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A Theorem of Pitman Type for Simple Random Walks on Z^d

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Introduction.

Pitman's theorem ([2]) for a one-dimensional Brownian motion $B(t)$ states that $B(t)-2M(t)$ is a Bessel process of index 3, where $B(0)=0$ is assumed and $M(t)$ denotes the minimum of $B(s)$, $0\leqq s\leqq t$. This theorem can be obtained, after a scaling limit, from a similar theorem for a coin-tossing random walk on $\boldsymbol{Z}$ which is easy to prove and may still be called Pitman's theorem. An extension of Pitman's theorem to higher dimensional random walks is the following: given a simple random walk $\boldsymbol{S}_n$ on the d-dimensional lattice $\boldsymbol{Z}^d$ starting at 0, let $S_n^{(i)}$ be the i-th coordinate of $\boldsymbol{S}_n$ and denote by $M_n^{(i)}$ the minimum of $S_k^{(i)}$, $0\leqq k\leqq n$. Then the process

$$\boldsymbol{S}_n-2\boldsymbol{M}_n=(S_n^{(1)}-2M_n^{(1)},\ S_n^{(2)}-2M_n^{(2)},\ \cdots,\ S_n^{(d)}-2M_n^{(d)}) \tag{1}$$

ought to be a Markov chain. Unlike the corresponding statement for a higher dimensional Brownian motion, the above statement for $d\geqq 2$ is not an immediate consequence of the one for $d=1$ since the coordinate processes of $\boldsymbol{S}_n$ are not independent (in the case $d\geqq 2$). The purpose of this paper is to prove that $\boldsymbol{S}_n-2\boldsymbol{M}_n$ is a Markov chain on the d-dimensional (sub-)lattice $\boldsymbol{Z}_+^d$ of points with nonnegative integral coordinates (Theorem 1). Although a straightforward method used in the case $d=1$ (see §2) may also be applied to the case $d\geqq 2$, the argument will be quite messy. In this paper we employ another method which is based on the following simple observation: the coordinate processes of a simple random walk on $\boldsymbol{Z}^d$ $(d\geqq 2)$ with *continuous* time are independent although this is not true for the case of discrete time.

§1. Statement of the result.

Given an integer $d\geqq 2$, we write $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_d$ for the d-dimensional

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unit vectors $(1, 0, \cdots, 0)$, $(0, 1, 0, \cdots, 0)$, $\cdots$, $(0, \cdots, 0, 1)$ spanning $\boldsymbol{Z}^d$. Given *positive* constants $p_i^{(\sigma)}$, $\sigma=\pm 1$, $1\leqq i\leqq d$, such that $\sum_{i=1}^d p_i=1$, where $p_i=p_i^{(+1)}+p_i^{(-1)}$, let $\boldsymbol{S}_n$ denote the position at time n $(=0, 1, 2, \cdots)$ of a particle performing the random walk on $\boldsymbol{Z}^d$, according to the following rule: the particle starts at 0, namely, $\boldsymbol{S}_0=0$, and when the first n positions $\boldsymbol{S}_k$, $0\leqq k\leqq n-1$, are fixed, the particle starts afresh at $\boldsymbol{S}_{n-1}$, jumping next to one of the $2d$ neighbours $\boldsymbol{S}_n=\boldsymbol{S}_{n-1}+\sigma\boldsymbol{e}_i$, $\sigma=\pm 1$, $1\leqq i\leqq d$, with probability $p_i^{(\sigma)}$ for landing at $\boldsymbol{S}_{n-1}+\sigma\boldsymbol{e}_i$. We are interested in the Markovian property of the process $\{\boldsymbol{S}_n-2\boldsymbol{M}_n;\ n\geqq 0\}$ defined by (1).

Before giving the definition of $\tilde{\boldsymbol{p}}(\boldsymbol{x}, \boldsymbol{y})$, $\boldsymbol{x}$, $\boldsymbol{y}\in\boldsymbol{Z}_+^d$, which is expected to be the transition function of $\boldsymbol{S}_n-2\boldsymbol{M}_n$, we introduce a transition function $p(x, y; \alpha)$ on $\boldsymbol{Z}_+$ with parameter $\alpha\in(0, 1)$ as follows: if $\alpha\neq 1/2$,

$$(2.a)\qquad p(x, y; \alpha)=\begin{cases}1 & \text{for } x=0,\ y=1\ ,\\ (1-\alpha)(1-\gamma^x)(1-\gamma^{x+1})^{-1} & \text{for } x\geqq 1,\ y=x-1\ ,\\ \alpha(1-\gamma^{x+2})(1-\gamma^{x+1})^{-1} & \text{for } x\geqq 1,\ y=x+1\ ,\\ 0 & \text{otherwise}\ ,\end{cases}$$

where $\gamma=(1-\alpha)\alpha^{-1}$; if $\alpha=1/2$,

$$(2.b)\qquad p(x, y; 1/2)=\begin{cases}1 & \text{for } x=0,\ y=1\ ,\\ 2^{-1}x(x+1)^{-1} & \text{for } x\geqq 1,\ y=x-1\ ,\\ 2^{-1}(x+2)(x+1)^{-1} & \text{for } x\geqq 1,\ y=x+1\ ,\\ 0 & \text{otherwise}\ .\end{cases}$$

We then define $\tilde{\boldsymbol{p}}(\boldsymbol{x}, \boldsymbol{y})$ for $\boldsymbol{x}=(x_1, x_2, \cdots, x_d)\in\boldsymbol{Z}_+^d$ and $\boldsymbol{y}=(y_1, y_2, \cdots, y_d)\in\boldsymbol{Z}_+^d$ by

$$(3)\qquad \tilde{\boldsymbol{p}}(\boldsymbol{x}, \boldsymbol{y})=\begin{cases}p_i p(x_i, y_i; p_i^{(+1)}/p_i) & \text{for } \boldsymbol{y}=\boldsymbol{x}+\sigma\boldsymbol{e}_i \text{ with some } i\ (1\leqq i\leqq d) \text{ and } \sigma=\pm 1\ ,\\ 0 & \text{otherwise}\ ,\end{cases}$$

or equivalently by

$$(3')\qquad \tilde{\boldsymbol{p}}(\boldsymbol{x}, \boldsymbol{y})=\boldsymbol{h}(\boldsymbol{x})^{-1}\boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{h}(\boldsymbol{y})\ ,$$

where $\boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y})$ is the (one-step) transition function of $\boldsymbol{S}_n$ and

$$\boldsymbol{h}(\boldsymbol{x})=\prod_{i=1}^d h_i(x_i)\ ,$$

$$h_i(x)=\begin{cases}\left|1-\left\{\dfrac{p_i^{(-1)}}{p_i^{(+1)}}\right\}^{x+1}\right| & \text{if } p_i^{(+1)}\neq p_i^{(-1)}\ ,\\ x+1 & \text{if } p_i^{(+1)}=p_i^{(-1)}\ .\end{cases}$$

Our result is now stated as follows.

THEOREM 1. *$\{S_n-2M_n; n\geqq 0\}$ is a Markov chain on $\boldsymbol{Z}_+^d$ with (one-step) transition function $\tilde{p}(\boldsymbol{x}, \boldsymbol{y})$.*

§ 2. Pitman's theorem in the case $d=1$.

For our proof of Theorem 1 we need its one-dimensional version (Proposition 2); of course, this is essentially due to Pitman [2]; however, since a detailed proof in the case of asymmetric random walks on $\boldsymbol{Z}$ seems to be found nowhere, we give it here.

Given positive constants α and β such that $\alpha+\beta=1$, we consider a random walk

$$\begin{cases} S_0=0 \\ S_n=X_1+\cdots+X_n\,, & (n\geqq 1)\,, \end{cases}$$

where X_k, $k\geqq 1$, are independent identically distributed random variables with $P\{X_k=1\}=\alpha$, $P\{X_k=-1\}=\beta$ $(k\geqq 1)$. We put $M_n=\min\{S_k;\ 0\leqq k\leqq n\}$.

PROPOSITION 2. *$\{S_n-2M_n; n\geqq 0\}$ is a Markov chain on $\boldsymbol{Z}_+$ with transition function $p(x, y; \alpha)$ given by* (2).

PROOF. We give the proof only in the case $\alpha>1/2$, since the proof in the case $\alpha\leqq 1/2$ is much easier. Define a random time $\tau=\min\{n\geqq 1;\ S_n=-1\}$ with the convention $\min\varnothing=\infty$ and let

$$w=\begin{cases} (0, S_1, \cdots, S_{\tau-1}, 1) & \text{if} \quad \tau<\infty\,, \\ (S_n, n\geqq 0) & \text{if} \quad \tau=\infty\,. \end{cases}$$

Note that the assumption $\alpha>1/2$ implies $\tau=\infty$ with positive probability. We regard w as a random variable taking values in $\mathscr{W}=\mathscr{W}'\cup\mathscr{W}''$, where $\mathscr{W}'$ is the space of finite sequences $w=(w(n), 0\leqq n\leqq l)$ in $\boldsymbol{Z}_+$ such that (i) $1\leqq l<\infty$, (ii) $w(0)=w(l-1)=0$, (iii) $w(l)=1$ and (iv) $w(n+1)-w(n)=\pm 1$ for each n; $\mathscr{W}''$ is the space of infinite sequences $w=(w(n), n\geqq 0)$ in $\boldsymbol{Z}_+$ such that (i) $w(0)=0$, (ii) $w(n+1)-w(n)=\pm 1$ for each n, and (iii) $w(n)\to\infty$ as $n\to\infty$.

Take independent copies $w_1, w_2, \cdots$ of w, write $w_k=(w_k(n), 0\leqq n\leqq l_k)$ or $w_k=(w_k(n), n\geqq 0)$ according as $w_k\in\mathscr{W}'$ or $w_k\in\mathscr{W}''$ and put $L_0=0$, $L_k=l_1+l_2+\cdots+l_k$, $k\geqq 1$, with the convention $l_k=\infty$ if $w_k\in\mathscr{W}''$. Since $\tau=\infty$ has positive probability, we have $N=\min\{k\geqq 1;\ L_k=\infty\}<\infty$ almost surely. Denote by $(\boldsymbol{W}, \boldsymbol{P})$ the probability space on which w_k, $k\geqq 1$, are defined. We then define a process $\{W_n,\ n\geqq 0\}$ on the probability space $(\boldsymbol{W}, \boldsymbol{P})$ by

$$W_n=k-1+w_k(n-L_{k-1}) \qquad \text{for} \quad L_{k-1}\leqq n<L_k\ , \quad k=1, 2, \cdots, N\ . \tag{4}$$

It is easy to see that the process $\{W_n,\ n\geqq 0\}$ is identical in law to $\{S_n-2M_n,\ n\geqq 0\}$.

Given $x(n)\in \boldsymbol{Z}_+$, $0\leqq n\leqq m$ $(m\geqq 1)$, such that $x(0)=0$, $x(1)=1$ and $|x(n)-x(n-1)|=1$ for $1\leqq n\leqq m$, we now compute $\boldsymbol{P}(\Lambda)$ where $\Lambda=\{W_n=x(n);\ 0\leqq n\leqq m\}$. Let $M_m^+=\min\{W_n;\ n\geqq m\}$ and put

$$\Lambda_x=\Lambda\cap\{M_m^+=x\}\ , \qquad x\in \boldsymbol{Z}_+\ .$$

$M_m^+=x$ $(0\leqq x\leqq x(m))$ implies $\max\{k;\ L_k\leqq m\}\leqq x$ and hence $m<L_{x+1}$. Therefore, we have

$$\boldsymbol{P}\{\Lambda\}=\sum_{x=0}^{x(m)}\boldsymbol{P}\{\Lambda_x\}=\sum_{x=0}^{x(m)}\sum_{k=0}^{x}\boldsymbol{P}\{\Lambda_x,\ L_k\leqq m<L_{k+1}\}\ .$$

From the definition (4) of W_n we can see that

$$M_m^+=x \text{ and } L_k\leqq m<L_{k+1} \implies L_{k+1}=\infty$$

provided that $k<x$. Therefore, if $0\leqq k<x$, then

$$\begin{aligned}\boldsymbol{P}\{\Lambda_x,\ L_k\leqq m<L_{k+1}\}&=\boldsymbol{P}\{\Lambda_x,\ L_k\leqq m,\ L_{k+1}=\infty\}\\&=\pi(x(0),\ x(1),\ \cdots,\ x(m))\left(\frac{\beta}{\alpha}\right)^k\xi(x(m))\ ,\end{aligned}$$

where $\pi(x(0),\ x(1),\ \cdots,\ x(m))=\prod_{n=1}^m p(x(n-1),\ x(n))$, $p(x(n-1),\ x(n))$ being equal to α or β according as $x(n)-x(n-1)$ is 1 or -1; $\xi(x(m))$ is the probability that the random walk $\{x(m)+S_n,\ n\geqq 0\}$ starting at $x(m)$ hits x but does not hit $x-1$. It is easy to compute $\xi(x(m))$, namely, we have $\xi(x(m))=\gamma^{x(m)-x}(1-\gamma)$, where $\gamma=\beta/\alpha$ (for example, see [1], p. 314). If $k=x$, we have

$$\begin{aligned}\boldsymbol{P}\{\Lambda_x,\ L_k\leqq m<L_{k+1}\}&=\boldsymbol{P}\{\Lambda_x,\ L_x\leqq m,\ L_{x+1}=\infty\}+\boldsymbol{P}\{\Lambda_x,\ L_x\leqq m<L_{x+1}<\infty\}\\&=\pi(x(0),\ x(1),\ \cdots,\ x(m))\left(\frac{\beta}{\alpha}\right)^x\xi(x(m))\\&\quad+\pi(x(0),\ x(1),\ \cdots,\ x(m))\left(\frac{\beta}{\alpha}\right)^x\eta(x(m))\ ,\end{aligned}$$

where $\eta(x(m))$ is the probability that the random walk $\{x(m)+S_n,\ n\geqq 0\}$ hits $x-1$, namely, $\eta(x(m))=\gamma^{x(m)-x+1}$. Therefore, we finally obtain

$$
\begin{aligned}
(5) \qquad P\{\Lambda\} &= \sum_{x=0}^{x(m)} P\{\Lambda_x\} \\
&= \sum_{x=0}^{x(m)} \sum_{k=0}^{x} \pi(x(0), x(1), \cdots, x(m)) \gamma^{k+x(m)-x}(1-\gamma) \\
&\quad + \sum_{x=0}^{x(m)} \pi(x(0), x(1), \cdots, x(m)) \gamma^{x(m)+1} \\
&= \pi(x(0), x(1), \cdots, x(m)) \sum_{x=0}^{x(m)} \gamma^{x} \\
&= \prod_{n=1}^{m} \tilde{p}(x(n-1), x(n)) ,
\end{aligned}
$$

where

$$
\tilde{p}(x, y) = p(x, y) \frac{h(y)}{h(x)} , \qquad y - x = \pm 1
$$

$$
(h(x) = (1-\gamma^{x+1})(1-\gamma)^{-1},\ x \in \boldsymbol{Z}_+) .
$$

Since the above $\tilde{p}(x, y)$ coincides with $p(x, y; \alpha)$ defined by (2) provided that $y - x = \pm 1$, Proposition 2 follows from (5).

§ 3. Proof of Theorem 1.

For each i $(1 \leqq i \leqq d)$, let $S^{(i)}(t)$ be a random walk on $\boldsymbol{Z}$ with *continuous* time, starting at 0 $(S^{(i)}(0) = 0)$ and having generator G_i where

$$
G_i f(x) = p_i^{(+1)} f(x+1) + p_i^{(-1)} f(x-1) - p_i f(x) .
$$

We assume that $\{S^{(i)}(t), t \geqq 0\}$, $1 \leqq i \leqq d$, are defined on a common probability space and that they are *independent*. Then it is easy to see that

$$
\boldsymbol{S}(t) = (S^{(1)}(t), S^{(2)}(t), \cdots, S^{(d)}(t))
$$

is a random walk on $\boldsymbol{Z}^d$ with continuous time whose generator is $\boldsymbol{G}$:

$$
\boldsymbol{G} f(\boldsymbol{x}) = \sum_{i=1}^{d} \sum_{\sigma=\pm 1} p_i^{(\sigma)} f(\boldsymbol{x} + \sigma \boldsymbol{e}_i) - f(\boldsymbol{x}) .
$$

In other words, the coordinate processes of a simple random walk on $\boldsymbol{Z}^d$ with continuous time are independent while this is not true in the case of discrete time. This fact, in spite of being elementary and even obvious, can not be found easily in literature; according to K. Itô it was E. B. Dynkin who had referred to this fact. Denote by $T_1, T_2, \cdots$ the successive jumping times of $\boldsymbol{S}(t)$ and put $T_0 = 0$ for convention. Then the process $\{\boldsymbol{S}(T_n), n \geqq 0\}$ on $\boldsymbol{Z}^d$ with discrete time is identical in law to the random walk $\{S_n, n\geqq 0\}$ introduced in § 1. Moreover, putting $M^{(i)}(t)=\min\{S^{(i)}(s); 0\leqq s\leqq t\}$ and $\boldsymbol{M}(t)=(M^{(1)}(t), M^{(2)}(t), \cdots, M^{(d)}(t))$, we see that

$$\{\boldsymbol{S}_n-2\boldsymbol{M}_n, n\geqq 0\}\overset{d}{=}\{\boldsymbol{S}(T_n)-2\boldsymbol{M}(T_n), n\geqq 0\} , \tag{6}$$

where $\overset{d}{=}$ means the equality in distribution. On the other hand, since for each i the successive jumping times of $S^{(i)}(t)-2M^{(i)}(t)$ coincide with those of $S^{(i)}(t)$, Proposition 2 implies that each $S^{(i)}(t)-2M^{(i)}(t)$ is a continuous time Markov chain on $\boldsymbol{Z}_+$ with generator $\widetilde{G}_i$:

$$\widetilde{G}_i f(x)=p_i\{\sum_{y\in \boldsymbol{Z}_+} p(x, y; p_i^{(+1)}/p_i)f(y)-f(x)\} .$$

Therefore, the independence of $S^{(i)}(t)-2M^{(i)}(t)$, $1\leqq i\leqq d$, implies that $\boldsymbol{S}(t)-2\boldsymbol{M}(t)$ is a continuous time Markov chain on $\boldsymbol{Z}_+^d$ with generator $\widetilde{\boldsymbol{G}}$:

$$\widetilde{\boldsymbol{G}}f(\boldsymbol{x})=\sum_{i=1}^{d}\sum_{\sigma=\pm 1} p_i p(x_i, x_i+\sigma; p_i^{(+1)}/p_i)f(\boldsymbol{x}+\sigma\boldsymbol{e}_i)-f(\boldsymbol{x}) ,$$
$$\boldsymbol{x}=(x_1, \cdots, x_d) .$$

Observing $\boldsymbol{S}(t)-2\boldsymbol{M}(t)$ at its successive jumping times $T_1, T_2, \cdots$ we see that $\{\boldsymbol{S}(T_n)-2\boldsymbol{M}(T_n), n\geqq 0\}$ is a discrete time Markov chain on $\boldsymbol{Z}_+^d$ with (one-step) transition function $\tilde{\boldsymbol{p}}(\boldsymbol{x}, \boldsymbol{y})$ given by (3). This combined with (6) completes the proof of Theorem 1.

References

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