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Automorphisms of Unital C^*-Algebras Which are Strongly Morita Equivalent to Irrational Rotation C^*-Algebras

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Abstract. Let B be a unital C^*-algebra which is strongly Morita equivalent to an irrational rotation C^*-algebra. Then Rieffel showed that it is isomorphic to $A_\theta \otimes M_n$ where A_θ is an irrational rotation C^*-algebra and M_n is the $n\times n$ matrix algebra over $\boldsymbol{C}$. In the present paper we will show that for any automorphism α of $A_\theta\otimes M_n$ there are unitary elements $w\in A_\theta\otimes M_n$, $W\in M_n$ and an automorphism β of A_θ such that $\alpha=\mathrm{Ad}(w)\circ(\beta\otimes\mathrm{Ad}(W))$.

§1. Preliminaries.

For each irrational number $\theta\in\boldsymbol{R}$ let A_θ be an irrational rotation C^*-algebra by θ and for each $n\in\boldsymbol{N}$ let M_n be the $n\times n$ matrix algebra over $\boldsymbol{C}$. Let B be a unital C^*-algebra which is strongly Morita equivalent to A_η for some irrational number η. Then Rieffel [7] showed that there are an $n\in\boldsymbol{N}$ and an irrational number $\theta\in\boldsymbol{R}$ such that B is isomorphic to $A_\theta\otimes M_n$ where θ is an element in the orbit of η under the action of $GL(2, \boldsymbol{Z})$ on irrational numbers by linear fractional transformations and $GL(2, \boldsymbol{Z})$ is the group of all 2×2 matrices over $\boldsymbol{Z}$ with determinant 1 or -1. In what follows, we will study automorphisms of $A_\theta\otimes M_n$. Let u and v be generators of A_θ with $uv=e^{2\pi i\theta}vu$. Then $K_1(A_\theta)=\boldsymbol{Z}[u]\oplus\boldsymbol{Z}[v]$. Let τ be the unique tracial state on A_θ and p be a Rieffel projection in A_θ with $\tau(p)=\theta$. Then $K_0(A_\theta)=\boldsymbol{Z}[1]\oplus\boldsymbol{Z}[p]$ and $\tau_*(K_0(A_\theta))=\boldsymbol{Z}+\boldsymbol{Z}\theta$ where τ_* is the homomorphism of $K_0(A_\theta)$ into $\boldsymbol{R}$ induced by τ. Let Tr be the unique tracial state on M_n and tr be the unique tracial state on $A_\theta\otimes M_n$ defined by $\tau\otimes\mathrm{Tr}$. And let $\{e_{ij}: i, j=1, 2, \cdots, n\}$ be matrix units of M_n and U and V be the generators of M_n defined by $U=\sum_{j=1}^n e^{2\pi i(j/n)}e_{jj}$ and $V=e_{1n}+\sum_{j=2}^n e_{j\,j-1}$. Then $UV=e^{2\pi i(1/n)}VU$. And let I_n be the unit of M_n. Furthermore let A_θ^∞ be the dense $*$-subalgebra of smooth elements of A_θ with respect to the canonical action of the two dimensional torus.

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§2. Automorphisms of $A_\theta \otimes M_n$.

Let A be a C^*-algebra and B be a C^*-subalgebra of A. Let α be an automorphism of A. Let $\alpha|_B$ denote the monomorphism of B into A defined by $\alpha|_B(x)=\alpha(x)$ for any $x \in B$.

LEMMA 1. *Let α be an automorphism of $A_\theta \otimes M_n$. Then there is a unitary element $w \in A_\theta \otimes M_n$ such that $(\mathrm{Ad}(w^*)\circ\alpha)|_{A_\theta \otimes Ce_{jj}}$ is an automorphism of $A_\theta \otimes Ce_{jj}$ for $j=1, 2, \cdots, n$.*

PROOF. Since $\mathrm{tr}\circ\alpha$ is a tracial state on $A_\theta \otimes M_n$, by the uniqueness of the tracial state on $A_\theta \otimes M_n$, $\mathrm{tr}(\alpha(1\otimes e_{jj}))=\mathrm{tr}(1\otimes e_{jj})$ for $j=1, 2, \cdots, n$. Hence by Rieffel [7, 2.5. Corollary] there is a partial isometry $w_j \in A_\theta \otimes M_n$ such that $w_j^* w_j = 1\otimes e_{jj}$ and $w_j w_j^* = \alpha(1\otimes e_{jj})$ for $j=1, 2, \cdots, n$. We define $w=\sum_{j=1}^n w_j$. Then w is a unitary element in $A_\theta \otimes M_n$ such that $w(1\otimes e_{jj})w^* = \alpha(1\otimes e_{jj})$. Thus for any $x \in A_\theta$ and $j=1, 2, \cdots, n$,

$$\begin{aligned}(\mathrm{Ad}(w^*)\circ\alpha)(x\otimes e_{jj}) &= (\mathrm{Ad}(w^*)\circ\alpha)(1\otimes e_{jj})(\mathrm{Ad}(w^*)\circ\alpha)(x\otimes I_n)(\mathrm{Ad}(w^*)\circ\alpha)(1\otimes e_{jj}) \\ &= (1\otimes e_{jj})(\mathrm{Ad}(w^*)\circ\alpha)(x\otimes I_n)(1\otimes e_{jj}) .\end{aligned}$$

Since $(1\otimes e_{jj})(A_\theta \otimes M_n)(1\otimes e_{jj}) = A_\theta \otimes Ce_{jj}$ for $j=1, 2, \cdots, n$, we obtain that $(\mathrm{Ad}(w^*)\circ\alpha)(x\otimes e_{jj}) \in A_\theta \otimes Ce_{jj}$ for any $x \in A_\theta$ and $j=1, 2, \cdots, n$. Hence $(\mathrm{Ad}(w^*)\circ\alpha)|_{A_\theta \otimes Ce_{jj}}$ is an automorphism of $A_\theta \otimes Ce_{jj}$ for $j=1, 2, \cdots, n$.

Q.E.D.

Let $M_n(A_\theta)$ be the $n\times n$ matrix algebra over A_θ. We identify $M_n(A_\theta)$ with $A_\theta \otimes M_n$. Let $A_\theta^\infty \otimes M_n$ denote the $n\times n$ matrix algebra over A_θ^∞, i.e., $M_n(A_\theta^\infty)$.

COROLLARY 2. *Let α be an automorphism of $A_\theta \otimes M_n$ with $\alpha(A_\theta^\infty \otimes M_n) = A_\theta^\infty \otimes M_n$. Then there is a unitary element $w \in A_\theta^\infty \otimes M_n$ such that $(\mathrm{Ad}(w^*)\circ\alpha)|_{A_\theta \otimes Ce_{jj}}$ is an automorphism of $A_\theta \otimes Ce_{jj}$.*

PROOF. By the assumptions, $\alpha(1\otimes e_{jj}) \in A_\theta^\infty \otimes M_n$ for $j=1, 2, \cdots, n$. Since $\mathrm{tr}(\alpha(1\otimes e_{jj}))=\mathrm{tr}(1\otimes e_{jj})$, $[\alpha(1\otimes e_{jj})]=[1\otimes e_{jj}]$ in $K_0(A_\theta^\infty)$ for $j=1, 2, \cdots, n$. Hence $(1\otimes e_{jj})(A_\theta^\infty)^n$ is stably isomorphic to $\alpha(1\otimes e_{jj})(A_\theta^\infty)^n$ as a finitely generated projective right A_θ^∞-module. However the same result as Rieffel [7, 2.2. Theorem] holds for A_θ^∞, that is, A_θ^∞ has cancellation. Thus there is a partial isometry $w_j \in A_\theta^\infty \otimes M_n$ such that $w_j^* w_j = 1\otimes e_{jj}$ and $w_j w_j^* = \alpha(1\otimes e_{jj})$ for $j=1, 2, \cdots, n$. Therefore if we repeat the same discussion as Lemma 1, we obtain the conclusion. Q.E.D.

Now let α be an automorphism of $A_\theta \otimes M_n$. We suppose that $\alpha|_{A_\theta \otimes Ce_{jj}}$ is an automorphism of $A_\theta \otimes Ce_{jj}$ for $j=1, 2, \cdots, n$.

Since $A_\theta \otimes Ce_{jj}$ is isomorphic to A_θ, there is an automorphism β_j of A_θ such that $\alpha(x\otimes e_{jj})=\beta_j(x)\otimes e_{jj}$ for $j=1, 2, \cdots, n$. Furthermore we have the following lemma.

LEMMA 3. *Let α be an automorphism of $A_\theta\otimes M_n$ satisfying the above condition and β_j, $j=1, 2, \cdots, n$, be as above. Then there are unitary elements $y_j \in A_\theta$ such that*

$$\beta_{j+1}(x)=y_j\beta_j(x)y_j^* \qquad \textit{for} \quad j=1, 2, \cdots, n-1$$

and

$$\beta_1(x)=y_n\beta_n(x)y_n^*$$

for any $x\in A_\theta$. In particular if $\alpha(A_\theta^\infty\otimes M_n)=A_\theta^\infty\otimes M_n$, $y_j\in A_\theta^\infty$ for $j=1, 2, \cdots, n$.

PROOF. Let $V=e_{1n}+\sum_{j=2}^n e_{j\,j-1}$. Then $(1\otimes V)(1\otimes e_{jj})(1\otimes V)^*=1\otimes e_{j+1\,j+1}$ for $j=1, 2, \cdots, n-1$. Since $\alpha(1\otimes V)(1\otimes e_{jj})=(1\otimes e_{j+1\,j+1})\alpha(1\otimes V)$, we obtain that

$$\alpha(1\otimes V)=y_n\otimes e_{1n}+\sum_{j=2}^n y_{j-1}\otimes e_{j\,j-1}$$

for some $y_j\in A_\theta$ ($j=1, 2, \cdots, n$). Since $\alpha(1\otimes V)$ is a unitary element in $A_\theta\otimes M_n$, y_j, $j=1, 2, \cdots, n$, are unitary elements in A_θ. Since $\alpha(1\otimes V)\times(\beta_j(x)\otimes e_{jj})=(\beta_{j+1}(x)\otimes e_{j+1\,j+1})\alpha(1\otimes V)$, we obtain that

$$\beta_1(x)y_n=y_n\beta_n(x)$$

and

$$\beta_j(x)y_{j-1}=y_{j-1}\beta_{j-1}(x) \qquad \text{for} \quad j=2, 3, \cdots, n\,.$$

Therefore we get the conclusion. In particular if $\alpha(A_\theta^\infty\otimes M_n)=A_\theta^\infty\otimes M_n$, $\alpha(1\otimes V)\in A_\theta^\infty\otimes M_n$. Hence $y_j\in A_\theta^\infty$ for $j=1, 2, \cdots, n$. Q.E.D.

COROLLARY 4. *Let α be an automorphism of $A_\theta\otimes M_n$. Then there are a unitary element $w\in A_\theta\otimes M_n$ and an automorphism β of A_θ such that $(\mathrm{Ad}(w^*)\circ\alpha)(x\otimes e_{jj})=\beta(x)\otimes e_{jj}$ for any $x\in A_\theta$ and $j=1, 2, \cdots, n$. In particular if $\alpha(A_\theta^\infty\otimes M_n)=A_\theta^\infty\otimes M_n$, $w\in A_\theta^\infty\otimes M_n$ and $\beta(A_\theta^\infty)=A_\theta^\infty$.*

PROOF. By Lemma 1 we can assume that α satisfies the assumptions of Lemma 3. Hence there are unitary elements $y_j\in A_\theta$ and automorphisms β_j of A_θ for $j=1, 2, \cdots, n$ such that

$$\alpha(x\otimes e_{jj})=\beta_j(x)\otimes e_{jj},\ \beta_{j+1}(x)=y_j\beta_j(x)y_j^* \quad \text{for} \quad j=1, 2, \cdots, n-1$$

and

$$\beta_1(x)=y_n\beta_n(x)y_n^*$$

for any $x\in A_\theta$. Let $\beta=\beta_1$ and $w_j=y_j\cdots y_2y_1$ for $j=1, 2, \cdots, n-1$. And let $w=1\otimes e_{11}+\sum_{j=1}^{n-1}w_j\otimes e_{j+1\,j+1}$. Then we obtain that $(\mathrm{Ad}(w^*)\circ\alpha)(x\otimes e_{jj})=\beta(x)\otimes e_{jj}$ for $j=1, 2, \cdots, n$. Furthermore we suppose that $\alpha(A_\theta^\infty\otimes M_n)=A_\theta^\infty\otimes M_n$. Then by Corollary 2 and Lemma 3 we can easily see that $w\in A_\theta^\infty\otimes M_n$ and $\beta(A_\theta^\infty)=A_\theta^\infty$. Q.E.D.

LEMMA 5. *Let α be an automorphism of $A_\theta\otimes M_n$. We suppose that there is an automorphism β of A_θ such that $\alpha(x\otimes e_{jj})=\beta(x)\otimes e_{jj}$ for any $x\in A_\theta$ and $j=1, 2, \cdots, n$. Then there is a unitary element $W\in M_n$ such that $\alpha=\beta\otimes\mathrm{Ad}(W)$.*

PROOF. In the same way as the proof of Lemma 3 we can show that $\alpha(1\otimes V)=y_n\otimes e_{1n}+\sum_{j=2}^{n}y_{j-1}\otimes e_{j\,j-1}$ where y_j, $j=1, 2, \cdots, n$, are unitary elements in A_θ, and that $\beta(x)y_j=y_j\beta(x)$ for any $x\in A_\theta$ and $j=1, 2, \cdots, n$. Hence y_j, $j=1, 2, \cdots, n$, are in $A_\theta\cap A_\theta'$. Since $A_\theta\cap A_\theta'=\boldsymbol{C}1$, y_j, $j=1, 2, \cdots, n$, are in $\boldsymbol{C}1$. Thus there is a unitary element $Y\in M_n$ such that $\alpha(1\otimes V)=1\otimes Y$. Let $U=\sum_{j=1}^{n}e^{2\pi i(j/n)}e_{jj}$. Since $UV=e^{2\pi i(1/n)}$, we get $UY=e^{2\pi i(1/n)}YU$. Hence U and V (or Y) generate M_n. Since $\alpha(1\otimes U)=1\otimes U$ and $\alpha(1\otimes V)=1\otimes Y$, $\alpha|_{C1\otimes M_n}$ is an automorphism of $\boldsymbol{C}1\otimes M$. Hence there is a unitary element $W\in M_n$ such that $\alpha|_{C1\otimes M_n}=\mathrm{Ad}(1\otimes W)$. Thus we obtain that $\alpha=\beta\otimes\mathrm{Ad}(W)$. Q.E.D.

THEOREM 6. *Let α be an automorphism of $A_\theta\otimes M_n$. Then there are a unitary element $w\in A_\theta\otimes M_n$, an automorphism β of A_θ and a unitary element $W\in M_n$ such that $\alpha=\mathrm{Ad}(w)\circ(\beta\otimes\mathrm{Ad}(W))$. In particular if $\alpha(A_\theta^\infty\otimes M_n)=A_\theta^\infty\otimes M_n$, then there are a unitary element $w\in A_\theta^\infty\otimes M_n$, an automorphism β of A_θ with $\beta(A_\theta^\infty)=A_\theta^\infty$ and a unitary element $W\in M_n$ such that $\alpha=\mathrm{Ad}(w)\circ(\beta\otimes\mathrm{Ad}(W))$.*

PROOF. This is trivial by Corollary 4 and Lemma 5. Q.E.D.

Let α be an automorphism of $A_\theta\otimes M_n$ and α_* be the automorphism of $K_1(A_\theta\otimes M_n)$ induced by α. Since $K_1(A_\theta\otimes M_n)$ is isomorphic to $\boldsymbol{Z}^2$, we can regard α_* as an element of $GL(2, \boldsymbol{Z})$.

COROLLARY 7. *With the above assumptions let α be an automorphism of $A_\theta\otimes M_n$ with $\alpha(A_\theta^\infty\otimes M_n)=A_\theta^\infty\otimes M_n$. Then $\alpha_*\in SL(2, \boldsymbol{Z})$.*

PROOF. By Theorem 6 there are a unitary element $w \in A_\theta^\infty \otimes M_n$, an automorphism β of A_θ with $\beta(A_\theta^\infty) = A_\theta^\infty$ and a unitary element $W \in M_n$ such that $\alpha = \mathrm{Ad}(w) \circ (\beta \otimes \mathrm{Ad}(W))$. Since $K_1(A_\theta) \cong \boldsymbol{Z}^2$, β_* can be regarded as an element of $GL(2, \boldsymbol{Z})$. Then by Cuntz, Elliott, Goodman and Jørgensen [2], $\beta_* \in SL(2, \boldsymbol{Z})$. And $\alpha_* = \beta_*$ on $\boldsymbol{Z}^2$ since $\alpha = \mathrm{Ad}(w) \circ (\beta \otimes \mathrm{Ad}(W))$. Thus $\alpha_* \in SL(2, \boldsymbol{Z})$. Q.E.D.

For any s and $t \in \boldsymbol{R}$ let $\beta_{(s,t)}$ be the automorphism of A_θ defined by $\beta_{(s,t)}(u) = e^{2\pi i s} u$ and $\beta_{(s,t)}(v) = e^{2\pi i t} v$, and for any $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \boldsymbol{Z})$ let β_g be the automorphism of A_θ defined by $\beta_g(u) = u^a v^c$ and $\beta_g(v) = u^b v^d$.

COROLLARY 8. *Let α be an automorphism of $A_\theta \otimes M_n$ with $\alpha(A_\theta^\infty \otimes M_n) = A_\theta^\infty \otimes M_n$. Let θ have the generic Diophantine property. Then there are unitary elements $w \in A_\theta^\infty \otimes M_n$, $W \in M_n$, $z \in A_\theta^\infty$ and $s, t \in \boldsymbol{R}$, $g \in SL(2, \boldsymbol{Z})$ such that*

$$\alpha = \mathrm{Ad}(w) \circ ((\mathrm{Ad}(z) \circ \beta_{(s,t)} \circ \beta_g) \otimes \mathrm{Ad}(W)) .$$

PROOF. This is trivial by Theorem 6 and Elliott [3]. Q.E.D.

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