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Lipschitz Classes and Fourier Series of Stochastic Processes

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§ 1. Introduction.

Let $f(t)\in L^1(T)$, $T=(-\pi, \pi)$, be a 2π-periodic function and write, for a positive integer j,

$$\Delta_h^{(j)}f(t)=\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}f(t+kh)\ , \tag{1.1}$$

$$L^{(j)}(h,\ t;\ f)=h^{-1}\int_0^h \Delta_u^{(j)}f(t)du\ . \tag{1.2}$$

Kinukawa [4] has discussed the problem to characterize the Lipschitz class of $f(t)$ satisfying

$$ {}_aA_{p,j,\alpha}^{\circ}(f)=\left(\int_0^1 h^{-1}dh\left\{\int_T[h^{-\alpha}|\Delta_h^{(j)}f(t)|]^a dt\right\}^{p/a}\right)^{1/p}<\infty\ , \qquad (\alpha,\ a,\ p>0) \tag{1.3}$$

in terms of Fourier coefficients of the functions of the class. He also discussed a more general class of $f(t)$ for which

$$ {}_aA_{p,j,\alpha}(f)=\left(\int_0^1 h^{-1}dh\left\{\int_T[h^{-\alpha}|L^{(j)}(h,\ t;\ f)|]^a dt\right\}^{p/a}\right)^{1/p}<\infty\ , \tag{1.4}$$

generalizing a Yadav's result on absolute convergence of Fourier series.

We are interested in a more general Lipschitz class for a later purpose.

Throughout this paper, $\phi(t)$ is either identically one on $[0, 1]$ or a nonnegative nondecreasing function such that $\phi(0)=0$ and $t^{-1}\phi(t)$ is nonincreasing on $(0, 1]$.

We introduce, for a nonnegative integer l,

$$ {}_aA_{p,j,\alpha}^{l,\phi}(f)=\left(\int_0^1 h^{-l-1}[\phi(h)]^{-1}dh\left\{\int_T[h^{-\alpha}|L^{(j)}(h,\ t;\ f)|]^a dt\right\}^{p/a}\right)^{1/p} \qquad (\alpha,\ a,\ p>0)\ . \tag{1.5}$$

Our main purpose is to study on the class of stochastic processes which

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is similar to the above classes of functions and generalize the author's some previous results on almost sure absolute convergence and some sample properties of stochastic processes.

Let $X(t, \omega)$, $t \in \boldsymbol{R}^1$, $\omega \in \Omega$, Ω being a given probability space, be a measurable stochastic process on $(\boldsymbol{R}^1 \times \Omega)$. Since we are interested in some local sample properties of $X(t, \omega)$, we suppose, for simplicity, $X(t, \omega)$ is 2π-periodic, namely

$$E|X(t+2\pi, \omega)-X(t, \omega)|=0$$

for all $t \in \boldsymbol{R}^1$ and hence $X(t+2\pi, \omega)=X(t, \omega)$ for almost all t, with probability one (almost surely).

Suppose for $1 \leqq r, s$

$$X(t, \omega) \in L^{s,r}(T \times \Omega) , \tag{1.6}$$

that is,

$$\int_T \|X(t, \cdot)\|_r^s dt < \infty , \tag{1.7}$$

where $\|X(t, \cdot)\|_r = [E|X(t, \omega)|^r]^{1/r}$.

We remark that (1.6) implies

$$X(t, \omega) \in L^\theta(T) , \qquad \theta = \min(r, s) \tag{1.8}$$

almost surely, that is, $X(t, \omega)$, as a function of t, belongs to $L^\theta(T)$ almost surely ([3]).

§ 2. Lemmas.

We shall begin with the following lemma.

LEMMA 1. *Let* $a>0$, $-\infty<b<\infty$ *and* $0<c\leqq 1$. *Let* $r_n \geqq 0$, $n=1, 2, \cdots$, *and* p_n, $n=1, 2, \cdots$, *be such that* $0<p_n \leqq p_{n+1}$, $0<p_{2n} \leqq Kp_n$. *We have*

$$\sum_{n=1}^\infty n^b p_n \Big(\sum_{k=n}^\infty r_k^a\Big)^c \geqq K \sum_{n=1}^\infty n^{b+c} p_n r_n^{ac} . \tag{2.1}$$

K's are constants which are independent of $\{p_n\}$ *and* $\{r_n\}$, *and may be different and depend on other parameters in* (2.1).

Throughout this paper, we use K for constants which may be different on each occurrence. The lemma for particular choices of a, b, c and p_n is familiar. We, just for completeness, give the proof of it.

PROOF. The proof is carried out by Riemann's condensation method as is usually done for the known special cases.

The left hand side of (2.1) is

$$\sum_{k=0}^{\infty}\sum_{n=2^k}^{2^{k+1}-1} n^b p_n\Big(\sum_{\nu=n}^{\infty} r_\nu^a\Big)^c \geqq \sum_{k=0}^{\infty} 2^{(b+1)k} p_{2^k}\Big(\sum_{\nu=2^k+1}^{2^{k+2}-1} r_\nu^a\Big)^c$$

$$\geqq \sum_{k=0}^{\infty} 2^{(b+c)k} p_{2^k} \sum_{\nu=2^k+1}^{2^{k+2}-1} r_\nu^{ac}$$

by the Hölder inequality, and the last one, because of the condition on p_n, is

$$\geqq K\sum_{k=0}^{\infty}\sum_{\nu=2^k+1}^{2^{k+2}-1} \nu^{b+c} p_\nu r_\nu^{ac} = K\sum_{\nu=2}^{\infty} \nu^{b+c} p_\nu r_\nu^{ac} ,$$

from which (2.1) easily follows.

Now we introduce the following three quantities for any sequence $\{r_n, -\infty<n<\infty\}$ of complex numbers. Let a, α be positive numbers, l be a nonnegative integer and j be a positive integer.

$$(2.2)\qquad {}_aB_{p,\alpha}^{l,\phi}(\{r_n\}) = \Big\{\sum_{k=1}^{\infty} k^{p\alpha+l-1}[\phi(k^{-1})]^{-1}\Big(\sum_{|n|\geqq k} |r_n|^a\Big)^{p/a}\Big\}^{1/p} ,$$

$$(2.3)\qquad {}_aC_{p,j,\alpha}^{l,\phi}(\{r_n\}) = \Big\{\sum_{k=1}^{\infty} k^{p\alpha-pj+l-1}[\phi(k^{-1})]^{-1}\Big(\sum_{|n|\leqq k} |r_n|^a|n|^{aj}\Big)^{p/a}\Big\}^{1/p} ,$$

$$(2.4)\qquad {}_a\tilde{A}_{p,j,\alpha}^{l,\phi}(\{r_n\}) = \Big\{\int_0^1 t^{-p\alpha-l-1}[\phi(t)]^{-1}\Big[\sum_{n=-\infty}^{\infty}\Big| r_n t^{-1}\int_0^t (1-e^{inu})^j du\Big|^a\Big]^{p/a} dt\Big\}^{1/p} .$$

These are defined for the case $l=0$, $\phi(t)\equiv 1$ by Kinukawa [4] who proved the following lemma for this special case.

LEMMA 2. *Let $\{r_n, -\infty<n<\infty\}$ be any sequence of complex numbers, l be nonnegative and a, p, $\alpha>0$.*

(i) *Suppose $0<p<a$. If ${}_aB_{p,\alpha}^{l,\phi}(\{r_n\})$ is finite, then, for any positive integer j such that $j>\alpha+(l+1)/p$ when $\phi(t)\not\equiv 1$, and $j>\alpha+l/p$ when $\phi(t)\equiv 1$, we have*

$$(2.5)\qquad {}_aC_{p,j,\alpha}^{l,\phi}(\{r_n\}) \leqq K_a B_{p,\alpha}^{l,\phi}(\{r_n\}) .$$

(ii) *Suppose $0<p<a$. If ${}_aC_{p,j,\alpha}^{l,\phi}(\{r_n\})$ is finite for some positive integer j, then*

$$(2.6)\qquad {}_aB_{p,\alpha}^{l,\phi}(\{r_n\}) \leqq K_a C_{p,j,\alpha}^{l,\phi}(\{r_n\}) .$$

(iii) *If ${}_a\tilde{A}_{p,j,\alpha}^{l,\phi}(\{r_n\})$ is finite for some positive integer j, then*

$$(2.7)\qquad {}_aC_{p,j,\alpha}^{l,\phi}(\{r_n\}) \leqq K_a \tilde{A}_{p,j,\alpha}^{l,\phi}(\{r_n\}) .$$

(iv) *If* ${}_aB^{l,\phi}_{p,\alpha}(\{r_n\})$ *and* ${}_aC^{l,\phi}_{p,j,\alpha}(\{r_n\})$ *are finite for some positive integer* j*, then*

$$ {}_a\tilde{A}^{l,\phi}_{p,j,\alpha}(\{r_n\})\leqq K[{}_aB^{l,\phi}_{p,\alpha}(\{r_n\})+{}_aC^{l,\phi}_{p,j,\alpha}(\{r_n\})]\ . \tag{2.8}$$

Here K's are constants independent of $(\{r_n\})$, *and may depend on other parameters involved in the above inequalities.*

The proof is similarly performed by Riemann's condensation argument as in Kinukawa's paper. The proof will be given here for completeness. We first note that for any $\beta\geqq 1$,

$$\phi(t)\leqq\phi(\beta t)\leqq\beta\phi(t)\ . \tag{2.9}$$

PROOF OF (i).

$$\begin{aligned}[{}_aB^{l,\phi}_{p,\alpha}(\{r_n\})]^p&=\sum_{m=1}^{\infty}\sum_{k=2^{m-1}}^{2^m-1}k^{p\alpha+l-1}[\phi(k^{-1})]^{-1}(\sum_{|n|\geqq k}|r_n|^a)^{p/a}\\ &\geqq K\sum_{m=1}^{\infty}2^{m(p\alpha+l)}[\phi(2^{-m})]^{-1}(\sum_{|n|\geqq 2^m}|r_n|^a)^{p/a}\\ &\geqq K\sum_{m=1}^{\infty}2^{mjp}\cdot 2^{m(p\alpha-pj+l)}[\phi(2^{-m})]^{-1}(\sum_{2^{m+1}\geqq|n|\geqq 2^m}|r_n|^a)^{p/a}\ . \end{aligned}\tag{2.10}$$

Since $2^{-\nu}[\phi(2^{-\nu})]^{-1}$ is nonincreasing, we have

$$\begin{aligned}&\sum_{\nu=m}^{\infty}2^{\nu(p\alpha-pj+l)}[\phi(2^{-\nu})]^{-1}\\ &\leqq 2^{-m}[\phi(2^{-m})]^{-1}\sum_{\nu=m}^{\infty}2^{\nu(p\alpha-pj+l+1)}\\ &\leqq[\phi(2^{-m})]^{-1}2^{m(p\alpha-pj+l)}\end{aligned}$$

when $\phi(t)\not\equiv 1$. Note that $p\alpha-pj+l+1<0$. Therefore, when $\phi(t)\not\equiv 1$, the right hand side of (2.10) is

$$\begin{aligned}&\geqq K\sum_{m=1}^{\infty}2^{mjp}(\sum_{2^{m+1}>|n|\geqq 2^m}|r_n|^a)^{p/a}\sum_{\nu=m}^{\infty}2^{\nu(p\alpha-pj+l)}[\phi(2^{-\nu})]^{-1}\\ &=K\sum_{\nu=1}^{\infty}2^{\nu(p\alpha-pj+l)}[\phi(2^{-\nu})]^{-1}\sum_{m=1}^{\infty}2^{mjp}(\sum_{2^{m+1}>|n|\geqq 2^m}|r_n|^a)^{p/a}\end{aligned}$$

which, because of Jensen's inequality, is

$$\begin{aligned}&\geqq K\sum_{\nu=1}^{\infty}2^{\nu(p\alpha-pj+l)}[\phi(2^{-\nu})]^{-1}\Big(\sum_{m=1}^{\nu}\sum_{2^{m+1}>|n|\geqq 2^m}|r_n|^a|n|^{aj}\Big)^{p/a}\\ &\geqq K\sum_{k=1}^{\infty}k^{p\alpha-pj+l-1}[\phi(k^{-1})]^{-1}(\sum_{2k\geqq|n|\geqq 2}|r_n|^a|n|^{aj})^{p/a}\ .\end{aligned}$$

From this (2.5) follows.

When $\phi(t)\equiv 1$, the proof goes through (actually more simply) with $p\alpha-pj+l<0$.

PROOF OF (ii).

$$[{}_aC^{l,\phi}_{p,j,\alpha}(\{r_n\})]^p \geqq \sum_{m=1}^{\infty} \sum_{k=2^{m-1}}^{2^m-1} k^{p\alpha-pj+l-1}[\phi(k^{-1})]^{-1}\Big(\sum_{|n|\leqq k}|r_n|^a|n|^{aj}\Big)^{p/a}$$

which, because of (2.9) with $\beta=2$, $t=2^{-m}$, is

$$\geqq K\sum_{m=1}^{\infty} 2^{m(p\alpha-pj+l)}[\phi(2^{-m})]^{-1}\Big(\sum_{|n|\leqq 2^{m-1}}|r_n|^a|n|^{aj}\Big)^{p/a} .$$

Since $\sum_{k=1}^{m} 2^{k(p\alpha+l)}[\phi(2^{-k})]^{-1}\leqq K2^{m(p\alpha+l)}[\phi(2^{-m})]^{-1}$,

$$\begin{aligned}[{}_aC^{l,\phi}_{p,j,\alpha}(\{r_n\})]^p &\geqq K\sum_{m=1}^{\infty} 2^{-mpj}\Big(\sum_{2^{m-2}<|n|\leqq 2^{m-1}}|r_n|^a|n|^{aj}\Big)^{p/a}\sum_{k=1}^{m} 2^{k(p\alpha+l)}[\phi(2^{-k})]^{-1}\\ &=K\sum_{k=1}^{\infty} 2^{k(p\alpha+l)}[\phi(2^{-k})]^{-1}\sum_{m=k}^{\infty} 2^{-mpj}\Big(\sum_{2^{m-2}<|n|\leqq 2^{m-1}}|r_n|^a|n|^{aj}\Big)^{p/a}\end{aligned}$$

which, because of Jensen's inequality, is

$$\begin{aligned}&\geqq K\sum_{k=1}^{\infty} 2^{k(p\alpha+l)}[\phi(2^{-k})]^{-1}\Big(\sum_{m=k}^{\infty} 2^{-maj}\sum_{2^{m-2}<|n|\leqq 2^{m-1}}|r_n|^a|n|^{aj}\Big)^{p/a}\\ &\geqq K\sum_{k=1}^{\infty} 2^{k(p\alpha+l)}[\phi(2^{-k})]^{-1}\sum_{m=k}^{\infty}\Big(\sum_{2^{m-2}<|n|\leqq 2^{m-1}}|r_n|^a\Big)^{p/a}\\ &\geqq K\sum_{\nu=1}^{\infty} \nu^{p\alpha+l-1}[\phi(\nu^{-1})]^{-1}\Big(\sum_{|n|\geqq\nu}|r_n|^a\Big)^{p/a} .\end{aligned}$$

This proves (ii).

REMARK. No assumption is seemingly made on j in (ii). However actually j is restricted as $j>\alpha+l/p$, which is seen from the finiteness of ${}_aC^{l,\phi}_{p,l,\alpha}(\{r_n\})$ since $\sum k^{p\alpha-pj+l-1}[\phi(k^{-1})]^{-1}$ should be finite.

PROOF OF (iii).

$$[{}_a\tilde{A}^{l,\phi}_{p,j,\alpha}(\{r_n\})]^p \geqq \sum_{k=1}^{\infty}\int_{[(k+1)j]^{-1}}^{(kj)^{-1}} t^{-p\alpha-l-1}[\phi(t)]^{-1}dt\Big[\sum_{|n|\leqq k}|r_n|^a\Big|\frac{1}{t}\int_0^t(1-e^{inu})^j du\Big|^a\Big]^{p/a} .$$

Now for $|n|\leqq k$, $0<t<(kj)^{-1}$,

$$\begin{aligned}\Big|\frac{1}{t}\int_0^t(1-e^{inu})^j du\Big| &= \Big|\frac{2^j}{t}\int_0^t e^{inju/2}\Big(\sin\frac{nu}{2}\Big)^j du\Big|\\ &\geqq \frac{2^j}{t}\Big|\int_0^t \cos\frac{nju}{2}\Big(\sin\frac{nu}{2}\Big)^j du\Big|\\ &\geqq 2^j t^{-1}\cos 1\int_0^t\Big(\sin\frac{|n|u}{2}\Big)^j du\\ &\geqq K(|n|t)^j .\end{aligned}$$

Hence

$$[{}_a\tilde{A}^{l,\phi}_{p,j,\alpha}(\{r_n\})]^p \geqq K\sum_{k=1}^{\infty}\int_{[(k+1)j]^{-1}}^{(kj)^{-1}} t^{-p\alpha-l-1+pj}[\phi(t)]^{-1}dt(\sum_{|n|\leq k}|r_n|^a|n|^{aj})^{p/a}$$

which, (2.9) being noted, is seen to be

$$\geqq K\sum_{k=1}^{\infty}[\phi(k^{-1})]^{-1}(\sum_{|n|\leq k}|r_n|^a|n|^{aj})^{p/a}\int_{[(k+1)j]^{-1}}^{(kj)^{-1}} t^{-p\alpha-l-1+pj}dt$$

$$\geqq K\sum_{k=1}^{\infty} k^{p\alpha-pj+l-1}[\phi(k^{-1})]^{-1}(\sum_{|n|\leq k}|r_n|^a|n|^{aj})^{p/a} .$$

This proves (iii).

PROOF OF (iv). We first remark that the series $\sum|r_n|^a$ should converge, which is implied by the finiteness of ${}_aB^{l,\phi}_{p,\alpha}(\{r_n\})$.

We then have

$$\left[\sum_{n=-\infty}^{\infty}|r_n|^a\left|\frac{1}{t}\int_0^t(1-e^{inu})^j du\right|^a\right]^{p/a}$$

$$\leqq K\left[\sum_{n=-\infty}^{\infty}|r_n|^a\left(\frac{1}{t}\int_0^t\left|\sin\frac{nu}{2}\right|^j du\right)^a\right]^{p/a}$$

$$\leqq K[\sum_{|n|\leq 1/t}|r_n|^a(|n|t)^{aj}]^{p/a}+K(\sum_{|n|>1/t}|r_n|^a)^{p/a} , \tag{2.11}$$

whether p/a is larger than 1, or not.

Now

$$[{}_aC^{l,\phi}_{p,j,\alpha}(\{r_n\})]^p \geqq K\sum_{k=1}^{\infty} k^{p\alpha+l-1}[\phi(k^{-1})]^{-1}(\sum_{|n|\leq 1/t}|r_n|^a|nt|^{aj})^{p/a}$$

for $(k+1)^{-1}\leqq t<k^{-1}$, and this is not less than

$$K\sum_{k=1}^{\infty}\int_{(k+1)^{-1}}^{k^{-1}} t^{-p\alpha-l-1}[\phi(t)]^{-1}(\sum_{|n|\leq 1/t}|r_n|^a|nt|^{aj})^{p/a}dt . \tag{2.12}$$

In a similar way

$$[{}_aB^{l,\phi}_{p,\alpha}(\{r_n\})]^p \geqq K\sum_{k=1}^{\infty} k^{p\alpha+l-1}[\phi(k^{-1})]^{-1}(\sum_{|n|\geq k}|r_n|^a)^{p/a}$$

$$\geqq K\int_0^1 t^{-p-\alpha l-1}[\phi(t)]^{-1}(\sum_{|n|>1/t}|r_n|^a)^{p/a}dt . \tag{2.13}$$

Hence, for any $p>0$, (2.12), (2.13) and (2.11) give

$${}_aB^{l,\phi}_{p,\alpha}(\{r_n\})+{}_aC^{l,\phi}_{p,j,\alpha}(\{r_n\})$$

$$\geqq K\{[{}_aB^{l,\phi}_{p,\alpha}(\{r_n\})]^p+[{}_aC^{l,\phi}_{p,j,\alpha}(\{r_n\})]^p\}^{1/p}$$

$$\geqq K\left\{\int_0^1 t^{-p\alpha-l-1}[\phi(t)]^{-1}\left[\sum_{n=-\infty}^{\infty}|r_n|^a\left|\frac{1}{t}\int_0^t(1-e^{inu})^j du\right|^a\right]^{p/a}dt\right\}^{1/p} .$$

Thus the proof is complete.

§ 3. Almost sure absolute convergence of the Fourier series of a stochastic process.

Let $X(t, \omega)$ be a 2π-periodic stochastic process belonging to $L^{s,r}(T\times\Omega)$, $1\leqq r, s$. Then from (1.8), $X(t, \omega)\in L^{\theta}(T)$, $\theta=\min(r, s)$, almost surely and we may define the Fourier series of $X(t, \omega)$

(3.1) $$X(t, \omega) \sim \sum_{n=-\infty}^{\infty} C_n(\omega)e^{int} ,$$

where

(3.2) $$C_n(\omega)=\frac{1}{2\pi}\int_T X(t, \omega)e^{-int}dt$$

almost surely

Writing, for a positive integer j and $h>0$,

(3.3) $$M^{*(j)}_{s,r}(h)=\sup\left[\frac{1}{2\pi}\int_T \|\Delta^{(j)}_u X(t, \cdot)\|^s_r dt\right]^{1/s} ,$$

the author [3] has shown

THEOREM A. *Let $\phi(t)$ be as in Section 1 and let $1\leqq r, s\leqq 2$. If, for some positive integer j and some nonnegative integer l,*

(3.4) $$\sum_{n=1}^{\infty} n^{l-1+1/\theta}[\phi(n^{-1})]^{-1}M^{*(j)}_{s,r}(n^{-1})<\infty ,$$

then

(3.5) $$\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} |n|^l[\phi(|n|^{-1})]^{-1}|C_n(\omega)|<\infty ,$$

almost surely.

We remark that the condition (3.4) is equivalent to

(3.6) $$\int_0^1 t^{-l-1/\theta-1}[\phi(t)]^{-1}M^{*(j)}_{s,r}(t)dt<\infty$$

as is easily seen.

As generalizations of $L^{(j)}$ and ${}_aA^{l,\phi}_{p,j,\alpha}$ for a function, we define the following quantities, for α, a, $p>0$, a positive integer j, and a nonnegative integer l, and for $X(t, w)\in L^{s,r}(T\times\Omega)$, $1\leqq s, r$:

$$(3.7)\qquad L^{(j)}(h, t, \omega; X)=\frac{1}{h}\int_0^h \Delta_u^{(j)} X(t, \omega)du ,$$

$$(3.8)\qquad {}_aA_{p,j,\alpha,r}^{l,\phi}(X)=\left(\int_0^1 h^{-l-1}[\phi(h)]^{-1}dh\cdot\left\{\int_T [h^{-\alpha}\|L^{(j)}(h, t, \cdot; X)\|_r]^a dt\right\}^{p/a}\right)^{1/p} .$$

We agree to call the class of $X(t, \omega)$ for which ${}_aA_{p,j,\alpha,r}^{l,\phi}(X)<\infty$ the class ${}_aA_{p,j,\alpha,r}^{l,\phi}$.

THEOREM 1. *Let* $1\leqq s\leqq r$, $0<p\leqq s\leqq 2$, $1/s+1/s'=1$. *If, for some positive integer* j *and some nonnegative integer* l *and with* $\alpha=1/p-1/s'$, $X(t, \omega)\in {}_sA_{p,j,\alpha,r}^{l,\phi}$, *then* ${}_{s'}B_{p,\alpha}^{l,\phi}(\{\|C_n(\cdot)\|_r\})<\infty$, $C_n(\omega)$ *being the Fourier coefficients of* $X(t, \omega)$.

COROLLARY 1. *Under the conditions of Theorem* 1, *we have*

$$(3.9)\qquad \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} |n|^l[\phi(|n|^{-1})]^{-1}|C_n(\omega)|^p<\infty$$

almost surely.

Theorem A with $1\leqq s\leqq r$ is a consequence of Corollary 1, since

$$(3.10)\qquad \left[\frac{1}{2\pi}\int_T \|L^{(j)}(h, t, \cdot; X)\|_r^s dt\right]^{1/s}\leqq\left\{\frac{1}{2\pi}\int_T\left[\frac{1}{h}\int_0^h \|\Delta_u^{(j)} X(t, \cdot)\|_r du\right]^s dt\right\}^{1/s}$$

which, by the Minkowski inequality, is not greater than

$$\frac{1}{h}\int_0^h\left[\frac{1}{2\pi}\int_T \|\Delta_u^{(j)} X(t, \cdot)\|_r^s dt\right]^{1/s} du\leqq M_{s,r}^{*(j)}(h) .$$

PROOF OF THEOREM 1. We first mention

$$(3.11)\qquad \Delta_h^{(j)} X(t, \omega) \sim \sum_{n=-\infty}^{\infty} C_n(\omega)(1-e^{inh})^j e^{int} .$$

We now have, with $\alpha=1/p-1/s'$,

$$\begin{aligned}
&[{}_{s'}\tilde{A}_{p,j,\alpha}^{l,\phi}(\{\|C_n(\cdot)\|_r\})]^p\\
&=\int_0^1 h^{-p\alpha-l-1}[\phi(h)]^{-1}dh\left[\sum_{n=-\infty}^{\infty}\|C_n(\cdot)\|_r^{s'}\frac{1}{h}\left|\int_0^h(1-e^{inu})^j du\right|^{s'}\right]^{p/s'}\\
&=\int_0^1 h^{-p\alpha-l-1}[\phi(h)]^{-1}dh\left\{\sum_{n=-\infty}^{\infty}\left[E\left|C_n(\omega)\frac{1}{h}\int_0^h(1-e^{inu})^j du\right|^r\right]^{s'/r}\right\}^{p/s'}
\end{aligned}$$

which, by the Minkowski inequality, is not greater than

$$\int_0^1 h^{-p\alpha-l-1}[\phi(h)]^{-1}dh\left\{E\left[\sum_{n=-\infty}^{\infty}|C_n(\omega)|^{s'}\frac{1}{h}\left|\int_0^h(1-e^{inu})^j du\right|^{s'}\right]^{r/s'}\right\}^{p/r} .$$

This, by the Hausdorff-Young inequality, is not greater than

$$K\int_0^1 h^{-p\alpha-l-1}[\phi(h)]^{-1}dh\left\{E\left[\int_T |L^{(j)}(h,\,t,\,\omega;\,X)|^s dt\right]^{r/s}\right\}^{p/r} .$$

Using the Minkowski inequality again, we see that the last one is not greater than

$$K\int_0^1 h^{-p\alpha-l-1}[\phi(h)]^{-1}dh\left[\int_T \|L^{(j)}(h,\,t,\,\cdot;\,X)\|_r^s dt\right]^{p/s} \tag{3.12}$$

which is finite by assumption. Thus ${}_{s'}\tilde{A}^{l,\phi}_{p,j,\alpha}(\{\|C_n(\cdot)\|_r\})$ is finite.

On the other hand, from Lemma 2 (iii) and (ii) with $\alpha=1/p-1/s'$, we have

$$\begin{aligned} {}_{s'}\tilde{A}^{l,\phi}_{p,j,\alpha}(\{\|C_n(\cdot)\|_r\}) &\geqq K_{s'}C^{l,\phi}_{p,j,\alpha}(\{\|C_n(\cdot)\|_r\}) \\ &\geqq K_{s'}B^{l,\phi}_{p,\alpha}(\{\|C(\cdot)\|_r\}) . \end{aligned} \tag{3.13}$$

This proves the theorem.

PROOF OF COROLLARY 1. By Theorem 1,

$$\sum_{n=1}^{\infty} n^{p\alpha+l-1}[\phi(n^{-1})]^{-1}[\sum_{|k|\geqq n}\|C_k(\cdot)\|_r^{s'}]^{p/s'}<\infty . \tag{3.14}$$

Noting $p\alpha+l-1=-p/s'+l$ and $p/s'\leqq 1$, we can, by Lemma 1, easily see that (3.14) implies

$$\sum_{n=-\infty}^{\infty}{}' |n|^l[\phi(|n|^{-1})]^{-1}\|C_n(\cdot)\|_r^p<\infty , \tag{3.15}$$

where $\sum'$ means that the term for $n=0$ is dropped out.

Now because of the Minkowski inequality, we have

$$\begin{aligned} &\left\{E\left(\sum_{n=-\infty}^{\infty}{}' |n|^l[\phi(|n|^{-1})]^{-1}|C_n(\omega)|^p\right)^{r/p}\right\}^{p/r} \\ &\leqq \sum_{n=-\infty}^{\infty}{}' \{E(|n|^l[\phi(|n|^{-1})]^{-1})^{r/p}|C_n(\omega)|^r\}^{p/r} \\ &= \sum_{n=-\infty}^{\infty}{}' |n|^l[\phi(|n|^{-1})]^{-1}\|C_n(\cdot)\|_r^p \end{aligned}$$

which is finite by (3.14). Therefore

$$\sum_{n=-\infty}^{\infty}{}' |n|^l[\phi(|n|^{-1})]^{-1}|C_n(\omega)|^p<\infty$$

almost surely.

REMARK. We remark that the assumption of Theorem 1 necessarily implies $j>(l+1)/p-1/s'$.

This is immediate. Because from the proof of Theorem 1, ${}_{s'}C^{l;\phi}_{p,j,\alpha}(\{\|C_n(\cdot)\|_r\})$ is finite which implies in turn $\sum k^{p\alpha-pj+l-1}[\phi(k^{-1})]^{-1}<\infty$. From this $p\alpha-kj+l$ should be negative.

§ 4. Contraction and shrivel.

For two functions $f(t)$ and $g(t)$ defined on a domain D in $\boldsymbol{R}^1$, Beurling [1] introduced the idea of contraction: If $|g(t_1)-g(t_2)|\leqq|f(t_1)-f(t_2)|$ holds for all $t_1, t_2\in D$, he said that $g(t)$ is a contraction of $f(t)$.

We now suppose $D=\boldsymbol{R}^1$ and $f(t)$ and $g(t)$ are 2π-periodic. Yadav [6] generalized the condition of contraction, saying that $g(t)$ is a shrivel of $f(t)$ of order j, j being a positive integer, if

$$|\Delta_h^{(j)}g(t)|\leqq K|\Delta_h^{(j)}f(t)| \tag{4.1}$$

for all $0<h\leqq 1$, $t\in\boldsymbol{R}^1$, where K is a constant independent of t and h. As a matter of fact, he used the more general condition

$$\int_T|L^{(j)}(h, t; g)|^2dt\leqq K\int_T|L^{(j)}(h, t; f)|^2dt\ , \tag{4.2}$$

for all $0<h\leqq 1$, K being a constant independent of h.

We define the similar contraction condition for a stochastic process.

We shall say that a 2π-periodic stochastic process $X(t, \omega)$ is a shrivel of $f(t)$ in mean of order $j\,(\geqq 1)$ if

$$\|\Delta_h^{(j)}X(t, \cdot)\|_r\leqq K|\Delta_h^{(j)}f(t)| \tag{4.3}$$

for all $t\in T$, $0<h\leqq 1$, K being a constant independent of h and t. More generally as a generalization of (4.2), we, in what follows, are interested in the shrivel condition

$$\left\{\int_T\|L^{(j)}(h, t, \cdot; X)\|_r^s dt\right\}^{1/s}\leqq K\left\{\int_T|L^{(j)}(h, t; f)|^{s'}dt\right\}^{1/s'} \tag{4.4}$$

for $1\leqq s, r$, $1/s+1/s'=1$. When this is the case, $X(t, \omega)\in L^{s,r}(T\times\Omega)$ may be said an integrated mean shrivel of $f(t)$ of order j.

We now claim

THEOREM 2. *Let $0<p\leqq s\leqq 2$, $1\leqq s\leqq r$. Suppose $X(t, \omega)\in L^{s,r}(T\times\Omega)$ and $f(t)\in L^{s'}(T)$. If the shrivel condition* (4.3) *or more generally the integrated mean shrivel condition* (4.4) *holds for a positive integer j such*

that $j>\alpha+l/p$ *for* $\phi(t)\equiv 1$ *and* $j>\alpha+(l+1)/p$ *for* $\phi(t)\not\equiv 1$, *for some nonnegative integer* l, *with* $\alpha=1/p-1/s'$, *then* ${}_sB_{p,\alpha}^{l,\phi}(\{c_n\})<\infty$ *implies* ${}_{s'}B_{p,\alpha}^{l,\phi}(\{\|C_n(\cdot)\|_r\})<\infty$, *where* c_n *and* $C_n(\omega)$ *are Fourier coefficients of* $f(t)$ *and* $X(t,\omega)$ *respectively.*

COROLLARY 2. *Under the conditions of Theorem* 2, *if* ${}_sB_{p,\alpha}^{l,\phi}(\{c_n\})<\infty$, *then* (3.9) *holds almost surely.*

PROOF OF THEOREM 2. From (3.13) and (3.12),

$$\begin{aligned}&{}_{s'}B_{p,\alpha}^{l,\phi}(\{\|C_n(\cdot)\|_r\})\leqq K_{s'}\tilde{A}_{p,j,\alpha}^{l,\phi}(\{\|C_n(\cdot)\|_r\})\\ &\leqq K\left\{\int_0^1 h^{-p\alpha-l-1}[\phi(h)]^{-1}dh\left[\int_T\|L^{(j)}(h,t;X)\|_r^s dt\right]^{p/s}\right\}^{1/p}.\end{aligned}$$

By the condition (4.4), the last one is

$$\leqq K\left\{\int_0^1 h^{-p\alpha-l-1}[\phi(h)]^{-1}dh\left[\int_T|L^{(j)}(h,t,f)|^{s'}dt\right]^{p/s'}\right\}^{1/p}$$

which, because of the Hausdorff-Young theorem, is

$$\begin{aligned}&\leqq K\left\{\int_0^1 h^{-p\alpha-l-1}[\phi(h)]^{-1}\left[\sum_{n=-\infty}^{\infty}|c_n|^s\frac{1}{h}\left|\int_0^h(1-e^{inu})^j du\right|\right]^{p/s}\right\}^{1/p}\\ &\leqq K_s\tilde{A}_{p,j,\alpha}^{l,\phi}(\{c_n\}),\end{aligned}$$

which, from Lemma 2 (i) and (ii), is not greater than $K_sB_{p,\alpha}^{l,\phi}(\{c_n\})<\infty$. This proves the theorem.

The proof of Corollary 2 has been given in the course of the proof of Corollary 1.

§ 5. Sample properties of stochastic processes.

Let us denote by Λ_ϕ the Lipschitz class of 2π-periodic functions $f(t)$ such that

$$\sup_{t,|h|\leqq\delta}|f(t+h)-f(t)|\leqq K\phi(\delta)$$

if $\phi(t)$ is not identically one. We agree that Λ_ϕ means the class of continuous functions, if $\phi(t)\equiv 1$.

Let $X(t,\omega)$ be a 2π-periodic stochastic process which is stochastically continuous, namely $P(|X(t+h,\omega)-X(t,\omega)|>\varepsilon)\to 0$ (as $h\to 0$) for every $\varepsilon>0$ and every t. P is the probability measure on Ω. We know [2, Lemma 6.1] that in this case, the defining limit relation holds uniformly and the $(C,1)$ means $\sigma_n(t,\omega)$ of the Fourier series of $X(t,\omega)$ converges uniformly in probability to $X(t,\omega)$.

Now we take $p=1$ in Corollary 1 or 2. Then under the conditions of Theorems 1 and 2 with $p=1$,

$$\sum_{n=-\infty}^{\infty}{}' |n|^l[\phi(n^{-1})]^{-1}|C_n(\omega)|<\infty \tag{5.1}$$

almost surely. (5.1) of course implies $\sum_{n=-\infty}^{\infty}|C_n(\omega)|<\infty$ almost surely. Define

$$X_0(t, \omega)=\sum_{n=-\infty}^{\infty} C_n(\omega)e^{int} . \tag{5.2}$$

Then keeping the above facts in mind, we have that $X_0(t, \omega)=X(t, \omega)$ almost surely for each t, namely $X_0(t, \omega)$ is a modification of $X(t, \omega)$. We see from (5.1) that $X_0(t, \omega)$ has the l-th derivative which belongs to Λ_ϕ. Therefore we can claim the following theorems.

THEOREM 3. *Let $1\leqq s\leqq 2$, $s\leqq r$. If $X(t, \omega)\in {}_sA^{l;\phi}_{1;j,1/s,r}$ for some positive integer j such that $j>1/s+l$ for $\phi(t)\equiv 1$ and $j>1/s+l+1$ for $\phi(t)\not\equiv 1$, l being some nonnegative integer. Then there is a modification $X_0(t, \omega)$ of $X(t, \omega)$, which has the l-th derivative belonging to Λ_ϕ.*

THEOREM 4. *Let $1\leqq s\leqq 2$, $s\leqq r$ and let a 2π-periodic process $X(t, \omega)\in L^{s,r}(T\times\Omega)$. If there is a 2π-periodic function $\in L^{s}(T)$ such that the mean shrivel condition (4.4) is satisfied for some positive integer $j>1/s+l$ for $\phi(t)\equiv 1$ and $j>1/s+l+1$ for $\phi(t)\not\equiv 1$, then there is a modification $X_0(t, \omega)$, which has the l-th derivative belonging to Λ_ϕ.*

For details of arguments proving these theorems, see [2, Section 6].

References

[1] A. BEURLING, On the spectral synthesis of bounded functions, Acta Math., **81** (1949), 225–238.

[2] T. KAWATA, Absolute convergence of Fourier series and its applications, Tôhoku Math. J., **35** (1983), 459–474.

[3] T. KAWATA, Lipschitz classes of periodic stochastic processes and Fourier series, Tokyo J. Math., **8** (1985), 277–301.

[4] M. KINUKAWA, Lipschitz classes and Fourier series, Sûgaku, **24** (1973), 51–53, Math. Soc. Japan (in Japanese).

[5] M. KINUKAWA, Some generalizations of contraction theorems, Pacific J. Math., **109** (1983), 121–134.

[6] B. S. YADAV, Contractions of functions and their Fourier series, Pacific J. Math., **31** (1969), 827–832.

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