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A Partial Order of Knots

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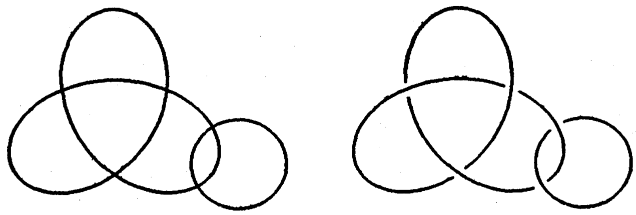
Abstract. In this paper we introduce a new partial order $\leqq$ on the set of all classical knots. We show for example that every nontrivial knot $\geqq$ the trefoil knot.

§1. Introduction.

Throughout this paper we work in the piecewise linear category. A *link* is the image of an embedding of a disjoint union of unordered and unoriented circles into S^3. A *knot* is a link with one component.

DEFINITION 0. We say that links L_1 and L_2 are *equivalent*, denoted by $L_1=L_2$, if there exists a homeomorphism of S^3 onto itself which maps L_1 onto L_2, where the homeomorphism is not required to be orientation preserving. Each equivalence class of links is called a *link type*. A *link projection* is the image of a map of a finite disjoint union of circles into S^2 whose multiple points are transverse double points only. A *link diagram* is a link projection together with an over/under information at each double point. See Fig. 1-1.



link projection link diagram

FIGURE 1-1

Regarding S^2 as the canonical subspace of S^3, we can suppose that a link diagram represents a link type. From a link projection $\hat{L}$ which

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has n double points, 2^n link diagrams arise. Hence at most 2^{n-1} link types arise from $\hat{L}$. We denote the set of all link types which arise from $\hat{L}$ by LINK($\hat{L}$). If a link type L_0 is an element of LINK($\hat{L}$), then we say that $\hat{L}$ is a *projection of* L_0. We denote the set of all projections of a link type L_0 by PROJ(L_0).

DEFINITION 1. For link types L_1 and L_2, we say that L_1 is a *minor* of L_2, denoted by $L_1 \leqq L_2$ or $L_2 \geqq L_1$, if $\mathrm{PROJ}(L_1) \supset \mathrm{PROJ}(L_2)$. If L_1 is a minor of L_2, then we say that L_2 *majorizes* L_1.

For the definitions of the other standard terms in knot theory, we refer to [3] and [1]. We denote the set of all μ-component link types by $\mathfrak{L}^\mu$, and the set of all μ-component prime alternating link types by $\mathfrak{PAL}^\mu$. From now on, we do not distinguish a link from its link type, and a diagram from the link type represented by it so long as no confusion arises.

In this paper we show the following results.

PROPOSITION 1. *For each natural number μ, the pair $(\mathfrak{PAL}^\mu, \leqq)$ is a partially ordered set. That is, for L_1, L_2, $L_3 \in \mathfrak{PAL}^\mu$, the following hold*:

(1) $L_1 \leqq L_1$ *(the reflexive law)*,

(2) *if $L_1 \leqq L_2$ and $L_2 \leqq L_3$, then $L_1 \leqq L_3$ (the transitive law)*,

(3) *if $L_1 \leqq L_2$ and $L_2 \leqq L_1$, then $L_1 = L_2$ (the antisymmetric law)*.

THEOREM 0. *For each natural number μ, every μ-component link majorizes the μ-component trivial link.*

THEOREM 1. *Every nontrivial knot majorizes the trefoil knot.*

THEOREM 2. *For a knot K, the following* (1) *and* (2) *are equivalent*:

(1) *The knot K majorizes the figure eight knot.*

(2) *The knot K has a prime factor which is not equivalent to any of the* (2, p)*-torus knots with $p \geqq 3$.*

THEOREM 3. *For a knot K, the following* (1) *and* (2) *are equivalent*:

(1) *The knot K majorizes the three-twist knot* ($=5_2$ *in the knot table in* [3]).

(2) *The knot K has a prime factor which is not equivalent to any of the* (2, p)*-torus knots with $p \geqq 3$, and the figure eight knot.*

THEOREM 4. *For a knot K, the following* (1) *and* (2) *are equivalent*:

(1) *The knot K majorizes the* (2, 5)*-torus knot* ($=5_1$ *in the knot table in* [3]).

(2) *The knot K has a prime factor which is not equivalent to any of the pretzel knots $L(p_1, p_2, p_3)$ with p_1, p_2 and p_3 odd integers.*

The Hasse diagram is a well-known method to illustrate a partially ordered set. Summarizing all the results stated above, we have a part of the Hasse diagram of $(\mathfrak{PAL}^1, \leqq)$ in Fig. 1-2, where a line segment means that the knot at the upper end is a successor of the knot at the lower end.

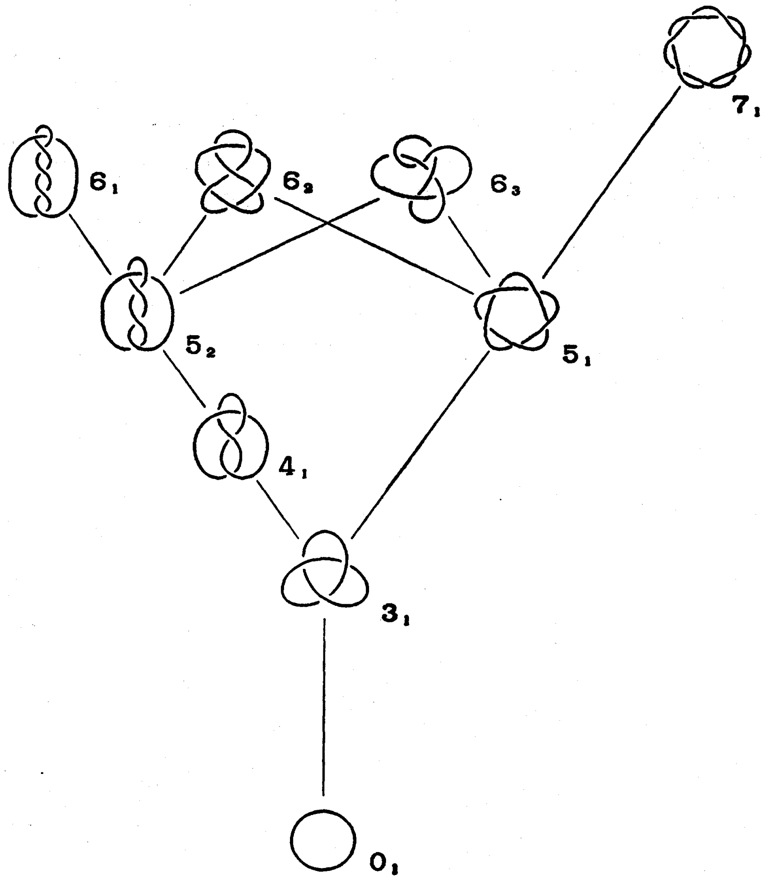


FIGURE 1-2

We will discuss on the two-component links $(\mathfrak{L}^2, \leqq)$ in the forthcoming paper [5].

§ 2. Basic properties.

PROOF OF THEOREM 0. We show that every link projection is a projection of the trivial link. We give an order to the immersed circles arbitrarily, choose an arbitrary base point that is not a crossing point for each immersed circle, and give an orientation also arbitrarily for each immersed circle. We trace the immersed circles in their given order and from their base points in the direction specified by their orientation. We add an over/under information for each crossing point of the projection so that every crossing point may be first traced as an over-crossing. Let us call this process of adding over/under informations *descending*. This process produces the trivial link diagram. See Fig. 2-1. $\square$

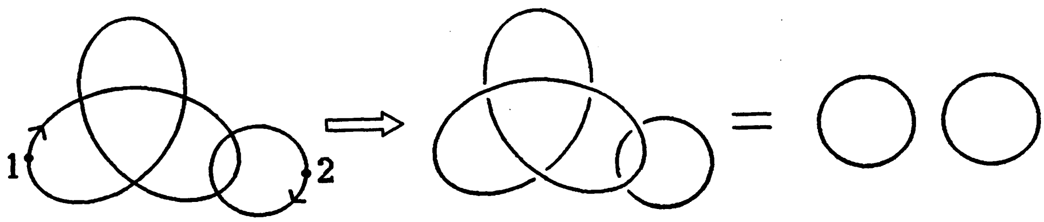


FIGURE 2-1

PROPOSITION 0. *Let L_1 and L_2 be links. If $L_1 \leqq L_2$ then it holds that $c(L_1) \leqq c(L_2)$,* $\text{bridge}(L_1) \leqq \text{bridge}(L_2)$ *and* $\text{braid}(L_1) \leqq \text{braid}(L_2)$, *where* $c(L)$, $\text{bridge}(L)$ *and* $\text{braid}(L)$ *denote the minimum number of crossings, the bridge index and the braid index of L, respectively.*

We note that we can define these numerical invariants of a link from the set of projections of the link, hence the proof of Proposition 0 is clear.

For each natural number μ, the pair $(\mathfrak{L}^{\mu}, \leqq)$ is a preordered set, i.e., the reflexive law and the transitive law hold. Now we prove the antisymmetric law for the prime alternating links.

PROOF OF PROPOSITION 1. Let $\tilde{L}_1$ be a diagram of L_1 which has the minimum number of crossings of L_1. Then the result in Murasugi [2] and Thistlethwaite [6] tells us that $\tilde{L}_1$ is alternating. By Proposition 0, $c(L_1)=c(L_2)$. Therefore by changing appropriate crossings of $\tilde{L}_1$, $\tilde{L}_1$ turns into a diagram $\tilde{L}_2$ of L_2 which has the minimum number of crossings of L_2. Again, the result in [2] and [6] tells us that $\tilde{L}_2$ is also alternating. This shows $L_1=L_2$. $\square$

§ 3. Proof of Theorem 1.

We say that a double point P of a link projection $\hat{L}$ is *nugatory* if $\hat{L}-P$ is disconnected. We say that a link projection $\hat{L}$ is *reduced* if $\hat{L}$ has no nugatory double points.

LEMMA 1. *For any link projection $\hat{L}$, there is a reduced link projection $\hat{L}'$ such that* LINK($\hat{L}$) = LINK($\hat{L}'$).

PROOF. Let $\hat{L}_1$ and $\hat{L}_2$ be link projections related as illustrated in Fig. 3-1, where $\hat{T}_1$ and $\hat{T}_2$ are parts of the link projection, then it is clear that LINK($\hat{L}_1$) = LINK($\hat{L}_2$). Therefore we can eliminate nugatory double points and obtain the desired link projection. □

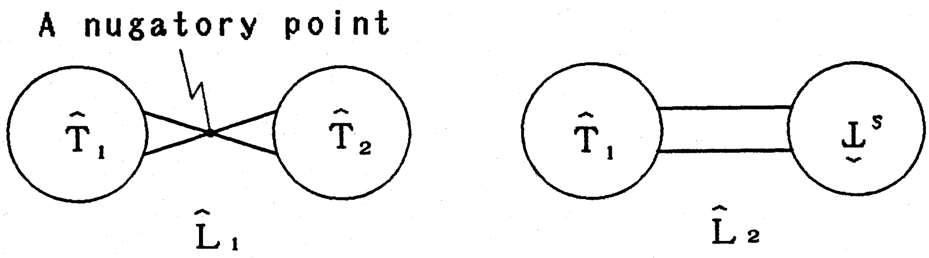


FIGURE 3-1

LEMMA 2. *Let $\hat{A}$ be the image of a map of a closed interval into $S^2 = \partial B^3$ whose multiple points are only transverse double points different from the end points. Then there is a properly embedded arc A in B^3 which satisfies the following conditions*:

(1) *The pair (B^3, A) is homeomorphic to the standard ball pair (B^3, B^1).*

(2) *The arc A is disjoint from the center $\{0\}$ of B^3 such that the restriction of the natural projection $B^3 - \{0\} \to \partial B^3 = S^2$ to A is a map with its multiple points transverse double points only whose image is equal to $\hat{A}$.*

This result follows by the descending process. See Fig. 3-2.

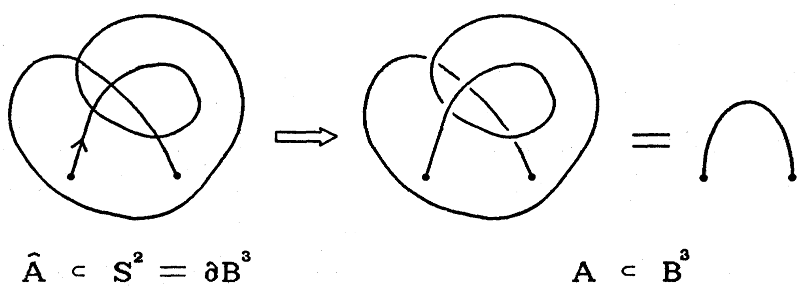
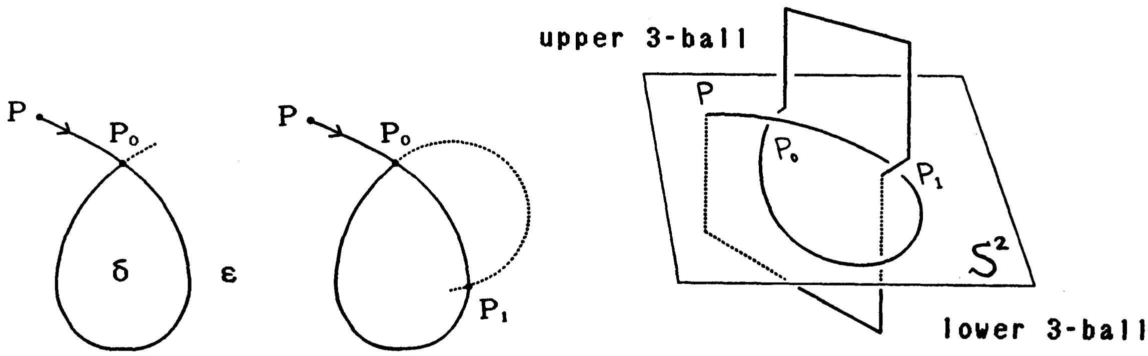


FIGURE 3-2

PROOF OF THEOREM 1. By Lemma 1, it is sufficient to show that every reduced projection $\hat{K}$ of a nontrivial knot K is a projection of the trefoil knot. We start from an arbitrary point P of $\hat{K}$ that is not a crossing point, and trace $\hat{K}$ in an arbitrarily chosen direction. The traced line crosses itself in time. Let P_0 be the first such point. We have traced out a simple closed curve which starts and ends at P_0. This simple closed curve bounds two disks on S^2. The disk which does not contain the point P looks like a tear drop. We denote this disk by δ. We denote the other disk, which looks like a heart, by ε. See Fig. 3.3 (a). We go on the tracing. As $\hat{K}$ is reduced, we must return into the disk δ. We denote the point at which we first return into δ by P_1. See Fig. 3-3 (b). We are ready to add over/under informations to $\hat{K}$. First we add over/under informations by the descending process which starts from P in the previously chosen direction, and then we change over/under at P_1. Applying twice Lemma 2 and its proof to the upper and the lower three-balls bounded by S^2 in S^3, we can deform the knot obtained above to the knot in Fig. 3-3 (c), which is a trefoil knot as desired. □



(a) (b) (c)

FIGURE 3-3

§4. Proof of Theorem 2.

We prepare some notions which are similar to the notions defined in §1. A *tangle* is the image of an embedding of a finite disjoint union of closed intervals into the three-ball $D^2\times[0, 1]$ which maps the boundary into $\partial D^2\times\{0\}$ and the interior into the interior of $D^2\times[0, 1]$. Two tangles T_1 and T_2 are said to be *equivalent*, denoted by $T_1=T_2$, if there exists an isotopic deformation of $D^2\times[0, 1]$ fixed on $\partial(D^2\times[0, 1])$ which deforms T_1 to T_2. A *tangle projection* is the image of a proper map of a finite disjoint union of closed intervals into the disk D^2, whose multiple points are only transverse double points in the interior. A *tangle diagram* is a tangle projection together with an over/under information at each double point. Regarding D^2 as $D^2\times\{0\}$, we can suppose that a tangle diagram represents an equivalence class of tangles. We do not distinguish a tangle from its equivalence class, and a tangle diagram from the equivalence class represented by the diagram so long as no confusion arises. For a tangle projection $\hat{T}$, we denote the set of all tangles which arise from $\hat{T}$ by $\mathrm{TANG}(\hat{T})$. If a tangle T_0 is an element of $\mathrm{TANG}(\hat{T})$, then we say that $\hat{T}$ is a *projection of* T_0. A *sub-arc* of a circle (resp. a closed interval) is a subspace of the circle (resp. the closed interval) which is homeomorphic to the closed interval.

LEMMA 3. *If $\hat{J}_1$ and $\hat{J}_2$ are link projections (resp. tangle projections) related as in Fig. 4-1, i.e., $\hat{J}_2$ is obtained from $\hat{J}_1$ by eliminating a part of $\hat{J}_1$ that is the image of a sub-arc of a circle (resp. a closed interval) which is itself a simple closed curve, then* $\mathrm{LINK}(\hat{J}_1)\supset\mathrm{LINK}(\hat{J}_2)$ *(resp.* $\mathrm{TANG}(\hat{J}_1)\supset\mathrm{TANG}(\hat{J}_2)$).

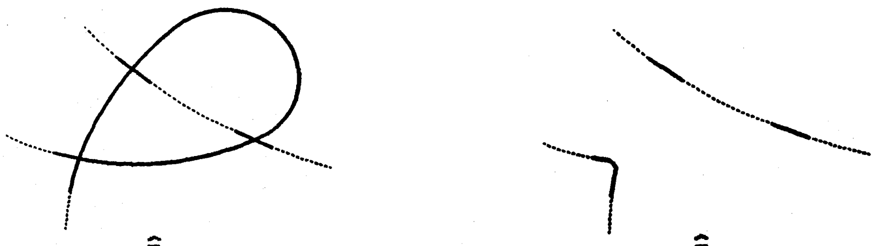


FIGURE 4-1

The proof is clear.

A *self-crossing point* is a double point of a link projection (resp. a tangle projection) whose preimage is contained in a circle (resp. a closed interval). A *mutual crossing point* is a double point of a link projection (resp. a tangle projection) which is not a self-crossing point. A *component of a link projection* (resp. *a tangle projection*) is the image of a circle (resp. a closed interval).

LEMMA 4. *Let T be the tangle represented by a tangle diagram $\tilde{T}$ which has the following two properties*:

(1) *$\tilde{T}$ has no self-crossing points,*

(2) *there is an ordering of the components of $\tilde{T}$ such that at each mutual crossing point of $\tilde{T}$, the former component is over the latter component.*

Let $\hat{T}$ be a tangle projection which has the pairwise same end points as $\tilde{T}$, where "pairwise same" means that the two end points of a com-

ponent of $\hat{T}$ *are the end points of a component of* $\tilde{T}$, *and vice versa. Then* $\hat{T}$ *is a projection of* T.

This result follows by the descending process specified by the order in $\tilde{T}$. See Fig. 4-2.

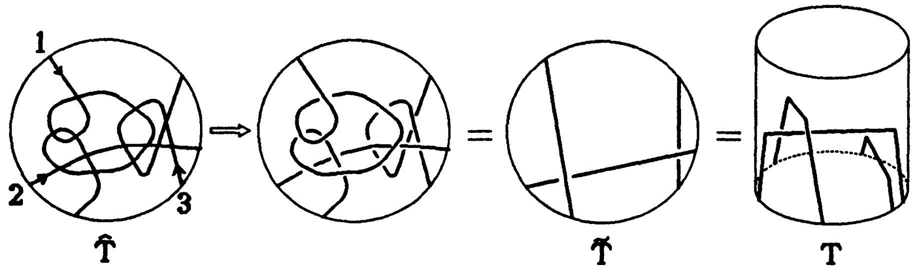


FIGURE 4-2

For points P_1 and P_2 on a component C of a tangle projection that are not self-crossing points, we denote by $\overline{P_1P_2}/C$ the image of the subarc of the closed interval with end points P_1 and P_2 which is contained in C. In the case that the component C is evidently known, we denote this by $\overline{P_1P_2}$. For $\overline{P_1P_2}$, we denote by $\overrightarrow{P_1P_2}$ the over/under informations of the crossing points of $\overline{P_1P_2}$ specified by the descending process which starts from P_1 and ends at P_2. For two parts $\overline{P_1P_2}$ and $\overline{P_3P_4}$ of a tangle projection with $\overline{P_1P_2}\cap\overline{P_3P_4}$ composed of finite points, we denote by $\overline{P_1P_2}<\overline{P_3P_4}$ the over/under informations of the crossing points $(\overline{P_1P_2}\cap\overline{P_3P_4})-\{P_1, P_2, P_3, P_4\}$ specified by the rule that $\overline{P_3P_4}$ is over $\overline{P_1P_2}$. We abbreviate $\overrightarrow{P_1P_2}$, $\overrightarrow{P_3P_4}$ and $\overline{P_1P_2}<\overline{P_3P_4}$ to $\overrightarrow{P_1P_2}<\overrightarrow{P_3P_4}$.

LEMMA 5. *Let* T *be the tangle in Fig.* 4-3. *Let* $\hat{T}$ *be a tangle projection which has the pairwise same end points as* T. *If there exist mutual crossing points of* $\hat{T}$, *then* $\hat{T}$ *is a projection of* T.

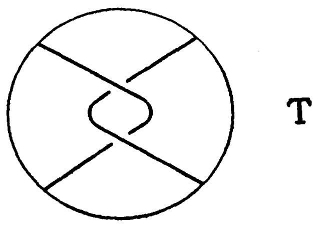


FIGURE 4-3

PROOF. Let us denote the end points of $\hat{T}$ by P_1, P_2, P_3 and P_4 as in Fig. 4-4. We trace the component $\overline{P_1P_2}$ of $\hat{T}$ from P_1 and denote the first mutual crossing point which we come across by P_0. By Lemma 3, it is sufficient to check the case that $\overline{P_1P_0}$ is simple. See Fig. 4-4.

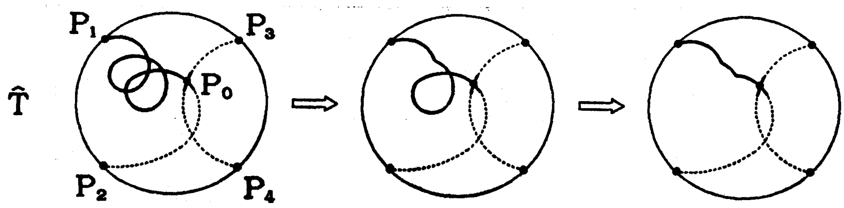


FIGURE 4-4

We consider the following two cases according to the sign of P_0 as in Fig. 4-5.

We add the following over/under informations to $\hat{T}$:

Case 1. $\overrightarrow{P_1P_2}$. $\overrightarrow{P_0P_2}<\overrightarrow{P_3P_4}$. $\overline{P_1P_2}$ is over at P_0.

Case 2. $\overrightarrow{P_2P_1}$. $\overrightarrow{P_2P_0}>\overrightarrow{P_4P_3}$. $\overline{P_1P_2}$ is under at P_0.

Let N be a sufficiently small regular neighborhood of $\partial D^2 \cup \overline{P_1P_0}$ in D^2, and let the disk D_0 be the closure of D^2-N. To see that the tangle obtained above is T, we first deform $\overline{P_0P_2}$ into $N\times[0, 1]$, and then apply Lemma 4 and its proof to D_0 to obtain the tangle in Fig. 4-6, which is T as desired. □

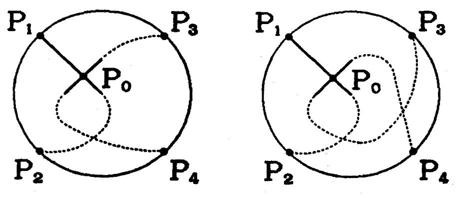


FIGURE 4-5

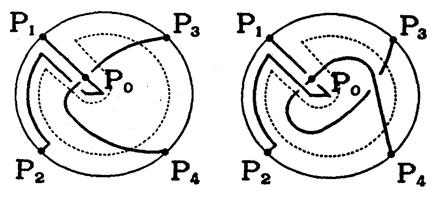


FIGURE 4-6

PROOF OF THEOREM 2. Let $\hat{K}_0$ be the knot projection in Fig. 4-7. Then

$$\mathrm{LINK}(\hat{K}_0) \ni \text{the } (2, p_1)\text{-torus knot} \# \cdots \# \text{the } (2, p_n)\text{-torus knot},$$

where "#" denotes the connected sum of knots, but the figure eight knot is not an element of LINK($\hat{K}_0$). This shows that the condition (2) is necessary for the condition (1).

Suppose that the knot K satisfies the condition (2). Let $\hat{K}$ be a reduced projection of K. We must show that $\hat{K}$ is a projection of the figure eight knot. We specify an arbitrary orientation of $\hat{K}$. We choose

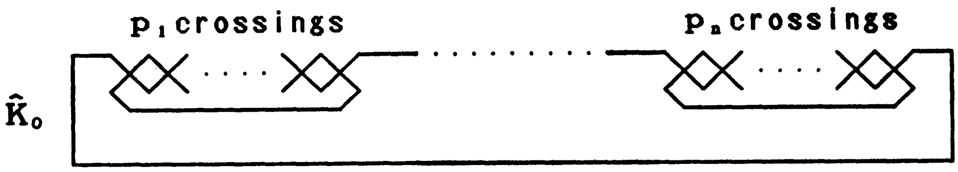


FIGURE 4-7

an arbitrary tear drop disk δ, the heart disk ε and the crossing point P_0 as in the proof of Theorem 1. We trace $\partial\delta$ from P_0 in the direction specified by the orientation of $\hat{K}$, and name the crossing points on $\partial\delta$ $P_0, P_1, \cdots$ and P_n respectively in the order of their appearance. It is clear that n is even and $n \geqq 2$ as $\hat{K}$ is reduced. We trace $\hat{K}$ from P_0 in the direction specified by the orientation of $\hat{K}$ into the disk ε and rename the crossing points on $\partial\delta$ $Q_0=P_0$, $Q_1, \cdots$ and Q_n respectively in the order of their appearance. We obtain a permutation $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ of $(1, 2, \cdots, n)$ such that $Q_i=P_{\sigma(i)}$ for $i \in \{1, 2, \cdots, n\}$. For i and j with $0 \leqq i < j \leqq n$ or $n \geqq i > j = 0$, we denote by $\overline{Q_iQ_j}$ the image of the sub-arc of the circle with starting point Q_i and terminal point Q_j with respect to the orientation of $\hat{K}$ which has the finite points of intersection with $\partial\delta$. See for example Fig. 4-8.

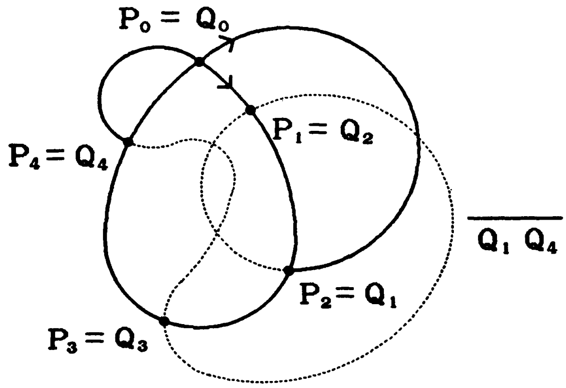


FIGURE 4-8

For $\overline{Q_iQ_j}$, we denote by $\overrightarrow{Q_iQ_j}$ the over/under informations of the crossing points of $\overline{Q_iQ_j}$ specified by the descending process which starts from Q_i and ends at Q_j. For the images M_1 and M_2 of sub-arcs of the circle with $M_1 \cap M_2$ composed of finite points, we denote by $M_1 < M_2$ the over/under informations of the crossing points $M_1 \cap M_2$ except their end points specified by the rule that M_2 is over M_1. We abbreviate $\overrightarrow{Q_iQ_j}$, $\overrightarrow{Q_kQ_l}$ and $\overline{Q_iQ_j} < \overline{Q_kQ_l}$ to $\overrightarrow{Q_iQ_j < Q_kQ_l}$.

Case 1. There exists an odd number i such that $\sigma(i)>\sigma(i+1)$.

We add the following over/under informations: $\overrightarrow{Q_0Q_i}>\partial\delta>\overrightarrow{Q_iQ_{i+1}}>\overrightarrow{Q_{i+1}Q_0}$, at Q_0, Q_i and Q_{i+1} as in Fig. 4-9 (a).

We note that $\overline{Q_iQ_{i+1}}$ is lying on δ. Applying Lemma 2 and its proof to $\overline{Q_0Q_i}$ and $\overline{Q_{i+1}Q_0}$ in the upper and the lower three-balls, and Lemma 4 and its proof to $\overline{Q_iQ_{i+1}}$ on δ, we obtain the figure eight knot as in Fig. 4-9 (b).

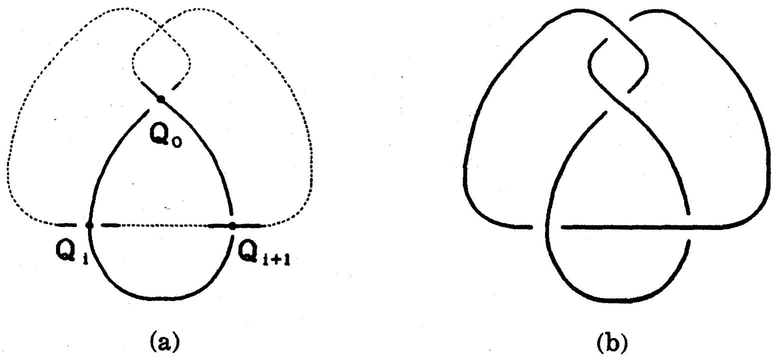


FIGURE 4-9

Case 2. $\sigma(i)<\sigma(i+1)$ for all odd number i, and there exists an even number m such that $\sigma(m)>\sigma(m+1)$.

Let j be the least such even number. We further refine on Case 2 as follows.

Case 2a. $\sigma(1)<\sigma(j+1)$. We add the following over/under informations: $\overrightarrow{Q_1Q_j}>\partial\delta>\overrightarrow{Q_0Q_1}>\overrightarrow{Q_jQ_{j+1}}>\overrightarrow{Q_{j+1}Q_0}$, at Q_0, Q_1, Q_j and Q_{j+1} as in Fig. 4-10 (a).

We note that $\overline{Q_0Q_1}$ and $\overline{Q_jQ_{j+1}}$ are lying on ε. Applying Lemma 2, Lemma 4 and their proofs, we obtain the figure eight knot as in Fig. 4-10 (b).

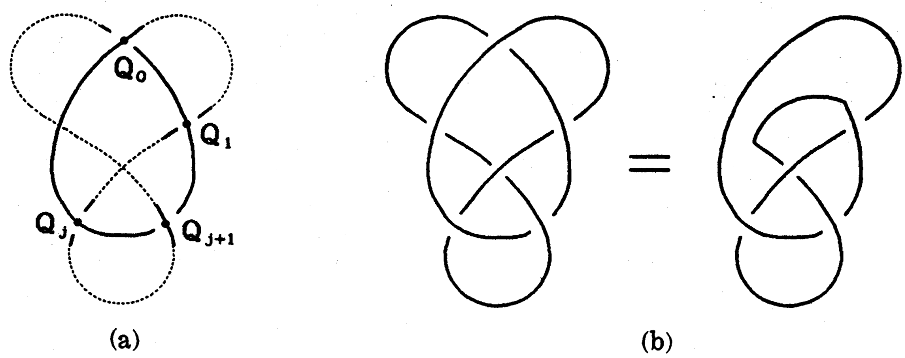
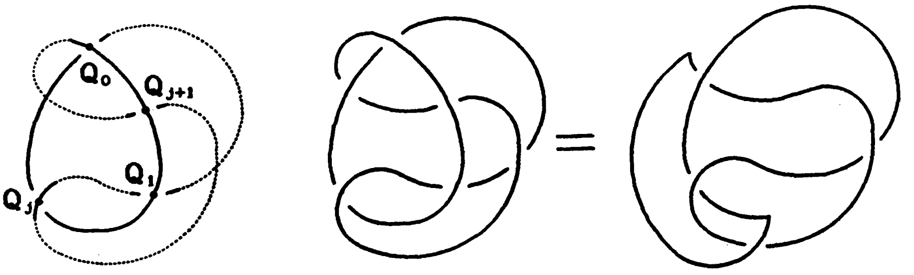


FIGURE 4-10

Case 2b. $\sigma(1)>\sigma(j+1)$. We note that $\sigma(1)<\sigma(j)$. We add the following: $\overrightarrow{Q_1Q_j}>\partial\delta>\overrightarrow{Q_jQ_{j+1}}>\overrightarrow{Q_0Q_1}>\overrightarrow{Q_{j+1}Q_0}$, at Q_0, Q_1, Q_j and Q_{j+1} as in Fig. 4-11 (a).

We note that $\overline{Q_0Q_1}$ and $\overline{Q_jQ_{j+1}}$ are lying on ε. Applying Lemma 2, Lemma 4 and their proofs, we obtain the figure eight knot as in Fig. 4-11 (b).



(a) (b)

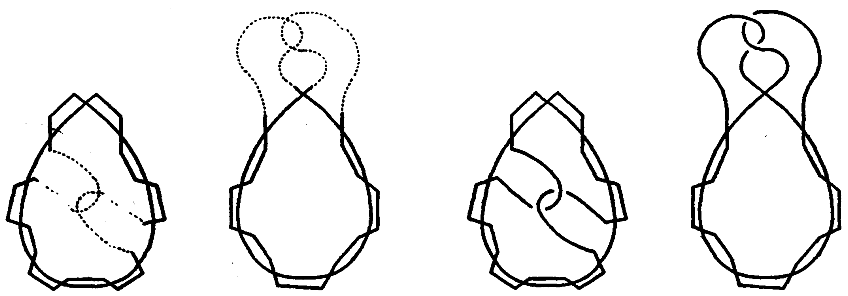
FIGURE 4-11

Case 3. $\sigma(i)=i$ for all i. We refine on Case 3 as follows.

Case 3a. There are numbers i and j with $0\leqq i<j\leqq n$ such that $(\overline{Q_iQ_{i+1}}\cap\overline{Q_jQ_{j+1}})-\partial\delta\neq\varnothing$ (here we suppose $n+1=0$). In this case, for the convenience of the proof, we add over/under informations to $\hat{K}$ step by step.

Step 1. For $k\neq i$, j, $\overline{Q_kQ_{k+1}}>\overline{Q_iQ_{i+1}}$ and $\overline{Q_kQ_{k+1}}>\overline{Q_jQ_{j+1}}$. For $k\neq i$, j, $m\neq i$, j with $k<m$, $\overrightarrow{Q_kQ_{k+1}}>\overrightarrow{Q_mQ_{m+1}}$.

We deform $\overline{Q_kQ_{k+1}}$ into a sufficiently small regular neighborhood of $\partial\delta$ in S^2. See for example, Fig. 4-12 (a) and (b).



(a) (b) (c) (d)

FIGURE 4-12

Step 2. To the crossing points $\overline{Q_iQ_{i+1}}\cap\overline{Q_jQ_{j+1}}-\partial\delta$, we apply Lemma 5 on δ or ε to hook them. See, for example, Fig. 4-12 (c) and (d).

As the projections are on S^2, we obtain the form illustrated in Fig. 4-13 (a).

Step 3. Now it is easy to make the figure eight knot by adding over/under informations to the crossing points on $\partial\delta$. See Fig. 4-13 (b).

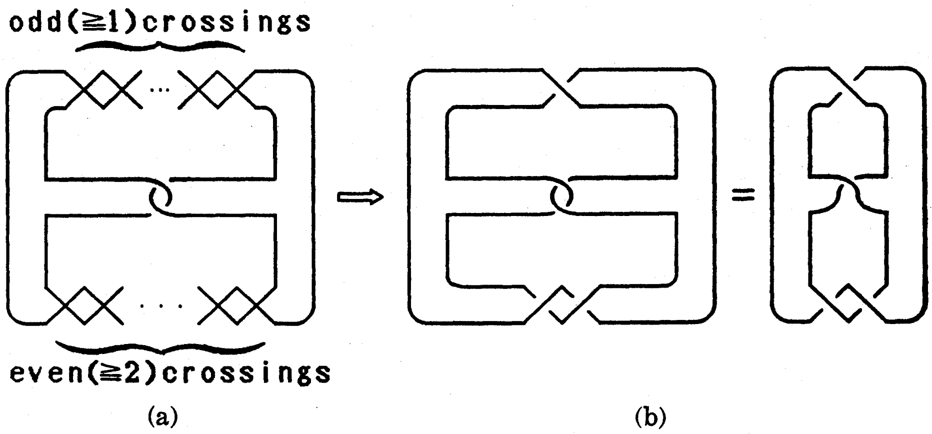


FIGURE 4-13

Case 3b. $\overline{Q_iQ_{i+1}} \cap \overline{Q_jQ_{j+1}} - \partial\delta = \emptyset$ for every $i \neq j$, $(n+1=0)$. In this case, we can decompose the knot K as follows:

$$K = \text{the } (2, m)\text{-torus knot} \# K_0 \# K_1 \# \cdots \# K_n ,$$

where $m \leqq n+1$ and K_i is the knot which corresponds to $\overline{Q_iQ_{i+1}}$. By the uniqueness of the prime decomposition of knots [4], some K_i must satisfy the condition (2). Therefore the result follows by the induction on the crossing number of $\hat{K}$. □

§ 5. Proof of Theorem 3.

LEMMA 6. *Let T_1 and T_2 be the tangles in Fig.* 5-1 *which have the pairwise same end points each other. Let $\hat{T}$ be a tangle projection which has the pairwise same end points as T_1. If there exist at least*

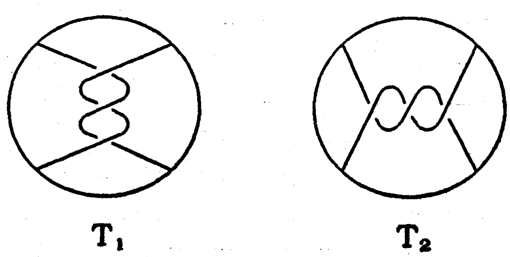


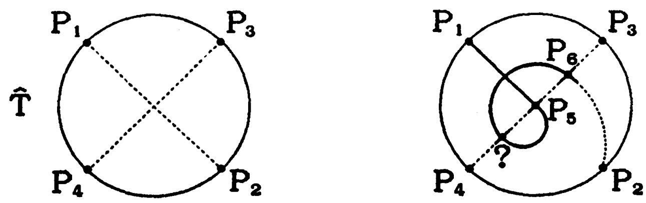
FIGURE 5-1

three mutual crossing points of $\hat{T}$, then at least one of the following holds:

(1) $\hat{T}$ *is a projection of* T_1.

(2) $\hat{T}$ *is a projection of* T_2.

PROOF. Let us denote the end points of $\hat{T}$ by P_1, P_2, P_3 and P_4 as in Fig. 5-2 (a). We trace the component $\overline{P_1P_2}$ from P_1 and denote the first and the second mutual crossing points which we come across by P_5 and P_6 respectively. By Lemma 3, we may suppose that both $\overline{P_1P_5}$ and $\overline{P_5P_6}/\overline{P_1P_2}$ are simple. Then $\overline{P_1P_6}$ is also simple, otherwise it contradicts the choice of P_5 and P_6. See Fig. 5-2 (b).



(a) (b)

FIGURE 5-2

Let us denote the sign of P_5 and P_6 as in Fig. 5-3.

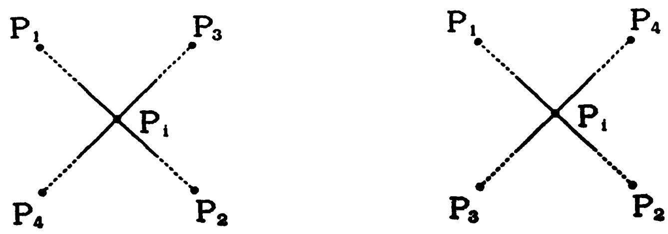


FIGURE 5-3

We consider the following eight cases.

Case 1. $\mathrm{sign}(P_5)=\mathrm{sign}(P_6)=+1$ and $\overline{P_3P_5}\subset\overline{P_3P_6}$.

Case 2. $\mathrm{sign}(P_5)=\mathrm{sign}(P_6)=+1$ and $\overline{P_3P_5}\supset\overline{P_3P_6}$.

Case 3. $\mathrm{sign}(P_5)=+1$, $\mathrm{sign}(P_6)=-1$ and $\overline{P_3P_5}\subset\overline{P_3P_6}$.

Case 4. $\mathrm{sign}(P_5)=+1$, $\mathrm{sign}(P_6)=-1$ and $\overline{P_3P_5}\supset\overline{P_3P_6}$.

Case 5. $\mathrm{sign}(P_5)=-1$, $\mathrm{sign}(P_6)=+1$ and $\overline{P_3P_5}\subset\overline{P_3P_6}$.

Case 6. $\mathrm{sign}(P_5)=-1$, $\mathrm{sign}(P_6)=+1$ and $\overline{P_3P_5}\supset\overline{P_3P_6}$.

Case 7. $\mathrm{sign}(P_5)=\mathrm{sign}(P_6)=-1$ and $\overline{P_3P_5}\subset\overline{P_3P_6}$.

Case 8. $\mathrm{sign}(P_5)=\mathrm{sign}(P_6)=-1$ and $\overline{P_3P_5}\supset\overline{P_3P_6}$.

We add over/under informations as follows:

Case 1. $\overrightarrow{P_6P_2}>\overrightarrow{P_1P_6}>\overrightarrow{P_3P_5}>\overrightarrow{P_6P_4}>\overrightarrow{P_5P_6}/\overline{P_3P_4}$. $\overline{P_3P_4}$ is over $\overline{P_1P_2}$ at P_5

and P_6.

Case 3 and Case 7. $\overrightarrow{P_3P_4}>\overrightarrow{P_6P_2}$. $\overrightarrow{P_1P_2}$. $\overline{P_3P_4}$ is over $\overline{P_1P_2}$ at P_5. $\overline{P_1P_2}$ is over $\overline{P_3P_4}$ at P_6.

Case 5. $\overrightarrow{P_3P_4}>\overrightarrow{P_6P_2}$. $\overrightarrow{P_1P_2}$. $\overline{P_1P_2}$ is over $\overline{P_3P_4}$ at P_5. $\overline{P_3P_4}$ is over $\overline{P_1P_2}$ at P_6.

We note that Case 2, Case 4, Case 6 and Case 8 are reversals of Case 1, Case 3, Case 5 and Case 7 respectively with respect to the roles of P_3 and P_4. Hence the additions of over/under informations are similar. Then we obtain the tangle T_1 or T_2 as in Fig. 5-4. □

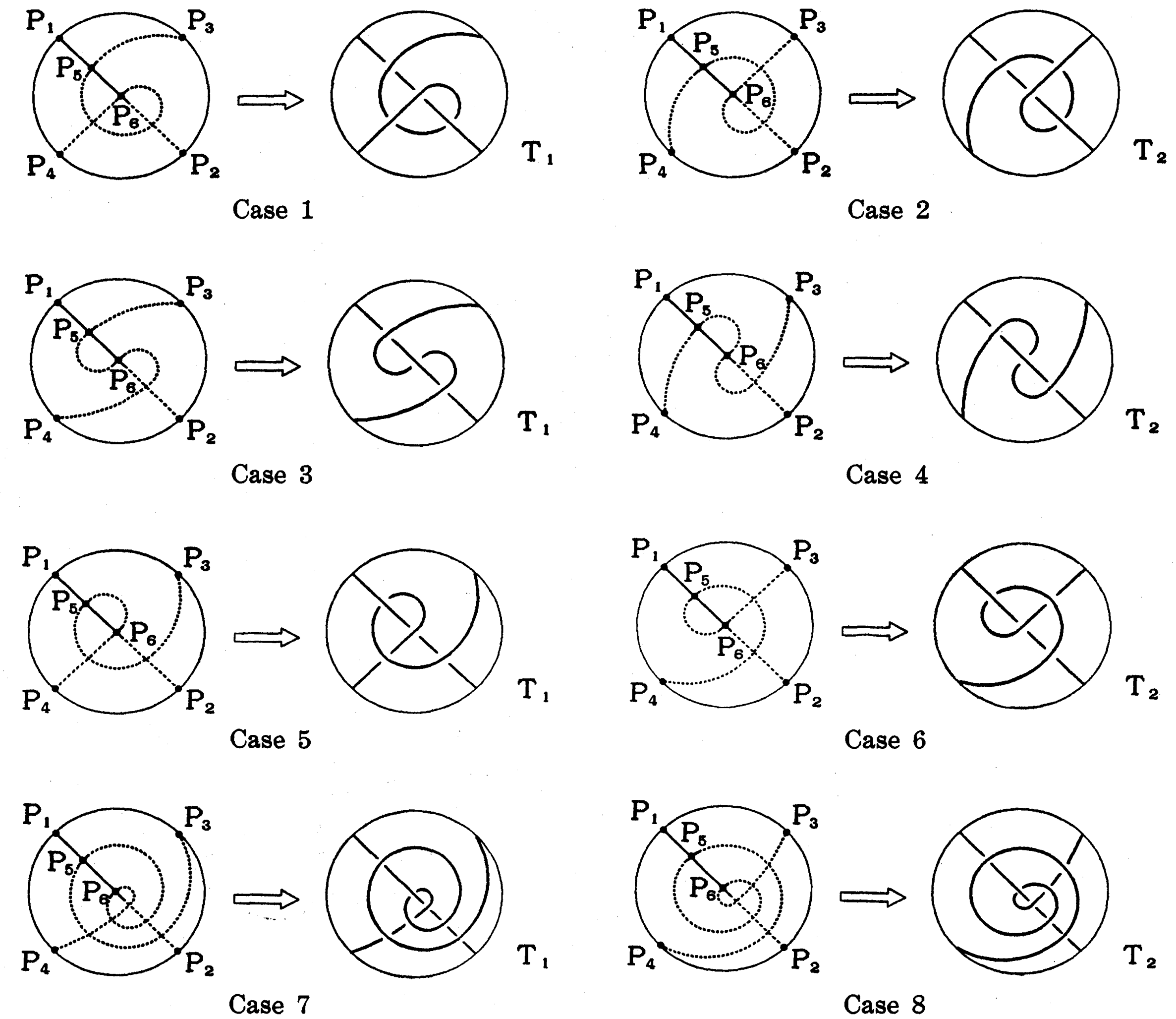


FIGURE 5-4

PROOF OF THEOREM 3. The proof of the necessity of the condition (2) for the condition (1) is similar to that of Theorem 2.

We show the sufficiency. Suppose that the knot K satisfies the condition (2). Let $\hat{K}$ be a reduced projection of K. We specify an arbitrary orientation of $\hat{K}$ and choose an arbitrary tear drop disk δ. We adopt the same notations as in the proof of Theorem 2.

Case 1. There exists an even number i such that $\sigma(i)>\sigma(i+1)$.

We add over/under informations to make the three-twist knot as follows: $\overrightarrow{Q_0Q_i}>\partial\delta>\overrightarrow{Q_iQ_{i+1}}>\overrightarrow{Q_{i+1}Q_0}$, at Q_0, Q_i and Q_{i+1} as in Fig. 5-5.

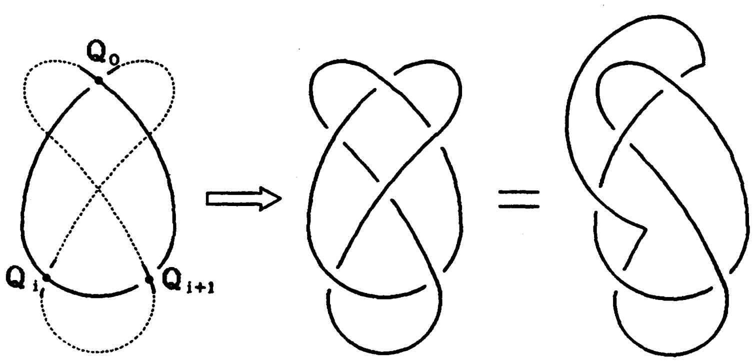


FIGURE 5-5

Case 2. $\sigma(i)<\sigma(i+1)$ for all even number i, and there exists an odd number m such that $\sigma(m)>\sigma(m+1)$.

Let j be the least such odd number. We refine on Case 2 as follows.

Case 2a. $j+1<n$ and $\sigma(j+2)<\sigma(j)$. We add over/under informations to make the three-twist knot as follows: $\overrightarrow{Q_0Q_j}>\partial\delta>\overrightarrow{Q_jQ_{j+1}}>\overrightarrow{Q_{j+1}Q_{j+2}}>\overrightarrow{Q_{j+2}Q_0}$, at Q_0, Q_j, Q_{j+1} and Q_{j+2} as in Fig. 5-6.

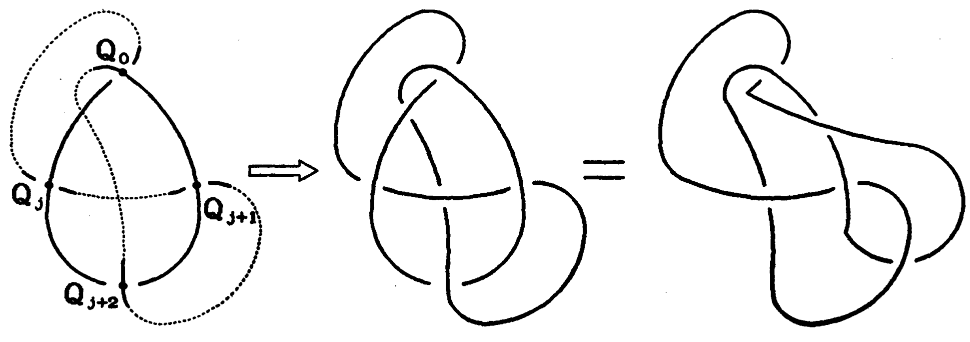


FIGURE 5-6

Case 2b. $j+1<n$ and $\sigma(j+2)>\sigma(j)$. We add over/under informations to make the three-twist knot as follows: $\partial\delta>\overrightarrow{Q_jQ_{j+1}}>\overrightarrow{Q_{j+1}Q_{j+2}}>\overrightarrow{Q_0Q_j}>\overrightarrow{Q_{j+2}Q_0}$, at Q_0, Q_j, Q_{j+1} and Q_{j+2} as in Fig. 5-7. We note that we can deform $\overline{Q_0Q_j}$ after deforming $\overline{Q_{j+2}Q_0}$ as in Fig. 5-7.

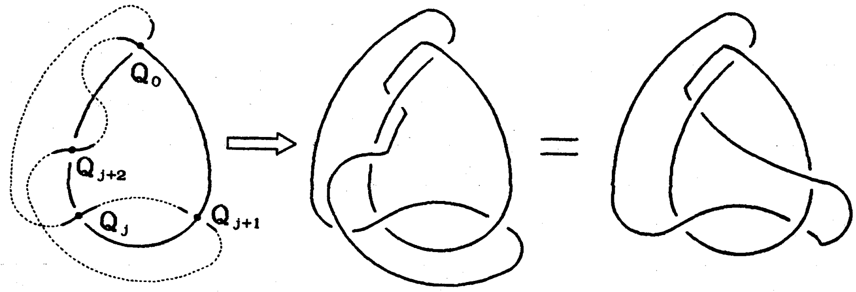
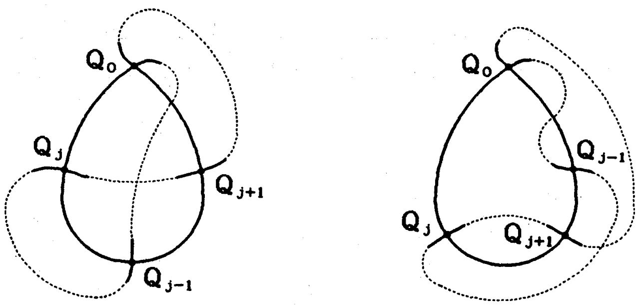


FIGURE 5-7

Case 2c. $j+1=n$, $j \geqq 3$ and $\sigma(j-1) > \sigma(j+1)$.

Case 2d. $j+1=n$, $j \geqq 3$ and $\sigma(j-1) < \sigma(j+1)$.

Case 2c and Case 2d are the reversals of Case 2a and Case 2b respectively. Hence the proof is similar. See Fig. 5-8.



Case 2c　　Case 2d

FIGURE 5-8

Case 2e. $j=1$, $n=2$ and $\overline{Q_0Q_1} \cap \overline{Q_2Q_0}$ has at least four points. We apply Lemma 6 to $\overline{Q_0Q_1} \cup \overline{Q_2Q_0}$ on ε to obtain the three-twist knot as in Fig. 5-9.

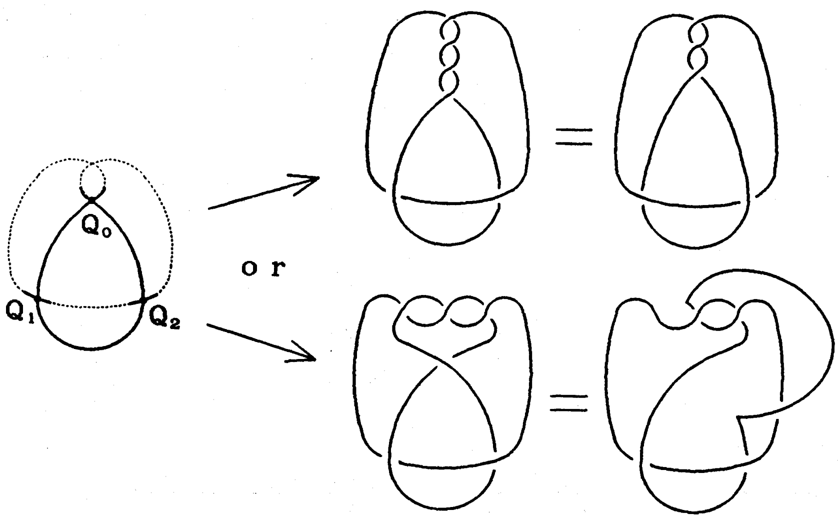


FIGURE 5-9

Case 2f. $j=1$, $n=2$ and $\overline{Q_0Q_1} \cap \overline{Q_2Q_0}$ consists of two points. In this case, we can decompose the knot K as follows: $K=K'\sharp K_0\sharp K_1\sharp K_2$, where K' is the trivial, the trefoil or the figure eight knot, and K_i is the knot which corresponds to $\overline{Q_iQ_{i+1}}$ $(2+1=0)$. Hence the result follows by the induction on the crossing number of $\hat{K}$.

Case 3. $\sigma(i)=i$ for all i.

In this case, the proof goes like the proof of Theorem 2. We note that we can produce the three-twist knot from the form illustrated in Fig. 4-13 (a). □

§6. Proof of Theorem 4.

We say that a link (resp. a tangle) projection $\hat{J}$ is *prime* if every simple closed curve on S^2 (resp. on the interior of D^2) which intersects with $\hat{J}$ transversally at two points bounds a trivial ball pair (B^2, B^1).

Let $\hat{T}_n$ be the tangle projection in Fig. 6-1 for each non-negative integer n.

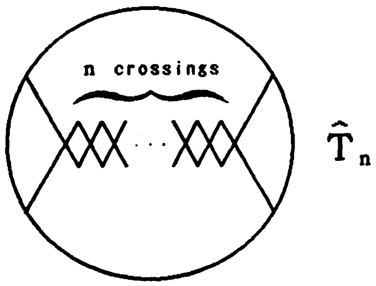


FIGURE 6-1

LEMMA 7. *Let T be the tangle in Fig. 6-2. Let $\hat{T}$ be a prime tangle projection which has the pairwise same end points as T. If $\hat{T}$ is not ambient isotopic rel. ∂D^2 to any of the tangle projections $\hat{T}_n$, then $\hat{T}$ is a projection of T.*

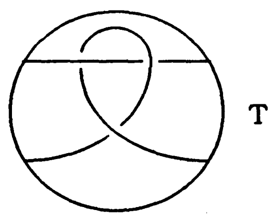


FIGURE 6-2

PROOF. In the case that one of the component of $\hat{T}$ has no self-crossing points, the proof is similar to that of Theorem 2. We consider the case that both components have self-crossing points. In this case we first simplify a component of $\hat{T}$, and obtain a new tangle projection $\hat{T}'$. By Lemma 3, it holds that $\mathrm{TANG}(\hat{T}) \supset \mathrm{TANG}(\hat{T}')$. Therefore it is sufficient to check the case that $\hat{T}'$ is of the form as illustrated in Fig. 6-3, where $\hat{t}_i$ is a sub-tangle projection. By assumption some $\hat{t}_i$ must have

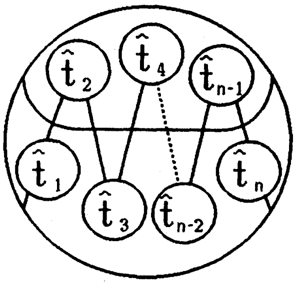


FIGURE 6-3

self-crossing points. As $\hat{T}$ is prime, there exists an eliminated part of $\hat{T}$ which intersects with $\hat{t}_i$. Then it is in any case straightforward to make the tangle T from $\hat{T}$ by careful determination of over/under informations. See for example, Fig. 6-4. □

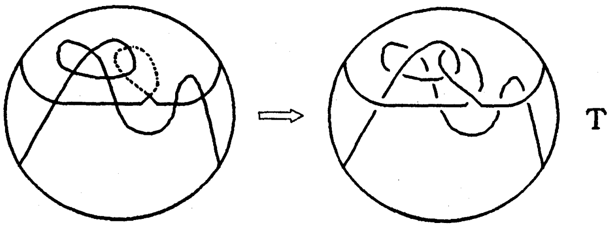


FIGURE 6-4

LEMMA 8. *Let T be the tangle in Fig.* 6-5. *Let $\hat{T}$ be a prime tangle projection which has the pairwise same end points as T. If $\hat{T}$ is not ambient isotopic rel. ∂D^2 to any of the tangle projections $\hat{T}_n$ in Fig.* 6-1, *then $\hat{T}$ is a projection of T.*

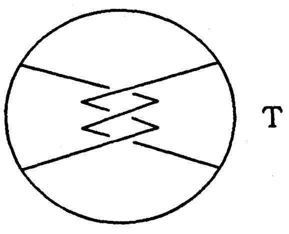


FIGURE 6-5

The proof is similar to that of Lemma 7.

PROOF OF THEOREM 4. Let $\hat{L}(p_1, p_2, p_3)$ be the knot projection in Fig. 6-6 for each triple of non-negative odd numbers p_1, p_2 and p_3. Then it is easily checked that LINK$(\hat{L}(p_1, p_2, p_3)) \not\ni$ the (2, 5)-torus knot. This shows the necessity of the condition (2) for the condition (1).

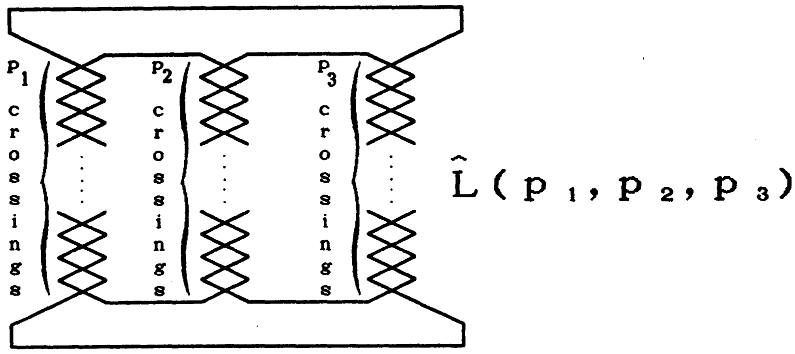
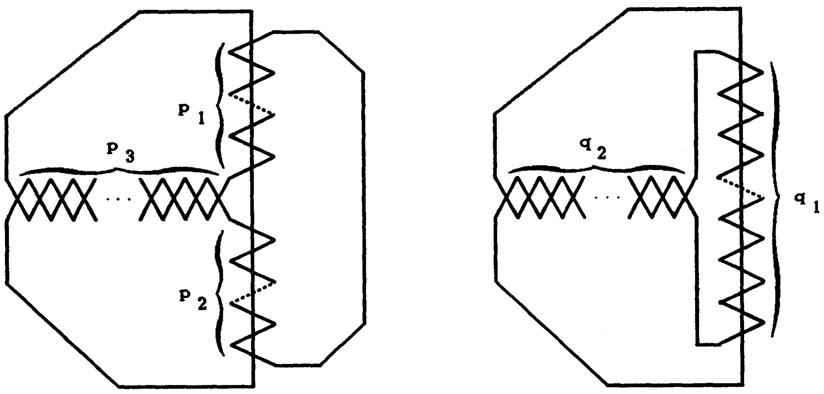


FIGURE 6-6

In order to show the converse, we show the following assertion.

ASSERTION. *Let $\hat{K}$ be a reduced prime knot projection. If* LINK$(\hat{K}) \not\ni$ *the* (2, 5)*-torus knot, then $\hat{K}$ is one of the knot projections in Fig.* 6-7 (a) *or* (b), *where p_1, p_2 and p_3 are non-negative odd numbers and q_1 and q_2 are non-negative even numbers.*



(a) (b)

FIGURE 6-7

It is easily seen that the knots which have these projections are the pretzel knots $L(r_1, r_2, r_3)$ with r_i odd numbers where $|r_i| \leqq p_i$ for $i=1, 2, 3$, or $|r_1| \leqq q_1+1$, $|r_2| \leqq q_2+1$ and $|r_3|=1$.

Now let us prove the assertion. Let $\hat{K}$ be a reduced prime knot projection. We choose a tear drop disk δ and adopt the same notations as in the proofs of the previous theorems, particularly the permutation $\sigma(i)$ of $i \in \{1, 2, \cdots, n\}$. For the convenience, we take a sufficiently small ε-neighbourhood of the point $P_0 = Q_0$ on S^2 and look at the complementary tangle projection $\hat{T}$. See Fig. 6-8.

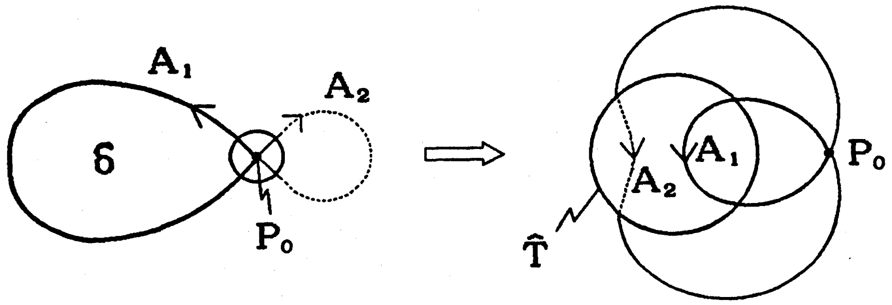


FIGURE 6-8

We list up the cases in which TANG($\hat{T}$) $\ni T_1$, where T_1 is the tangle in Fig. 6-9. We note that if TANG($\hat{T}$) $\ni T_1$, then LINK($\hat{K}$) $\ni$ the (2, 5)-torus knot.

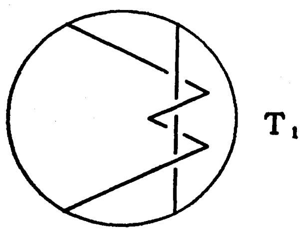


FIGURE 6-9

Case 1. There exists an even number i such that $\sigma(i) < \sigma(i+1)$. See Fig. 6-10.

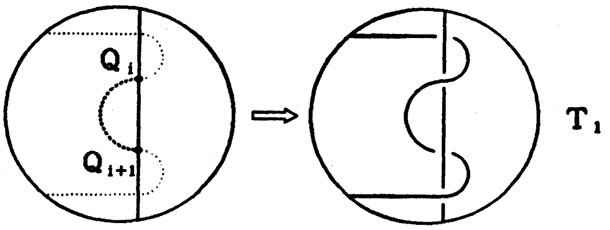


FIGURE 6-10

Case 2. There exist odd numbers i and j with $i < j$ such that $\sigma(i) <$

$\sigma(j)<\sigma(j-1)<\cdots<\sigma(i+2)<\sigma(i+1)$, and $\sigma(j)<\sigma(j+1)$. See Fig. 6-11.

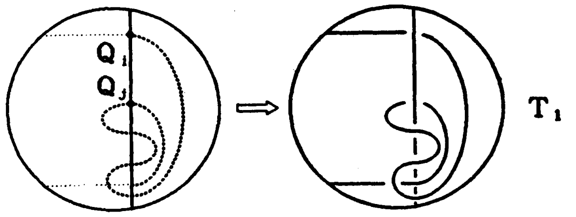


FIGURE 6-11

Case 3. There exist odd numbers i and j with $i<j$ such that $\sigma(j+1)<\sigma(i)<\sigma(j)<\sigma(j-1)<\cdots<\sigma(i+2)<\sigma(i+1)$. See Fig. 6-12.

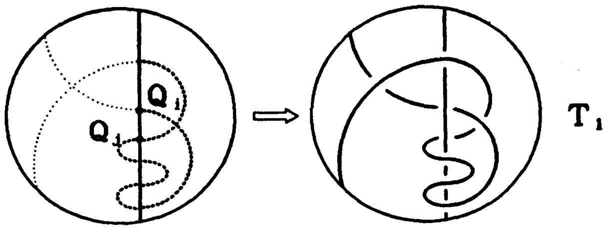


FIGURE 6-12

Case 4. There exist odd numbers i and j with $i<j$ such that $\sigma(j)<\sigma(i)<\sigma(j-1)<\cdots<\sigma(i+2)<\sigma(i+1)$, and $\sigma(i)<\sigma(i-1)<\sigma(i-2)<\cdots<\sigma(2)<\sigma(1)<\sigma(i+1)$. See Fig. 6-13.

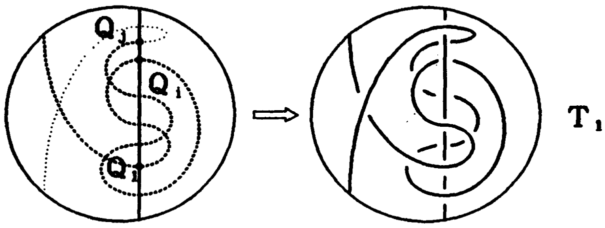


FIGURE 6-13

Case 5. There exist odd numbers i, j and k with $1<k<i<j$ such that $\sigma(j)<\sigma(i)<\sigma(j-1)<\cdots<\sigma(i+2)<\sigma(i+1)$, and $\sigma(i)<\sigma(i-1)<\cdots<\sigma(k+1)<\sigma(k)<\sigma(i+1)<\sigma(k-1)$. See Fig. 6-14.

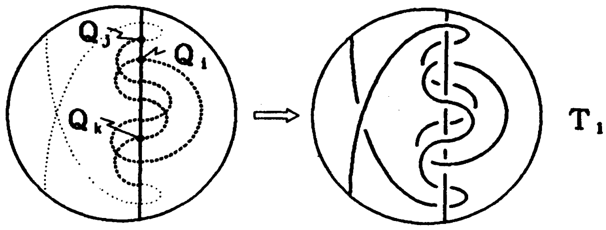


FIGURE 6-14

Case 6. There exists an odd number i such that $\sigma(i)<\sigma(n)<\sigma(1)<\sigma(i+1)$. See Fig. 6-15.

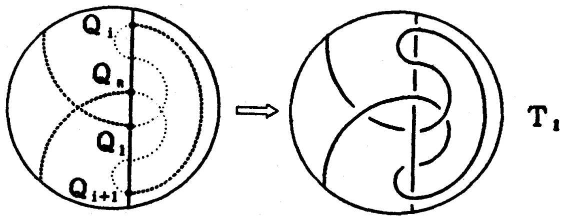
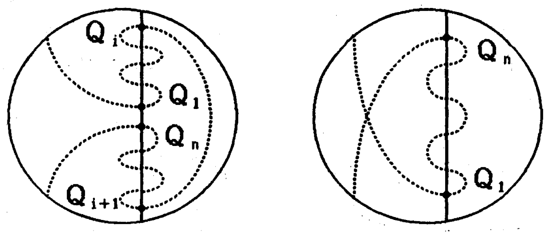


FIGURE 6-15

From the cases mentioned above, it follows that if $\mathrm{TANG}(\hat{T}) \not\ni T_1$, then one of the following two cases holds:

Case 7. There exists an odd number i such that $\sigma(i)<\sigma(i-1)<\cdots<\sigma(2)<\sigma(1)<\sigma(n)<\sigma(n-1)<\cdots<\sigma(i+2)<\sigma(i+1)$. See Fig. 6-16 (a).

Case 8. $\sigma(n)<\sigma(n-1)<\cdots<\sigma(2)<\sigma(1)$. See Fig. 6-16 (b).



(a) (b)

FIGURE 6-16

In these cases, if there exist numbers $i \neq j \in \{0, 1, \cdots, n\}$ with $\{i, j\} \neq \{0, n\}$ such that $\overline{Q_i Q_{i+1}} \cap \overline{Q_j Q_{j+1}} \neq \varnothing$, then an application of Lemma 5 ensures us that $\mathrm{TANG}(\hat{T}) \ni T_1$. See for example Fig. 6-17.

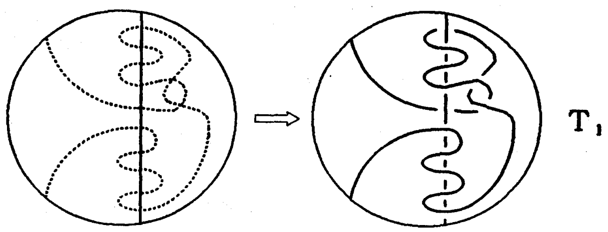


FIGURE 6-17

For the rest of the cases, the applications of Lemma 7 and Lemma 8 to the sub-tangles u in Fig. 6-18 establish the assertion since the tangles T_2 and T_3 in Fig. 6-19 can produce the (2, 5)-torus knot as in Fig. 6-20. $\square$

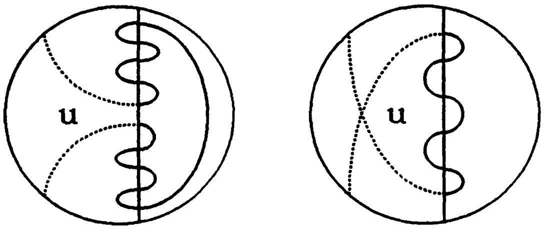


FIGURE 6-18

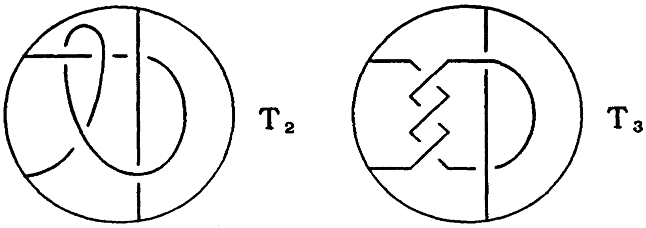


FIGURE 6-19

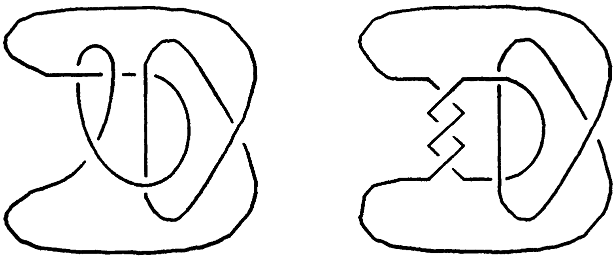


FIGURE 6-20

§ 7. Remarks and questions.

REMARK 1. For $\mu \geqq 2$, the pair $(\mathfrak{L}^{\mu}, \leqq)$ is not a partially ordered set.

For example, let L_1 be the two-component split link of two right-handed trefoil knots, and L_2 be the two-component split link of the right- and left-handed trefoil knots. Then $L_1 \neq L_2$, but $\mathrm{PROJ}(L_1) = \mathrm{PROJ}(L_2)$. Hence $L_1 \leqq L_2$ and $L_1 \geqq L_2$.

REMARK 2. In contrast to the example in Remark 1, the following hold: the square knot $\not\leqq$ the granny knot, and the square knot $\not\geqq$ the granny knot.

To see this, let $\hat{K}_1$ and $\hat{K}_2$ be the knot projections in Fig. 7-1. Then we can check the following:

$$\mathrm{LINK}(\hat{K}_1) = \{0_1, 3_1, 4_1, 5_1, 5_2, 6_3, \text{the square knot}, 8_{18}, 8_{20}\} \not\ni \text{the granny knot},$$

$$\mathrm{LINK}(\hat{K}_2) = \{0_1, 3_1, 4_1, 5_1, 5_2, 6_1, 6_2, \text{the granny knot}, 7_7, 8_{19}, 8_{21}, 9_{40}, 9_{42}, 9_{46}, 9_{47}\} \not\ni \text{the square knot}.$$

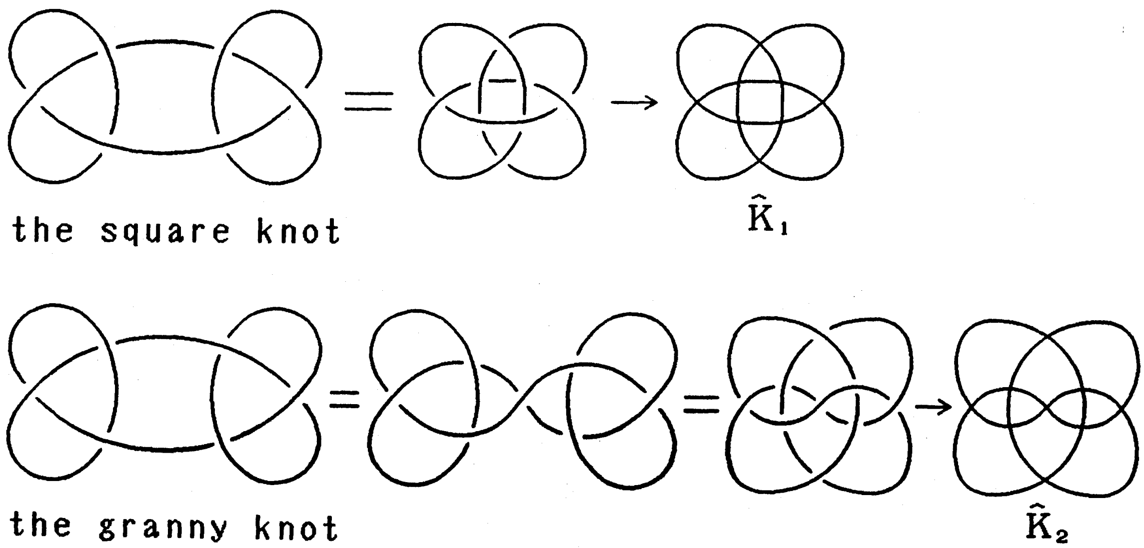


FIGURE 7-1

QUESTION 1. Is the pair $(\mathfrak{K}^1, \leqq)$ a partially ordered set?
This question is equivalent to the following:
Is the function $K \mapsto \mathrm{PROJ}(K)$ a complete knot invariant?

QUESTION 2. Does it hold that $K_1 \# K_2 \geqq K_1$ for $K_1, K_2 \in \mathfrak{K}^1$?

QUESTION 3. Let K_1 be a satelite knot with a companion knot K_2. Does it hold that $K_1 \geqq K_2$?

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