with* $\varphi(0)>0$.

PROOF. Suppose $\varphi(r)$ be an entire radially symmetric solution of (3.1) with $\varphi(0)=a>0$. So, φ satisfies (3.2).
Write the equation (3.2) (a) as

$$(3.3) \qquad (r\varphi')' = r\lambda \sinh \varphi .$$

By (3.2) (b) and (3.3),

$$(3.4) \qquad \varphi'(r) > 0 \quad \text{for} \quad r > 0$$

holds, and then, φ is an increasing function of r. In particular, $\sinh \varphi(r) \geqq \sinh a$ for $r > 0$. Substituting this into (3.3), we have

$$(r\varphi')' \geqq r\lambda \sinh a .$$

Integrating this twice, the inequality

$$(3.5) \qquad \varphi - a \geqq \frac{r^2}{4}\lambda \sinh a$$

holds, and hence φ tends to $+\infty$ as $r \to \infty$.

On the other hand,

$$\lambda \sinh \varphi \geqq \varphi''$$

because of (3.2) (a) and (3.4). Integrating this,

$$\begin{aligned}
\{\varphi'(r)\}^2 &= \int_0^r \{\varphi'(s)^2\}' ds = 2\int_0^r \varphi''(s)\varphi'(s) ds \\
&\leqq 2\lambda \int_0^r \sinh \varphi(s) \cdot \varphi'(s) ds = 2\lambda \int_a^{\varphi(r)} \sinh x \, dx \\
&= 2\lambda(\cosh \varphi(r) - \cosh a) \\
&\leqq 2\lambda(\cosh^2 \varphi(r) - 1) = 2\lambda \sinh^2 \varphi(r) ,
\end{aligned}$$

since $\cosh \varphi(r) \geqq 1$. Then,

$$\begin{aligned}
\frac{\varphi'}{r} &\leqq \frac{\sqrt{2\lambda}}{r} \sinh \varphi \\
&\leqq \frac{1}{2}\lambda \sinh \varphi \quad \text{for} \quad r > r_1 ,
\end{aligned}$$

where $r_1 = 2\sqrt{2/\lambda}$. Substituting this into (3.2) (a), we have

$$\varphi'' \geqq \frac{1}{2}\lambda \sinh \varphi \quad \text{for} \quad r > r_1 .$$

Thus, for $r > r_1$,

$$\begin{aligned}
\{\varphi'(r)\}^2 - \{\varphi'(r_1)\}^2 &= 2\int_{r_1}^r \varphi'(s)\varphi''(s) ds \\
&\geqq \lambda \int_{\varphi(r_1)}^{\varphi(r)} \sinh(x) dx \\
&= \lambda\{\cosh \varphi(r) - \cosh \varphi(r_1)\} .
\end{aligned}$$

Hence, there exists a positive number r_2 such that for $r \geqq r_2$,

$$\begin{aligned}\varphi'(r) &\geqq \sqrt{\lambda\{\cosh\varphi(r)-\cosh\varphi(r_1)\}+\{\varphi'(r_1)\}^2}\\ &\geqq C_1\exp\left(\frac{\varphi(r)}{2}\right),\end{aligned}$$

where C_1 is a positive constant. Integrating this inequality, we have

$$\exp\left(-\frac{\varphi(r)}{2}\right)\leqq -\frac{C_1}{2}r+C_2 \tag{3.6}$$

with some constant C_2. Here, $\lim_{r\to\infty}\varphi(r)=+\infty$ because of (3.5). Then, the left-hand side of (3.6) tends to 0 when $r\to+\infty$. This shows that r is bounded, and contradicts the assumption. □

COROLLARY 3.4. *Let φ be a non-trivial radially symmetric solution of* (3.1) *with $\varphi(0)>0$. Then, there exists a positive number R for which* $\lim_{r\to R}\varphi(r)=+\infty$.

PROOF. By (3.4), $\varphi(r)$ is an increasing function of r. On the other hand, φ is a solution of (3.2) (a) in a finite interval $[0, R)$ because of Lemma 3.2. Hence, φ tends to $+\infty$ as $r\to R$. □

PROOF OF PROPOSITION 3.1. Let ρ be an entire solution of (3.1) which is not identically 0. So, we can suppose $\rho(0)\neq 0$. Assume $\rho(0)=2a>0$ and take a radially symmetric solution of (3.1) with $\varphi(0)=a$. So, there exists a positive number R such that $\lim_{r\to R}\varphi(r)=+\infty$ because of Corollary 3.4.

Let $f=\varphi-\rho$, a function defined on $B_R=\{(u, v);\ r=\sqrt{u^2+v^2}<R\}$ with $\lim_{r\to R} f=+\infty$. Then, f takes a minimum at some point p in B_R. Assume $f(p)<0$. So,

$$\begin{aligned}\Delta f(p)&=\Delta\varphi(p)-\Delta\rho(p)\\ &=\lambda\{\sinh\varphi(p)-\sinh\rho(p)\}\\ &=2\lambda\cosh\frac{\varphi(p)+\rho(p)}{2}\sinh\frac{f(p)}{2}\\ &<0\,.\end{aligned}$$

This contradicts the fact that f takes its minimum at p. Hence $f=\varphi-\rho\geqq 0$ in B_R. In particular, $f(0)=\varphi(0)-\rho(0)=a-2a=-a\geqq 0$. This is impossible. Thus there exists no entire solution ρ of (3.1) which takes a positive value.

When $\rho(0)<0$, we have the same conclusion by considering $-\rho$ instead of ρ.

REMARK. In [8], Osserman showed the non-existence of entire solutions of $\Delta u \geqq f(u)$ on $\boldsymbol{R}^n$, where f is a positive, increasing function with large growth rate. Though our equation (2.4) does not satisfy his assumptions, almost all parts of his proof are valid for Proposition 3.1.

§ 4. Proof of the main theorem.

Let Σ be a complete space-like surface satisfying the assumptions of the theorem. Then,

$$2\varepsilon^{-1}|\Phi| \geqq e^{\sigma} \tag{4.1}$$

holds in the isothermal coordinates as in § 2.

Note that a complete space-like surface can be represented as an entire graph on the x^1x^2-plane in L^3. In particular, Σ must be simply connected. Thus, Σ is conformal to either the Poincaré disc $\boldsymbol{H}^2$ or the euclidean plane $\boldsymbol{R}^2$, since it is non-compact.

Assume Σ is conformal to $\boldsymbol{H}^2=(D, g_0)$, where $D=\{z \in \boldsymbol{C}; |z|<1\}$ and $g_0=4dzd\bar{z}/(1-|z|^2)^2$. So, (Σ, g) is isometric to $(D, g=e^{\sigma}dzd\bar{z})$ for some function σ on D. Here, the completeness of g implies

$$\lim_{(u,v)\to\partial D} e^{\sigma}=+\infty .$$

Therefore the function Φ is a non-vanishing holomorphic function on D which satisfies

$$\lim_{(u,v)\to\partial D} |\Phi|=+\infty \tag{4.2}$$

because of (4.1). Put $\Psi=\Phi^{-1}$. Then, Ψ is holomorphic on D and continuous on $\bar{D}$ with $\Psi|_{\partial D}=0$. Then, by Cauchy's formula,

$$\Psi(0)=-\frac{\sqrt{-1}}{2\pi}\int_{\partial D}\frac{\Psi(z)}{z}dz=0 .$$

This is impossible. Therefore Σ cannot be conformal to $\boldsymbol{H}^2$.

Hence Σ must be conformal to $\boldsymbol{R}^2$. Then we can take global coordinates (u, v) of Σ in which the first fundamental forms g is written as (2.1). So, σ in (2.1) satisfies the equation (2.4) and then, must be constant because of Lemma 3.1. Thus g is the flat metric and hence, Σ is congruent to the hyperbolic cylinder $C(H)$. This completes the proof of the theorem. □

REMARK 1. Let ρ be a radially symmetric solution of (2.4) on B_R

and consider a metric $g=|\Phi/H|e^{\rho}dzd\bar{z}$ on B_R. When $\rho(0)<0$, the metric g is not complete on B_R since $\lim_{r\to R}\rho(r)=-\infty$. On the other hand, g is a complete metric on B_R when $\rho(0)>0$. Then g, Φ and H give a complete space-like surface in L^3 with constant mean curvature H and given Φ. This surface has no umbilical points, but $\lim_{r\to R}(k_1-k_2)=0$ since $\lim_{r\to R}\rho(r)=+\infty$. So, the assumption (0.3) of the theorem is essential.

REMARK 2. For a surface in the enclidean space $\boldsymbol{R}^3$, the Gauss equation implies that $\Delta\rho=-\lambda\sinh\rho$ in the same situation in §2, where λ is a positive constant. For this equation, the maximum principle like as Proposition 3.1 does not hold. So the Gauss equation is expected to have non-trivial solutions. This is one of the reasons why there are counter-examples for Hopf conjecture; immersed tori in $\boldsymbol{R}^3$ with constant mean curvature [10].

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Present Address:
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY, KEIO UNIVERSITY
HIYOSHI, KOHOKU-KU, YOKOHAMA 223, JAPAN