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K-Groups and λ-Invariants of Algebraic Number Fields

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Dedicated to the late Professor Suguru Hamada

Introduction.

Let F be a totally real algebraic number field, O_F the integer ring of F and $K_m(O_F)$ Quillen's higher K-group of O_F for each non-negative integer m. According to Quillen [8], $K_m(O_F)$ is a finite abelian group for even $m=2n$ $(n\geqq 1)$. Let p be an odd prime number and F' a Galois p-extension of F. In this paper, we investigate whether the prime p divides the order of $K_{2n}(O_{F'})$. (The order of $K_2(O_{F'})$ has been treated by several authors [2], [4], [9].) We shall state our main theorem in §1. In §2, we prove group-theoretical lemmas on $\boldsymbol{Z}_p$-modules on which a finite group acts, whose order is prime to p.

In the final part §3, we prove our main theorem in using first a result of Soulé, according to which we translate the language of K-theory into that of Iwasawa theory, then a result of Iwasawa (Lemma 4), with the help of which we refine Kida's formula (Lemma 5), which leads immediately to our theorem.

§1. Main theorem.

Throughout the following, let p be a fixed odd prime number. For a finite algebraic number field F, we denote by F_∞ the cyclotomic $\boldsymbol{Z}_p$-extension of F.

THEOREM. *Let F be a totally real algebraic number field of finite degree, F' a Galois p-extension of F, ζ a primitive p-th root of* 1 *and n an odd positive integer. Let k denote $F(\zeta)$ and d the degree $(k:F)$. We assume that the μ-invariant μ_k of k_∞/k is zero. Then we have the*

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following:

(1) *We assume that* $n\not\equiv -1 \pmod d$. *If there exists a prime ideal* $\mathfrak{L}$ *of* F_∞ *which ramifies tamely in* F'_∞/F_∞, *then the prime* p *divides the order of the K-group* $K_{2n}(O_{F'})$.

(2) *We assume that* $n\equiv -1 \pmod d$. *If there exist two distinct prime ideals* $\mathfrak{L}_1$, $\mathfrak{L}_2$ *of* F_∞ *which ramify tamely in* F'_∞/F_∞, *then the prime* p *divides the order of* $K_{2n}(O_{F'})$.

(3) *We assume that* $d\neq 2$ *and that* F'/F *is unramified outside* p. *The prime* p *divides the order of* $K_2(O_F)$ *if and only if* p *divides the order of* $K_2(O_{F'})$.

(4) *We assume that* $d=2$ *and that at most one prime ideal ramifies tamely in* F'_∞/F_∞. *The prime* p *divides the order of* $K_2(O_F)$ *if and only if* p *divides the order of* $K_2(O_{F'})$.

REMARK. Let $\mathfrak{l}$ be a prime ideal of F and $\mathfrak{L}$ a prime ideal of F_∞ lying above $\mathfrak{l}$. Then if $\mathfrak{l}$ ramifies tamely in F'/F, $\mathfrak{L}$ ramifies tamely in F'_∞/F_∞.

§2. Group-theoretical lemmas.

Let G be a topological group and H_1, H_2 closed subgroups of G. We denote by (H_1, H_2) the topological commutator group of H_1 and H_2. The following two lemmas play important roles in this paper.

LEMMA 1. *Let* Δ *be a finite group whose order is prime to* p. *Let* G *be a finitely generated pro-p-group on which* Δ *acts. Let* N *be an open normal* Δ*-subgroup of* G *and* x *an element of* G *such that the coset* $\delta(x)N$ *coincides with* xN *for any element* δ *of* Δ. *Then there exists an element* y *in* xN *such that* $\delta(y)=y$ *for any element* δ *of* Δ.

PROOF. We put $N_0=N$ and $N_{i+1}=N_i^p(N_i, N)$. Then the system $\{N_i\}_{i=0}^\infty$ is a fundamental system of neighborhoods of unity. We put $x_0=x$ and $f(\delta)=\delta(x_0)^{-1}x_0N_1$ for each element $\delta\in\Delta$. Then the mapping $f\colon \Delta\to N_0/N_1$ is a 1-cocycle, where N_0/N_1 is a factor group of N_0 over N_1. Since the order of Δ is prime to p, the cohomology group $H^1(\Delta, N_0/N_1)$ is trivial. Hence there exists an element n_0 of N_0 such that $\delta(x_0)^{-1}x_0N_1=\delta(n_0)^{-1}n_0N_1$. We put $x_1=x_0n_0^{-1}$. Then we have $\delta(x_1)N_1=x_1N_1$. We repeat the above procedure and obtain x_i for $i=0, 1, 2, \cdots$. We put $y=\lim_i x_i$. Then we have $yN=xN$ and $\delta(y)=y$ for any element δ of Δ.

Now let E be a finitely generated free pro-p-group and G_0 a cyclic group of order d which acts on E. We assume that d divides $p-1$. Let

N be an open normal G_0-subgroup of E with $(E:N)=p^e$, e being a given positive integer. We put $\tilde{E}=(E, E)$, $\tilde{N}=(N, N)$, $X=E/\tilde{E}$ and $X'=N/\tilde{N}$. Let χ be a character (a homomorphism) of G_0 into $\boldsymbol{Z}_p^\times$ with the order d. We put

$$\varepsilon_i=\frac{1}{d}\sum_{g\in G_0}\chi(g)^i g^{-1}\in \boldsymbol{Z}_p[G_0]$$

for each integer i. We can consider X and X' as $\boldsymbol{Z}_p[G_0]$-modules in a natural way. Then we have the following:

LEMMA 2. *If G_0 acts on E/N trivially, then*

$$\mathrm{rank}_{\boldsymbol{Z}_p}\varepsilon_0 X'-1=p^e(\mathrm{rank}_{\boldsymbol{Z}_p}\varepsilon_0 X-1) \qquad \textit{and}$$
$$\mathrm{rank}_{\boldsymbol{Z}_p}\varepsilon_i X'=p^e(\mathrm{rank}_{\boldsymbol{Z}_p}\varepsilon_i X) \qquad \textit{for} \quad i=1, 2, \cdots, d-1\,.$$

PROOF. First, we prove our assertion for the case $e=1$. Let x, $y_1, \cdots, y_n$ be free generators of E. We may assume from Lemma 1 that $g(x)=x$ for every element $g\in G_0$ and that N contains $y_1, \cdots, y_n$. It is well known that

$$\{x^p, y_1, \cdots, y_n, xy_1x^{-1}, \cdots, xy_nx^{-1}, \cdots, x^{p-1}y_1x^{-(p-1)}, \cdots, x^{p-1}y_nx^{-(p-1)}\}$$

is a free generator system of N. We regard X and X' as $\boldsymbol{Z}_p$-modules. Then we have

$$\begin{aligned} X'&=\boldsymbol{Z}_p(\tilde{N}x^p)\oplus\Big(\bigoplus_{\substack{0\leq i\leq p-1\\ 1\leq j\leq n}}\boldsymbol{Z}_p(\tilde{N}x^i y_j x^{-i})\Big)\\ &=\boldsymbol{Z}_p(\tilde{N}x^p)\oplus\Big(\bigoplus_{j=1}^{n}\boldsymbol{Z}_p(\tilde{N}y_j)\Big)\oplus\Big(\bigoplus_{\substack{1\leq i\leq p-1\\ 1\leq j\leq n}}\boldsymbol{Z}_p(\tilde{N}x^i y_j x^{-i}y_j^{-1})\Big)\\ &=\boldsymbol{Z}_p(\tilde{N}x^p)\oplus\Big(\bigoplus_{j=1}^{n}\boldsymbol{Z}_p(\tilde{N}y_j)\Big)\oplus(\tilde{E}/\tilde{N})\,. \end{aligned}$$

Since $\boldsymbol{Z}_p(\tilde{N}x^p)\oplus\tilde{E}/\tilde{N}$ is a G_0-module and since d is prime to p, there exists a G_0-submodule $Y/\tilde{N}$ of $N/\tilde{N}$ such that $X'=\boldsymbol{Z}_p(\tilde{N}x^p)\oplus Y/\tilde{N}\oplus\tilde{E}/\tilde{N}$. Let $z_{i1}\tilde{N}, \cdots, z_{ir_i}\tilde{N}$ be a basis of $\varepsilon_i(Y/\tilde{N})$ for $0\leqq i\leqq d-1$. Then x, $z_{01}, \cdots$, $z_{0r_0}, \cdots, z_{d-1\,1}, \cdots, z_{d-1\,r_{d-1}}$ are free generators of E. Since we have

$$g(x^\nu z_{ij}x^{-\nu})\tilde{N}=x^\nu z_{ij}^{\chi(g)^i}x^{-\nu}\tilde{N}=(x^\nu z_{ij}x^{-\nu}\tilde{N})^{\chi(g)^i}$$

for any element $g\in G_0$, we have $\mathrm{rank}_{\boldsymbol{Z}_p}\varepsilon_i X'=p(\mathrm{rank}_{\boldsymbol{Z}_p}\varepsilon_i X)$ for $1\leqq i\leqq d-1$ and $\mathrm{rank}_{\boldsymbol{Z}_p}\varepsilon_0 X'-1=p(\mathrm{rank}_{\boldsymbol{Z}_p}\varepsilon_0 X-1)$.

Now, let e be any positive integer. There exists a sequence of subgroups of E

$$E=N_0\supset N_1\supset\cdots\supset N_e=N$$

such that each N_i/N_{i+1} is a cyclic group of order p. Hence induction shows our assertion.

§ 3. Proof of Theorem.

Let S be the set of prime ideals of F which ramify tamely in F'/F and S_0 the set of prime ideals of F lying above p. Let L be the maximal p-extension of k unramified outside $S\cup S_0$. As k/F is a Galois extension, L/F is a Galois extension. Since the degree $d=(k:F)$ is prime to p, there exists an intermediate field K between L and F such that $L=Kk$ and $K\cap k=F$. We notice that the Galois group $G(k/F)$ is isomorphic to $G(L/K)$ in a natural way and that $G(L/F)$ is a semi-direct product of $G(L/K)$ and $G(L/k)$. We put $G_0=G(L/K)$. Let $\chi:G(L/K)\to \boldsymbol{Z}_p^\times$ be the character such that $\zeta^g=\zeta^{\chi(g)}$ for all $g\in G(L/K)$. We define

$$\varepsilon_i=\frac{1}{d}\sum_{g\in G_0}\chi(g)^i g^{-1}\in \boldsymbol{Z}_p[G_0]$$

for each integer i. Let A_∞ be the p-part of the ideal class group of k_∞ and G_∞ the Galois group of k_∞ over F. Then G_∞ acts on A_∞ in a natural way. We put $A_\infty^-=\bigoplus_{i=1}^{d/2}\varepsilon_{2i-1}A_\infty$. Now, when F is replaced by F', the field k will be replaced by $k'=F'(\zeta)$, the p-part A_∞ of the ideal class group will be replaced by A'_∞ and the μ-invariant μ_k will be replaced by $\mu_{k'}$: similar notations will be used in the following. Let W_{p^n} be the group of p^n-th root of unity and $\mathscr{T}=\varprojlim W_{p^n}$ the Tate module. Thus $\mathscr{T}$ is a free $\boldsymbol{Z}_p$-module of rank 1, on which G_∞ acts in a natural way. If X is a G_∞-module which is also a $\boldsymbol{Z}_p$-module, we define, for each integer $\nu\geqq 0$, $X(\nu)=X\otimes_{\boldsymbol{Z}_p}\mathscr{T}\otimes_{\boldsymbol{Z}_p}\cdots\otimes_{\boldsymbol{Z}_p}\mathscr{T}$ (ν times), endowed with the diagonal action of G_∞. Soulé's theorem asserts that, for each odd positive integer ν, there exists a canonical surjective homomorphism

$$K_{2\nu}(O_F)(p)\longrightarrow (A_\infty^-(\nu))^{G_\infty}\qquad \text{(cf. [3] and [9])}\ ,$$

where $K_{2\nu}(O_F)(p)$ denotes the p-primary subgroup of $K_{2\nu}(O_F)$. (For a G_∞-module X, we denote as usual by X^{G_∞} the G_∞-invariant submodule.) This mapping is an isomorphism for $\nu=1$. Now, we have

$$A_\infty^-(\nu)^{G_\infty}=(A_\infty^-(\nu)^{G_0})^{G(k_\infty/k)}=((\varepsilon_{d-\nu}A_\infty)(\nu))^{G(k_\infty/k)}$$

for odd positive integer ν. Hence we see that $A_\infty^-(\nu)^{G_\infty}=0$ if and only if $\varepsilon_{d-\nu}A_\infty=0$. Therefore we have the following:

LEMMA 3. *Let ν be an odd positive integer. If $\varepsilon_{d-\nu}A_\infty \neq 0$, then p divides the order of $K_{2\nu}(O_F)$. Furthermore, $\varepsilon_{d-1}A_\infty \neq 0$ if and only if p divides the order of $K_2(O_F)$.*

Now, we assume, from now on, $\mu_k=0$. Then $\mu_{k'}=0$ follows from Iwasawa [5]. Furthermore, there exists a non-negative integer λ_i such that $\varepsilon_i A_\infty \cong (\boldsymbol{Q}_p/\boldsymbol{Z}_p)^{\lambda_i}$. Let k_∞^+ denote the maximal real subfield of k_∞, M the maximal p-extension of k_∞^+ unramified outside $S_0 \cup S$ and E the Galois group of M over k_∞^+. Let s be the number of prime ideals of F_∞ which lie above S. Then we have the following:

LEMMA 4 (cf. [6, Theorem 1 and the proof of Theorem 3]). *Let i be an odd integer such that $1 \leqq i \leqq d-1$. Let j be an integer such that $j \equiv 1-i \pmod{d}$. We put $X=E/(E, E)$. Then $\varepsilon_j X \cong \boldsymbol{Z}_p^{\lambda_i+s}$.*

REMARK. Let $\mathfrak{l}$ be a prime ideal in S and $\mathfrak{L}$ be a prime ideal of F_∞ lying above $\mathfrak{l}$. Since $\mathfrak{L}$ is tamely ramified in F'_∞/F_∞, $\mathfrak{L}$ splits in k_∞/F_∞.

Since M contains F', Lemma 2 and Lemma 4 yield the following lemma which is a refinement of Kida's formula (cf. [7]).

LEMMA 5. *We put $\varepsilon_i A_\infty = (\boldsymbol{Q}_p/\boldsymbol{Z}_p)^{\lambda_i}$ and $\varepsilon_i A'_\infty = (\boldsymbol{Q}_p/\boldsymbol{Z}_p)^{\lambda'_i}$. Then we have $\lambda'_1+s'-1=p^e(\lambda_1+s-1)$ and $\lambda'_i+s'=p^e(\lambda_i+s)$ for the odd integer i from 3 to $d-1$. Here, $p^e=(k'^+_\infty : k^+_\infty)=(E:E')$.*

Lemma 3 and Lemma 5 yield our theorem.

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