TOKYO J. MATH.
VOL. 12, NO. 1, 1989

Exponentially Bounded C-Semigroups and Integrated Semigroups

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Introduction.

Let X be a Banach space. We denote by $B(X)$ the set of all bounded linear operators from X into itself.

Let C be an injective operator in $B(X)$. We do not assume that the range $R(C)$ is dense in X. A family $\{S(t): t \geqq 0\}$ in $B(X)$ is called an *exponentially bounded C-semigroup on X*, if

(0.1) $\quad S(t+s)C=S(t)S(s) \quad$ for $t, s \geqq 0$ and $S(0)=C$,

(0.2) $\quad S(\cdot)x: [0, \infty) \to X$ is continuous for $x \in X$,

(0.3) $\quad$ there are $M \geqq 0$ and $a \in \boldsymbol{R} \equiv (-\infty, \infty)$ such that

$$\|S(t)\| \leqq Me^{at} \quad \text{for} \quad t \geqq 0 .$$

Let us define $L_\lambda \in B(X)$ for $\lambda > a$ by

$$L_\lambda x = \int_0^\infty e^{-\lambda t} S(t) x dt \quad \text{for} \quad x \in X .$$

Similarly as in the case of $\overline{R(C)} = X$ (see [4]), we see that L_λ is injective for $\lambda > a$ and the closed linear operator Z defined by

$$\begin{cases} D(Z) = \{x \in X: Cx \in R(L_\lambda)\} \\ Zx = (\lambda - L_\lambda^{-1} C)x \quad \text{for} \quad x \in D(Z) \end{cases} \tag{0.4}$$

is independent of $\lambda > a$. The operator Z will be called the *generator* of $\{S(t): t \geqq 0\}$.

Recently, Davies and Pang [4] introduced the notion of an exponentially bounded C-semigroup under the assumption that $R(C)$ is dense in X and gave a characterization of the generator of an exponentially bounded C-semigroup. (See [3] also.) Later, the authors [6, 9, 11] gave a characterization of the complete infinitesimal generator of an exponentially

Received March 28, 1988

bounded C-semigroup and then a unified treatment of the generation of semigroups of class $(C_{(k)})$ and that of semigroups of growth order α.

Let n be a positive integer. A family $\{U(t): t \geqq 0\}$ in $B(X)$ is called an *n-times integrated semigroup on* X (see [1]), if

(0.5) $U(\cdot)x: [0, \infty) \to X$ is continuous for $x \in X$,

(0.6) $$U(t)U(s)x = \frac{1}{(n-1)!}\left(\int_t^{s+t}(s+t-r)^{n-1}U(r)x\,dr - \int_0^s (s+t-r)^{n-1}U(r)x\,dr\right) \quad \text{for } x \in X \text{ and } s, t \geqq 0, \text{ and}$$

$U(0)=0$,

(0.7) $U(t)x=0$ for all $t>0$ implies $x=0$,

(0.8) there are $M \geqq 0$ and $\omega \in \boldsymbol{R}$ such that $\|U(t)\| \leqq Me^{\omega t}$ for $t \geqq 0$.

For convenience we call a semigroup of class (C_0) on X also *0-times integrated semigroup on X.*

It is known that if $\{U(t): t \geqq 0\}$ is an n-times integrated semigroup, then there exists a unique closed linear operator A such that $(\omega, \infty) \subset \rho(A)$ (the resolvent set of A) and

(0.9) $$R(\lambda : A)x\,(\equiv (\lambda - A)^{-1}x) = \int_0^\infty \lambda^n e^{-\lambda t} U(t)x\,dt \quad \text{for } x \in X \text{ and } \lambda > \omega.$$

The operator A is called the *generator* of $\{U(t): t \geqq 0\}$.

In §1 we derive some results on the generator of an exponentially bounded C-semigroup. Among others, we obtain that the generator Z has the following properties ([Proposition 1.4]):

(a_1) $\lambda - Z$ is injective for $\lambda > a$;

(a_2) $D((\lambda - Z)^{-m}) \supset R(C)$ for $\lambda > a$ and $m \geqq 1$;

(a_3) $$\|(\lambda - Z)^{-m}C\| \leqq \frac{M}{(\lambda - a)^m} \quad \text{for } \lambda > a \text{ and } m \geqq 1;$$

(a_4) $Cx \in D(Z)$ and $ZCx = CZx$ for $x \in D(Z)$.

In §2 we shall construct an exponentially bounded C-semigroup under the above conditions (a_1)-(a_4). Our Theorem 2.1 (the first main result) shows that if A is a closed linear operator satisfying (a_1)-(a_4) with Z replaced by A, then there exists an exponentially bounded C_1-semigroup on $\overline{D(A)}$ with generator $C_1^{-1}A_1C_1$, where $C_1 = C|_{\overline{D(A)}}$ and A_1 is the part of

A in $\overline{D(A)}$. This generalizes results in [4, 6] and will be applied to establish Theorem 3.1 (the second main result) in § 3 which clarifies the relations between exponentially bounded C-semigroups and integrated semigroups. Theorem 3.1 generalizes a result in [10].

§ 1. Exponentially bounded C-semigroups.

For simplicity, by a C-semigroup on X we mean an exponentially bounded C-semigroup on X.

Let $\{S(t): t\geqq 0\}$ be a C-semigroup on X with generator Z. Let us define linear operators G and $\mathfrak{A}$ by

$$(1.1)\qquad \begin{cases} D(G)=\left\{x\in R(C):\ \lim\limits_{t\to 0+}\dfrac{C^{-1}S(t)x-x}{t}\ \text{exists}\right\} \\ Gx=\lim\limits_{t\to 0+}\dfrac{C^{-1}S(t)x-x}{t} \qquad \text{for}\quad x\in D(G) \end{cases}$$

and

$$(1.2)\qquad \begin{cases} D(\mathfrak{A})=\left\{x\in X:\ \lim\limits_{t\to 0+}\dfrac{S(t)x-Cx}{t}\in R(C)\right\} \\ \mathfrak{A}x=C^{-1}\lim\limits_{t\to 0+}\dfrac{S(t)x-Cx}{t} \qquad \text{for}\quad x\in D(\mathfrak{A})\ , \end{cases}$$

respectively. ($\mathfrak{A}$ is the infinitesimal generator of $\{S(t): t\geqq 0\}$ in the sense of Da Prato [3].)

The relations among G, $\mathfrak{A}$ and Z are as follows.

PROPOSITION 1.1. *We obtain the following* (1.3) *and* (1.4):

(1.3) $G\subset\bar{G}\subset\mathfrak{A}=Z$, *where* $\bar{G}$ *denotes the closure of* G ;

(1.4) $C^{-1}GC=C^{-1}\bar{G}C=C^{-1}ZC=Z$.

PROOF. To show $\mathfrak{A}\subset Z$, let $x\in D(\mathfrak{A})$ and $\lambda>a$, where a is a constant in (0.3). By $dS(t)Cx/dt=S(t)C\mathfrak{A}x$ for $t\geqq 0$, we have

$$\begin{aligned} CL_\lambda(\lambda-\mathfrak{A})x&=L_\lambda C(\lambda-\mathfrak{A})x=\lambda L_\lambda Cx-\int_0^\infty e^{-\lambda t}\frac{dS(t)Cx}{dt}dt \\ &=C^2x\ , \qquad \text{i.e.,} \\ L_\lambda(\lambda-\mathfrak{A})x&=Cx \qquad \text{for}\quad x\in D(\mathfrak{A})\ \text{and}\ \lambda>a\ . \end{aligned}$$

This implies $\mathfrak{A}\subset Z$. Next, to show $Z\subset\mathfrak{A}$, let $x\in D(Z)$ and take $y\in X$ such that $Cx=L_\lambda y$, where $\lambda>a$. Noting $C^{-1}S(h)u=S(h)C^{-1}u$ for $u\in R(C)$ and $h>0$,

$$h^{-1}(S(h)x-Cx)=h^{-1}(C^{-1}S(h)L_\lambda y-L_\lambda y)$$
$$=h^{-1}(e^{\lambda h}-1)\int_h^\infty e^{-\lambda t}S(t)ydt-h^{-1}\int_0^h e^{-\lambda t}S(t)ydt$$
$$\to\lambda L_\lambda y-Cy=C(\lambda x-y)\in R(C) \qquad \text{as} \quad h\to 0+ .$$

This shows that $x\in D(\mathfrak{A})$ and $\mathfrak{A}x=\lambda x-y=Zx$. Therefore $Z\subset\mathfrak{A}$ and hence $\mathfrak{A}=Z$. Since $G\subset\mathfrak{A}$ and $\mathfrak{A}$ $(=Z)$ is closed, G is closable and $\bar{G}\subset\mathfrak{A}$. So we have (1.3).

To prove (1.4), let $x\in D(\mathfrak{A})$ first. Then $\lim_{t\to 0+}(C^{-1}S(t)Cx-Cx)/t=\lim_{t\to 0+}(S(t)x-Cx)/t=C\mathfrak{A}x$, and hence $Cx\in D(G)$ and $GCx=C\mathfrak{A}x$. Therefore $\mathfrak{A}\subset C^{-1}GC$. Now, we want to show that $C^{-1}ZC\subset Z$. To this end, let $x\in D(C^{-1}ZC)$, i.e., $Cx\in D(Z)$ and $ZCx\in R(C)$. Then

$$L_\lambda(\lambda-C^{-1}ZC)x=L_\lambda C^{-1}(\lambda-Z)Cx=C^{-1}L_\lambda(\lambda-Z)Cx=Cx ,$$

and hence $x\in D(Z)$ and $Zx=(\lambda-L_\lambda^{-1}C)x=C^{-1}ZCx$. Consequently, $C^{-1}ZC\subset Z$. Combining these with (1.3), we obtain (1.4). Q.E.D.

$\bar{G}$ is called the *complete infinitesimal generator* (c.i.g.) of $\{S(t): t\geqq 0\}$. The following example shows that "$\bar{G}=Z$" does not hold in general.

EXAMPLE. Let $X=C[0, 1]$, and define $C\in B(X)$ by

$$(Cx)(t)=\int_0^t x(s)ds , \quad 0\leqq t\leqq 1 , \quad \text{for } x\in C[0, 1] .$$

Then C is injective and $R(C)=\{x\in C^1[0, 1]: x(0)=0\}$ (and hence $R(C)$ is not dense in X). Consider the C-semigroup $\{S(t): t\geqq 0\}$ defined by $S(t)=C$ for all $t\geqq 0$. In this case, $D(Z)=X$ and $Zx=0$ for $x\in X$, but $D(G)\subset R(C)$ and hence $D(\bar{G})\subset\overline{R(C)}\neq X$. This shows $\bar{G}\neq Z$.

PROPOSITION 1.2. *We have the following* (1.5)-(1.7):

$$(1.5) \qquad \begin{cases}(\lambda-Z)L_\lambda x=Cx & \textit{for} \quad x\in X \textit{ and } \lambda>a\\ L_\lambda(\lambda-Z)x=Cx & \textit{for} \quad x\in D(Z) \textit{ and } \lambda>a ,\end{cases}$$

where a is a constant in (0.3);

$$(1.6) \qquad S(t)x\in D(Z) \textit{ and } ZS(t)x=S(t)Zx \qquad \textit{for} \quad x\in D(Z) \textit{ and } t\geqq 0 ;$$

$$(1.7) \qquad \int_0^t S(s)xds\in D(Z) \textit{ and } S(t)x-Cx=Z\int_0^t S(s)xds \qquad \textit{for} \quad x\in X \textit{ and } t\geqq 0 .$$

PROOF. (1.5) and (1.6) follow from the definition of Z. It is easily seen that $\int_0^t S(s)xds \in D(\mathfrak{A})$ and $S(t)x - Cx = \mathfrak{A}\int_0^t S(s)xds$ for $x \in X$ and $t \geqq 0$. By $Z = \mathfrak{A}$, we obtain (1.7). Q.E.D.

COROLLARY 1.3. *For every* $x \in C(D(Z))$, $u(t) \equiv C^{-1}S(t)x$ *is a unique* C^1*(continuously differentiable)-solution of the Cauchy problem*

$$\text{(CP)} \qquad \frac{du(t)}{dt} = Zu(t)\,, \quad t \geqq 0, \quad \textit{and} \quad u(0) = x\,.$$

Moreover, the $u(t)$ *satisfies* $\|u(t)\| \leqq Me^{at}\|C^{-1}x\|$, *where* M *and* a *are constants in* (0.3).

PROOF. Let $x \in C(D(Z))$ and put $u(t) = C^{-1}S(t)x$ for $t \geqq 0$. By (1.6) and (1.7) we have that $Zu(t) = ZS(t)y = S(t)Zy$ and

$$u(t) - x = \int_0^t Zu(s)ds \qquad \text{for} \quad t \geqq 0\,,$$

where y is an element in $D(Z)$ such that $x = Cy$. Therefore $u(t)$ is a C^1-solution of (CP), and $\|u(t)\| = \|S(t)y\| \leqq Me^{at}\|y\| = Me^{at}\|C^{-1}x\|$. To show the uniqueness, let $v(t)$ be a C^1-solution of (CP) and $s > 0$ be arbitrarily given. Then

$$\frac{d}{dt}S(s-t)v(t) = S(s-t)Zv(t) - ZS(s-t)v(t) = 0$$

for $0 \leqq t \leqq s$. Integrating this over $[0, s]$, we obtain $Cv(s) = S(s)x$, i.e., $v(s) = C^{-1}S(s)x = u(s)$ for every $s > 0$. Q.E.D.

PROPOSITION 1.4. *Z satisfies the following* (a_1)-(a_4):

(a_1) $\lambda - Z$ *is injective for* $\lambda > a$;

(a_2) $D((\lambda - Z)^{-m}) \supset R(C)$ *for* $\lambda > a$ *and* $m \geqq 1$;

(a_3) $\|(\lambda - Z)^{-m}C\| \leqq M/(\lambda - a)^m$ *for* $\lambda > a$ *and* $m \geqq 1$, *where* M *and* a *are constants in* (0.3);

(a_4) $Cx \in D(Z)$ *and* $ZCx = CZx$ *for* $x \in D(Z)$.

PROOF. (a_1) and (a_4) follow from (1.5) and (1.6), respectively. Next, using induction with respect to m, we obtain (a_2) and

$$(\lambda - Z)^{-m}Cx = \int_0^\infty \cdots \int_0^\infty e^{-\lambda(t_1 + \cdots + t_m)} S(t_1 + \cdots + t_m)xdt_1 \cdots dt_m$$

for $x \in X$, $\lambda > a$ and $m \geqq 1$. Combining this with (0.3) we get (a_3).

Q.E.D.

§ 2. Construction of C-semigroups.

Throughout this section A denotes a closed linear operator in X satisfying the following conditions (which correspond to (a_1)-(a_4) in Proposition 1.4):

(A_1) there exists an $a \in \boldsymbol{R}$ such that $\lambda - A$ is injective for $\lambda > a$;

(A_2) $D((\lambda - A)^{-m}) \supset R(C)$ for $\lambda > a$ and $m \geqq 1$;

(A_3) there exists an $M \geqq 0$ such that $\|(\lambda - A)^{-m}C\| \leqq M/(\lambda - a)^m$ for $\lambda > a$ and $m \geqq 1$;

(A_4) $Cx \in D(A)$ and $ACx = CAx$ for $x \in D(A)$.

It is easily seen that (A_4) is equivalent to the following (A_4'):

(A_4') $(\lambda - A)^{-1}Cx = C(\lambda - A)^{-1}x$ for $\lambda > a$ and $x \in D((\lambda - A)^{-1})$.

The purpose of this section is to construct a C-semigroup on $\overline{D(A)}$ under these conditions. Our idea for construction is based on that of [7].

Our theorem is the following which generalizes [4, Theorem 11] and [6, Theorem 2].

THEOREM 2.1. *Let A be a closed linear operator satisfying* (A_1)-(A_4). *Then for every $x \in \overline{D(A)}$, the limit*

$$S_1(t)x \equiv \lim_{n\to\infty} \left(1 - \frac{tA}{n}\right)^{-n} Cx$$

exists uniformly on every bounded subset of $[0, \infty)$. *The family $\{S_1(t): t \geqq 0\}$ has the following properties:*

(2.1) $S_1(t): \overline{D(A)} \to \overline{D(A)}$;

(2.2) $S_1(t+s)Cx = S_1(t)S_1(s)x$ *and* $S_1(0)x = Cx$ *for* $x \in \overline{D(A)}$ *and* $t, s \geqq 0$;

(2.3) $\|S_1(t)x\| \leqq Me^{at}\|x\|$ *for* $x \in \overline{D(A)}$ *and* $t \geqq 0$;

(2.4) $\|S_1(t)x - S_1(s)x\| \leqq Me^{|a|\max\{t,s\}}\|Ax\|\,|t-s|$ *for* $x \in D(A)$ *and* $t, s \geqq 0$, *and hence* $S_1(\cdot)x: [0, \infty) \to \overline{D(A)}$ *is continuous for* $x \in \overline{D(A)}$;

(2.5) $S_1(t)x - Cx = A\int_0^t S_1(s)x\,ds$ *for* $x \in \overline{D(A)}$ *and* $t \geqq 0$;

(2.6) $(\lambda - A)^{-1}Cx = \int_0^\infty e^{-\lambda t}S_1(t)x\,dt$ *for* $x \in \overline{D(A)}$ *and* $\lambda > a$.

Therefore, setting $C_1 = C|_{\overline{D(A)}}$, $\{S_1(t): t \geqq 0\}$ is a C_1-semigroup on the Banach space $\overline{D(A)}$.

Moreover, $C_1^{-1}A_1C_1$ *is the generator of the* C_1*-semigroup* $\{S_1(t): t \geqq 0\}$, *where* A_1 *denotes the part of* A *in* $\overline{D(A)}$.

Before proving this theorem we prepare two lemmas. We define a linear subset $\tilde{\Sigma}$ of X and a function $\tilde{N}(\cdot)$ on $\tilde{\Sigma}$ by

$$\tilde{\Sigma}=\{x \in \bigcap_{\lambda>a,m\geqq 0} D((\lambda-A)^{-m}): \sup_{\lambda>a,m\geqq 0} \|(\lambda-a)^m(\lambda-A)^{-m}x\|<\infty\}$$

and

$$\tilde{N}(x)=\sup_{\lambda>a,m\geqq 0} \|(\lambda-a)^m(\lambda-A)^{-m}x\| \qquad \text{for} \quad x \in \tilde{\Sigma} .$$

Obviously, $\|x\| \leqq \tilde{N}(x)$ for $x \in \tilde{\Sigma}$ and $\tilde{N}(\cdot)$ defines a norm on $\tilde{\Sigma}$. Our assumptions (A_2) and (A_3) imply

$$R(C) \subset \tilde{\Sigma} \text{ and } \tilde{N}(x) \leqq M\|C^{-1}x\| \qquad \text{for} \quad x \in R(C) . \tag{2.7}$$

LEMMA 2.2. *The following conditions* (b_1)-(b_3) *(which are stated in* [7, § 4]) *are satisfied with* $Y=\tilde{\Sigma}$ *and* $|||\cdot|||=\tilde{N}(\cdot)$:

(b_1) Y *is a normed space under a certain norm* $|||\cdot|||$ *which is stronger than the original norm* $\|\cdot\|$ *of* X;

(b_2) *there exists a real* ω *such that for* $\lambda>\omega$, $R(\lambda-A)$ *contains* Y, $R(\lambda)\equiv(\lambda-A)^{-1}$ *exists, and such that* Y *is invariant under* $R(\lambda)$;

(b_3) *there exists a constant* $M \geqq 0$ *such that*

$$\|R(\lambda)^m x\| \leqq M(\lambda-\omega)^{-m}|||x||| \qquad \text{for} \quad x \in Y,\ \lambda>\omega \text{ and } m \geqq 0 .$$

Moreover we have

$$\tilde{N}((\lambda-a)R(\lambda)x) \leqq \tilde{N}(x) \qquad \text{for} \quad x \in \tilde{\Sigma} \text{ and } \lambda>a . \tag{2.8}$$

PROOF. (b_1) is obvious. To prove (b_2) and (b_3), we first note that clearly $R(\lambda-A) \supset \tilde{\Sigma}$ and $R(\lambda)\equiv(\lambda-A)^{-1}$ exists for $\lambda>a$, and the following equality holds:

$$R(\lambda)^m R(\mu)^n x=\sum_{l=m-1}^{\infty} {}_lC_{m-1}(\mu-\lambda)^{l-m+1}R(\mu)^{l+n+1}x \tag{2.9}$$

$$\text{for } x \in \tilde{\Sigma},\ \mu>\lambda>a,\ m \geqq 1 \text{ and } n \geqq 0.$$

Indeed, since

$$\|{}_lC_{m-1}(\mu-\lambda)^{l-m+1}R(\mu)^{l+n+1}x\| \leqq {}_lC_{m-1}\left(\frac{\mu-\lambda}{\mu-a}\right)^{l-m+1}(\mu-a)^{-m-n}\tilde{N}(x) \quad \text{for } x \in \tilde{\Sigma} ,$$

the series of the right side in (2.9) is absolutely convergent with respect to the norm $\|\cdot\|$. Let $x \in \tilde{\Sigma}$. Then

$$(\lambda-A)\sum_{l=0}^{k}(\mu-\lambda)^{l}R(\mu)^{l+n+1}x=R(\mu)^{n}x-(\mu-\lambda)^{k+1}R(\mu)^{k+n+1}x$$

and

$$\|(\mu-\lambda)^{k+1}R(\mu)^{k+n+1}x\|\leqq\left(\frac{\mu-\lambda}{\mu-a}\right)^{k+1}(\mu-a)^{-n}\tilde{N}(x)\to 0 \qquad \text{as} \quad k\to\infty ,$$

which imply that (2.9) holds for $m=1$. The conclusion follows from the induction with respect to m.

Next, setting $n=0$ in (2.9), we obtain

$$R(\mu)^{m}x=\sum_{l=m-1}^{\infty}{}_{l}C_{m-1}(\lambda-\mu)^{l-m+1}R(\lambda)^{l+1}x \qquad \text{for} \quad x\in\tilde{\Sigma},\ \lambda>\mu>a \text{ and } m\geqq 1 .$$

Since $R(\lambda)$ is closed, for $x\in\tilde{\Sigma}$, $\lambda>\mu>a$, $m\geqq 1$ and $n\geqq 0$

$$R(\lambda)^{n}R(\mu)^{m}x=\sum_{l=m-1}^{\infty}{}_{l}C_{m-1}(\lambda-\mu)^{l-m+1}R(\lambda)^{l+n+1}x . \tag{2.10}$$

Now, let $\mu>a$ and $x\in\tilde{\Sigma}$. Then, by (2.9), for λ with $\mu>\lambda>a$ we have

$$\begin{aligned}&\|(\lambda-a)^{m}R(\lambda)^{m}(\mu-a)^{n}R(\mu)^{n}x\|\\&\leqq\left(\frac{\lambda-a}{\mu-a}\right)^{m}\sum_{l=m-1}^{\infty}{}_{l}C_{m-1}\left(1-\frac{\lambda-a}{\mu-a}\right)^{l-m+1}\|(\mu-a)^{l+n+1}R(\mu)^{l+n+1}x\|\\&\leqq\left(\frac{\lambda-a}{\mu-a}\right)^{m}\sum_{l=m-1}^{\infty}{}_{l}C_{m-1}\left(1-\frac{\lambda-a}{\mu-a}\right)^{l-m+1}\tilde{N}(x)=\tilde{N}(x) .\end{aligned}$$

In the same way, by (2.10), for $\lambda>\mu$

$$\|(\lambda-a)^{n}R(\lambda)^{n}(\mu-a)^{m}R(\mu)^{m}x\|\leqq\tilde{N}(x) .$$

Consequently, for $n\geqq 1$ and $\mu>a$,

$$\|(\lambda-a)^{m}R(\lambda)^{m}(\mu-a)^{n}R(\mu)^{n}x\|\leqq\tilde{N}(x) \qquad \text{for} \quad \lambda>a,\ m\geqq 1 \text{ and } x\in\tilde{\Sigma} .$$

Hence for every $\mu>a$ and $n\geqq 1$,

$$(\mu-a)^{n}R(\mu)^{n}x\in\tilde{\Sigma} \text{ and } \tilde{N}((\mu-a)^{n}R(\mu)^{n}x)\leqq\tilde{N}(x) \qquad \text{for} \quad x\in\tilde{\Sigma} .$$

In particular, $\tilde{\Sigma}$ is invariant under $R(\lambda)$ for $\lambda>a$ and $\|R(\lambda)^{m}x\|\leqq(\lambda-a)^{-m}\tilde{N}(x)$ for $x\in\tilde{\Sigma}$ (i.e., (b_2) and (b_3) hold with $Y=\tilde{\Sigma}$, $|||\cdot|||=\tilde{N}(\cdot)$, $\omega=a$ and $M=1$), and (2.8) holds. Q.E.D.

In view of Lemma 2.2, we may employ the results given in [7, § 4]. Also, by using the argument due to [2] and [5], (2.8) implies the following

Lemma 2.3 ([5]). *For λ, $\mu>0$ with $\lambda|a|\leqq 1/2$, $\mu|a|\leqq 1/2$ and n, $m\geqq 0$,*

$$\tilde{N}(J_\lambda^m x-J_\mu^n x)\leqq[\exp(2|a|(m\lambda+n\mu))]((m\lambda-n\mu)^2+m\lambda^2+n\mu^2)^{1/2}\tilde{N}(Ax)$$

for $x\in\tilde{\Sigma}_1\equiv\{x\in\tilde{\Sigma}: Ax\in\tilde{\Sigma}\}$, *where* $J_\lambda=(1-\lambda A)^{-1}$.

Proof of Theorem 2.1. First, let $x\in D(A)$. Since $Cx\in D(A)\cap\tilde{\Sigma}$ and $ACx=CAx\in\tilde{\Sigma}$ (and hence $Cx\in C(D(A))\subset\tilde{\Sigma}_1$) by (A_4) and (2.7), it follows from Lemma 2.3 and (2.7) that

$$\|J_\lambda^{[t/\lambda]}Cx-J_\mu^{[t/\mu]}Cx\|\leqq Me^{4|a|t}((\lambda+\mu)^2+t(\lambda+\mu))^{1/2}\|Ax\|$$

for $t\geqq 0$. Therefore the limit $\lim_{\lambda\to 0+}J_\lambda^{[t/\lambda]}Cx$ exists uniformly on every bounded subset of $[0,\infty)$. This remains true for every $x\in\overline{D(A)}$, because $\|J_\lambda^{[t/\lambda]}C\|$ are uniformly bounded on every bounded subset of $[0,\infty)$ as $\lambda\to 0+$.

Define $S_1(t)$ for $t\geqq 0$ by

$$S_1(t)x=\lim_{\lambda\to 0+}J_\lambda^{[t/\lambda]}Cx\ \left(=\lim_{n\to\infty}\left(1-\frac{tA}{n}\right)^{-n}Cx\right)\qquad\text{for}\quad x\in\overline{D(A)}\ .$$

Clearly (2.1) and (2.3) hold, and it follows from (2.3), (2.7) and [7, Theorem 4.6] that (2.4) and (2.6) hold. By Lemma 2.3 again, for $x\in D(A)$

$$\begin{aligned}&\|J_\lambda^{[(t+s)/\lambda]}C\cdot Cx-J_\lambda^{[t/\lambda]}C\cdot J_\lambda^{[s/\lambda]}Cx\|\\&\quad=\|J_\lambda^{[(t+s)/\lambda]}C^2x-J_\lambda^{[t/\lambda]+[s/\lambda]}C^2x\|\quad(\text{by }(A_4'))\\&\quad\leqq e^{4|a|(t+s)}(4\lambda^2+2\lambda(t+s))^{1/2}\tilde{N}(AC^2x)\to 0\qquad\text{as}\quad\lambda\to 0+\ ,\end{aligned}$$

which implies

$$S_1(t+s)Cx=S_1(t)S_1(s)x\qquad\text{for}\quad x\in D(A)\text{ and }t,\ s\geqq 0\ .$$

Therefore (2.2) holds. Next, we will prove that (2.5) holds. By virtue of [7, Lemma 4.5] and the closedness of A, we have

$$\left(1-\frac{tA}{n}\right)^{-n}x-x=\int_0^t\left(1-\frac{sA}{n}\right)^{-(n+1)}Axds=A\int_0^t\left(1-\frac{sA}{n}\right)^{-(n+1)}xds$$

for $x\in\tilde{\Sigma}_1$, $t\geqq 0$ and integer n with $n>|a|t$. In particular, the following holds:

$$\left(1-\frac{tA}{n}\right)^{-n}Cx-Cx=A\int_0^t\left(1-\frac{sA}{n}\right)^{-(n+1)}Cxds$$

for $x\in D(A)$, $t\geqq 0$ and integer n with $n>|a|t$. Letting $n\to\infty$, and noting that

$$\left\|\left(1-\frac{sA}{n}\right)^{-(n+1)}Cx-\left(1-\frac{sA}{n}\right)^{-n}Cx\right\|$$
$$=\frac{s}{n}\left\|\left(1-\frac{sA}{n}\right)^{-(n+1)}CAx\right\|\leqq Ms\left(1-\frac{s|a|}{n}\right)^{-(n+1)}\frac{\|Ax\|}{n}\to 0$$

as $n\to\infty$, the closedness of A implies

$$\int_0^t S_1(s)xds\in D(A)\quad\text{and}\quad S_1(t)x-Cx=A\int_0^t S_1(s)xds$$

for $x\in D(A)$ and $t\geqq 0$. These remain true for $x\in\overline{D(A)}$ by the closedness of A and (2.3).

Finally, we will prove that $C_1^{-1}A_1C_1$ is the generator of the C_1-semigroup $\{S_1(t)\colon t\geqq 0\}$ on $\overline{D(A)}$. To this end, let Z_1 be the generator of $\{S_1(t)\colon t\geqq 0\}$ and let $\mathfrak{A}_1$ be the operator defined by

$$\begin{cases} D(\mathfrak{A}_1)=\left\{x\in\overline{D(A)}:\ \lim_{t\to 0+}\dfrac{S_1(t)x-C_1x}{t}\in R(C_1)\right\} \\ \mathfrak{A}_1x=C_1^{-1}\lim_{t\to 0+}\dfrac{S_1(t)x-C_1x}{t}\qquad\text{for}\quad x\in D(\mathfrak{A}_1)\end{cases}$$

(see (1.2)). Then we have

$$A_1\subset Z_1\,. \tag{2.11}$$

In fact, let $x\in D(A_1)$. Noting $(\lambda-A)^{-k}CA_1x=A(\lambda-A)^{-k}Cx$ for every $k\geqq 0$ and $\lambda>a$, it follows that

$$A\left(1-\frac{tA}{n}\right)^{-n}Cx=\left(1-\frac{tA}{n}\right)^{-n}CA_1x$$

for $t\geqq 0$. Letting $n\to\infty$, by the closedness of A, we have

$$S_1(t)x\in D(A)\quad\text{and}\quad AS_1(t)x=S_1(t)A_1x\in\overline{D(A)}$$

and hence $S_1(t)x\in D(A_1)$ and $A_1S_1(t)x=AS_1(t)x=S_1(t)A_1x$ for $t\geqq 0$. Combining this with (2.6), we have

$$C_1x=(\lambda-A)\int_0^\infty e^{-\lambda t}S_1(t)xdt=\int_0^\infty e^{-\lambda t}S_1(t)(\lambda-A_1)xdt$$
$$=\mathscr{L}_\lambda(\lambda-A_1)x\qquad\text{for}\quad\lambda>a\,,$$

where $\mathscr{L}_\lambda z=\int_0^\infty e^{-\lambda t}S_1(t)zdt$ for $z\in\overline{D(A)}$ and $\lambda>a$. So, by the definition of generator Z_1, we get

$$x \in D(Z_1) \quad \text{and} \quad A_1x=(\lambda-\mathscr{L}_\lambda^{-1}C_1)x=Z_1x\ .$$

This proves (2.11). Next, let $x \in D(\mathfrak{A}_1)$. By (2.5)

$$A\left(t^{-1}\int_0^t S_1(s)xds\right)=\frac{S_1(t)x-C_1x}{t} \to C_1\mathfrak{A}_1x$$

as $t \to 0+$. Since $\lim_{t\to 0+} t^{-1}\int_0^t S_1(s)xds=C_1x$ and A is closed, we get

$$C_1x \in D(A) \quad \text{and} \quad AC_1x=C_1\mathfrak{A}_1x \in \overline{D(A)}\ .$$

This means that $C_1x \in D(A_1)$ and $A_1C_1x\ (=AC_1x)=C_1\mathfrak{A}_1x$, i.e., $x \in D(C_1^{-1}A_1C_1) \equiv \{z \in \overline{D(A)}: C_1z \in D(A_1) \text{ and } A_1C_1z \in R(C_1)\}$ and $\mathfrak{A}_1x=C_1^{-1}A_1C_1x$. Therefore we obtain

$$\mathfrak{A}_1 \subset C_1^{-1}A_1C_1\ . \tag{2.12}$$

But $\mathfrak{A}_1=Z_1=C_1^{-1}Z_1C_1 \supset C_1^{-1}A_1C_1$ by Proposition 1.1 and (2.11). Combining this with (2.12), we have that $Z_1=C_1^{-1}A_1C_1$. Q.E.D.

§ 3. *C*-semigroups and integrated semigroups.

The following theorem establishes the relations between C-semigroups and integrated semigroups.

THEOREM 3.1. *Let A be a closed linear operator in X with $\rho(A)\neq\emptyset$. Let $c \in \rho(A)$ and $n \geqq 0$ be an integer. The following* (i)-(iii) *are mutually equivalent*:

(i) *A is the generator of an $(n+1)$-times integrated semigroup $\{U(t): t\geqq 0\}$ on X satisfying $\|U(t+h)-U(t)\| \leqq K'he^{\omega'(t+h)}$ for t, $h \geqq 0$, where $K' \geqq 0$ and $\omega' \in \boldsymbol{R}$ are constants;*

(ii) *A is the generator of a C-semigroup $\{S(t): t\geqq 0\}$ on X with $C=R(c: A)^{n+1}$ satisfying $\|S(t+h)-S(t)\| \leqq Khe^{\omega(t+h)}$ for t, $h \geqq 0$, where $K \geqq 0$ and $\omega \in \boldsymbol{R}$ are constants;*

(iii) *There exist $M \geqq 0$ and $a \in \boldsymbol{R}$ such that $(a, \infty) \subset \rho(A)$ and*

$$\|R(\lambda: A)^m R(c: A)^n\| \leqq M/(\lambda-a)^m \qquad \text{for} \quad \lambda>a \text{ and } m \geqq 1\ .$$

In this case, we have for $t \geqq 0$

$$U(t)x=(c-A)^{n+1}\int_0^t\int_0^{t_1}\cdots\int_0^{t_n} S(t_{n+1})xdt_{n+1}\cdots dt_2dt_1 \tag{3.1}$$

for $x \in X$

$$\left(=\int_0^t\int_0^{t_1}\cdots\int_0^{t_n}S(t_{n+1})(c-A)^{n+1}x\,dt_{n+1}\cdots dt_2dt_1 \quad \textit{for } x\in D(A^{n+1})\right).$$

Moreover, if A is a closed linear operator in X with $\rho(A)\neq\emptyset$ satisfying the equivalent conditions above and A_1 is the part of A in $\overline{D(A)}$, then we obtain the following (c_1)-(c_3):

(c_1) *A_1 is the generator of a C_1-semigroup $\{S_1(t): t\geqq 0\}$ on $\overline{D(A)}$ with $C_1=R(c: A)^n|_{\overline{D(A)}}$;*

(c_2) ([1]) *A_1 is the generator of an n-times integrated semigroup $\{U_1(t): t\geqq 0\}$ on $\overline{D(A)}$;*

(c_3) $U_1(t)x=(c-A_1)^n\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}}S_1(t_n)x\,dt_n\cdots dt_2dt_1$ *for $x\in\overline{D(A)}$ and $t\geqq 0$*

$$\left(=\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}}S_1(t_n)(c-A_1)^n x\,dt_n\cdots dt_2dt_1 \quad \textit{for } x\in D(A_1^n) \textit{ and } t\geqq 0\right).$$

This is a generalization of [10, Theorem 1] (and [8, Theorem 4.6]). Indeed Theorem 3.1 leads to

COROLLARY 3.2 ([10, Theorem 1]). *Let A be a densely defined closed linear operator in X with $\rho(A)\neq\emptyset$. Let $c\in\rho(A)$ and $n\geqq 0$ be an integer. The following* (i′)-(iii′) *are equivalent:*

(i′) *A is the generator of an n-times integrated semigroup $\{\tilde{U}(t): t\geqq 0\}$ on X;*

(ii′) *A is the c.i.g. of a C-semigroup $\{\tilde{S}(t): t\geqq 0\}$ on X with $C=R(c: A)^n$;*

(iii′) *there exist $M\geqq 0$ and $a\in\boldsymbol{R}$ such that $(a, \infty)\subset\rho(A)$ and*

$$\|R(\lambda: A)^m R(c: A)^n\|\leqq\frac{M}{(\lambda-a)^m} \qquad \textit{for } \lambda>a \textit{ and } m\geqq 1\,.$$

In this case, we have for $t\geqq 0$

$$\tilde{U}(t)x=(c-A)^n\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}}\tilde{S}(t_n)x\,dt_n\cdots dt_2dt_1 \qquad \textit{for } x\in X \tag{3.2}$$

$$\left(=\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}}\tilde{S}(t_n)(c-A)^n x\,dt_n\cdots dt_2dt_1 \quad \textit{for } x\in D(A^n)\right).$$

To derive the corollary from Theorem 3.1 we note the following which will be easily proved:

(d_1) If Z is the generator of a C-semigroup $\{S(t): t\geqq 0\}$ on X and $P\in B(X)$ is an injective operator satisfying $S(t)P=PS(t)$ for $t\geqq 0$, then $\{S(t)P: t\geqq 0\}$ is a PC-semigroup on X and Z is the generator of $\{S(t)P: t\geqq 0\}$.

(d_2) If A is the generator of an n-times integrated semigroup $\{U(t): t\geqq 0\}$ on X and $V(t)$, $t\geqq 0$, are defined by $V(t)x=\int_0^t U(s)xds$ for $x\in X$, then $\{V(t): t\geqq 0\}$ is an $(n+1)$-times integrated semigroup on X satisfying $\|V(t+h)-V(t)\|\leqq Me^{\omega(t+h)}h$ for t, $h\geqq 0$, where $M\geqq 0$ and $\omega\in\boldsymbol{R}$ are some constants, and A is the generator of $\{V(t): t\geqq 0\}$.

PROOF OF COROLLARY 3.2. In this case, note that the c.i.g. A in (ii′) coincides with the generator of $\{\tilde{S}(t): t\geqq 0\}$ (see [4, Theorem 35]). Since $A_1=A$ and $C_1=R(c: A)^n$ by $\overline{D(A)}=X$, "(iii′)⇒(ii′)" and "(iii′)⇒(i′)" follow from Theorem 3.1 (c_1) and (c_2), respectively.

To prove "(ii′)⇒(iii′)" let A be the generator of a C-semigroup $\{\tilde{S}(t): t\geqq 0\}$ on X with $C=R(c: A)^n$ and let $\|\tilde{S}(t)\|\leqq\tilde{M}e^{at}$ for $t\geqq 0$. Define $S(t)$, $t\geqq 0$, by $S(t)=\tilde{S}(t)R(c: A)$. Since $\tilde{S}(t)R(c: A)=R(c: A)\tilde{S}(t)$ for $t\geqq 0$ by (1.6), it follows from (d_1) that $\{S(t): t\geqq 0\}$ is a C-semigroup on X with $C=R(c: A)^{n+1}$ and A is the generator of $\{S(t): t\geqq 0\}$. Moreover, $\|S(t+h)x-S(t)x\|=\left\|\int_t^{t+h}\tilde{S}(s)AR(c: A)xds\right\|$ (by (1.6) and (1.7)) $\leqq\tilde{M}\|AR(c: A)\|h\times e^{|a|(t+h)}\|x\|$ for $x\in X$ and t, $h\geqq 0$. Therefore A satisfies (ii) in Theorem 3.1, and hence (iii′) (= (iii) in Theorem 3.1) holds.

To show "(i′)⇒(iii′)", let us define $U(t)$, $t\geqq 0$, by $U(t)x=\int_0^t\tilde{U}(s)xds$ for $x\in X$. By (d_2), A satisfies (i) in Theorem 3.1 and then (iii′) holds. Moreover, by (3.1)

$$\begin{aligned}\int_0^t\tilde{U}(s)xds&=U(t)x=(c-A)^{n+1}\int_0^t\int_0^{t_1}\cdots\int_0^{t_n}\tilde{S}(t_{n+1})R(c: A)xdt_{n+1}\cdots dt_2dt_1\\&=\int_0^t\left[(c-A)^n\int_0^{t_1}\cdots\int_0^{t_n}\tilde{S}(t_{n+1})xdt_{n+1}\cdots dt_2\right]dt_1\quad\text{for } x\in X \text{ and } t\geqq 0,\end{aligned}$$

which implies (3.2). Q.E.D.

REMARKS. 1. Each of the equivalent conditions (i)-(iii) in Theorem 3.1 is equivalent to the following (iv) (see [1, Theorem 4.1]):

(iv) there exist $M\geqq 0$ and $a\in\boldsymbol{R}$ such that $(a, \infty)\subset\rho(A)$ and

$$\left\|\frac{[R(\lambda: A)/\lambda^n]^{(k)}}{k!}\right\|\leqq\frac{M}{(\lambda-a)^{k+1}}\qquad\text{for }\lambda>a\text{ and }k\geqq 0\,.$$

2. In the case of $\overline{D(A)}\neq X$ in Theorem 3.1, "generator" in (ii) can not be replaced by "c.i.g.". In fact, the operator A of Example 6.4 in [1] is the generator of a C-semigroup on $X(=E)$ with $C=R(c: A)$ satisfying $\|S(t+h)-S(t)\|\leqq Ke^{\omega(t+h)}$ for t, $h\geqq 0$, but it is not the c.i.g. of any C-semigroup on X with $C=R(c: A)$.

PROOF OF THEOREM 3.1. We start by showing "(iii)$\Rightarrow$(ii)". By virtue of Theorem 2.1, there exists a C_1-semigroup $\{S_1(t): t\geqq 0\}$ on $\overline{D(A)}$ with $C_1=R(c: A)^n|_{\overline{D(A)}}$ satisfying the following (3.3)-(3.6):

$$(3.3)\qquad S_1(t)x=\lim_{m\to\infty}\left(\frac{m}{t}\right)^m R(m/t: A)^m R(c: A)^n x \qquad \text{for } x\in\overline{D(A)} \text{ and } t\geqq 0\,;$$

$$(3.4)\qquad \|S_1(t)x\|\leqq Me^{at}\|x\| \qquad \text{for } x\in\overline{D(A)} \text{ and } t\geqq 0\,;$$

$$(3.5)\qquad \|S_1(t+h)x-S_1(t)x\|\leqq Me^{|a|(t+h)}h\|Ax\| \qquad \text{for } x\in D(A) \text{ and } t,\ h\geqq 0\,;$$

$$(3.6)\qquad R(\lambda: A)R(c: A)^n x=\int_0^\infty e^{-\lambda t}S_1(t)x\,dt \qquad \text{for } x\in\overline{D(A)} \text{ and } \lambda>a\,.$$

Let us define $S(t)\in B(X)$, $t\geqq 0$, by

$$S(t)x=S_1(t)R(c: A)x \qquad \text{for } x\in X\,.$$

Clearly, $\{S(t): t\geqq 0\}$ satisfies (3.7)-(3.9):

$$(3.7)\qquad \|S(t)\|\leqq M\|R(c: A)\|e^{at} \qquad \text{for } t\geqq 0\,;$$

$$(3.8)\qquad \|S(t+h)-S(t)\|\leqq M\|AR(c: A)\|e^{|a|(t+h)}h \qquad \text{for } t,\ h\geqq 0\,;$$

$$(3.9)\qquad S(t)S(s)=S(t+s)R(c: A)^{n+1} \qquad \text{for } t,\ s\geqq 0 \text{ and } S(0)=R(c: A)^{n+1}\,.$$

Therefore $\{S(t): t\geqq 0\}$ is a C-semigroup on X with $C=R(c: A)^{n+1}$. Now, let Z be the generator of $\{S(t): t\geqq 0\}$. We want to show

$$(3.10)\qquad A\subset Z\,.$$

To this end, let $x\in D(A)$ and $\lambda>a$. Then by the closedness of A

$$\begin{aligned} L_\lambda Ax&=\int_0^\infty e^{-\lambda t}S(t)Ax\,dt=\int_0^\infty e^{-\lambda t}S_1(t)R(c: A)Ax\,dt\\ &=\int_0^\infty e^{-\lambda t}AS_1(t)R(c: A)x\,dt=A\int_0^\infty e^{-\lambda t}S_1(t)R(c: A)x\,dt=AL_\lambda x\,. \end{aligned}$$

Combining this with

$$R(c: A)^{n+1}x=(\lambda-A)\int_0^\infty e^{-\lambda t}S_1(t)R(c: A)x\,dt=(\lambda-A)L_\lambda x \qquad \text{(by (3.6))}\,,$$

we have $Cx\ (=R(c: A)^{n+1}x)=L_\lambda(\lambda-A)x\ (\in R(L_\lambda))$, i.e.,

$$x\in D(Z) \quad\text{and}\quad Zx=(\lambda-L_\lambda^{-1}C)x=Ax\,.$$

Therefore we obtain (3.10). Since $\lambda-Z$ is injective for $\lambda>a$ (by Proposition 1.4 (a_1)), (3.10) and $(a, \infty)\subset\rho(A)$ imply $Z=A$.

Next, to prove "(ii)$\Rightarrow$(i)" let A be the generator of a C-semigroup $\{S(t): t\geqq 0\}$ on X with $C=R(c: A)^{n+1}$ satisfying $\|S(t+h)-S(t)\|\leqq Ke^{\omega(t+h)}h$ for t, $h\geqq 0$. Let us define $V_k(t)$, $k\geqq 0$, by $V_0(t)=S(t)$ and

$$V_k(t)x=\int_0^t\int_0^{t_1}\cdots\int_0^{t_{k-1}}S(t_k)x\,dt_k\cdots dt_2dt_1 \qquad \text{for} \quad x\in X \text{ and } t\geqq 0 .$$

Similarly as in [10, Lemma], for $k=1, 2, \cdots, n+1$ we have the following (3.11)-(3.13):

$$\text{(3.11)} \qquad V_k(t)x\in D(A^k) \quad \text{and} \quad \int_0^t (c-A)^{k-1}V_{k-1}(s)x\,ds\in D(A) \qquad \text{for} \quad x\in X \text{ and } t\geqq 0 ;$$

$$\text{(3.12)} \quad (c-A)^kV_k(t)\in B(X) , \qquad \|(c-A)^kV_k(t)\|\leqq K_ke^{b_kt} \quad \text{and}$$
$$\|(c-A)^kV_k(t+h)-(c-A)^kV_k(t)\|\leqq M_ke^{a_k(t+h)}h \qquad \text{for} \quad t,\ h\geqq 0 ,$$

where K_k, M_k, a_k and b_k are nonnegative constants, and hence $(c-A)^kV_k(\cdot)x: [0, \infty)\to X$ is continuous for $x\in X$;

$$\text{(3.13)} \qquad (c-A)^kV_k(t)=c(c-A)^{k-1}V_k(t)-(c-A)^{k-1}V_{k-1}(t) + \frac{t^{k-1}}{(k-1)!}(c-A)^{k-1}C \qquad \text{for} \quad t\geqq 0 .$$

Now, define $U(t)$, $t\geqq 0$, by $U(t)x=(c-A)^{n+1}V_{n+1}(t)x$ for $x\in X$. Then, by (3.12), $U(t)\in B(X)$ and $\|U(t+h)-U(t)\|\leqq K'e^{\omega'(t+h)}h$ for t, $h\geqq 0$, where $K'=M_{n+1}$ and $\omega'=a_{n+1}$. Clearly, (0.5), (0.7) and (0.8) are satisfied. Similarly as in the proof of [10, (ii)$\Rightarrow$(i) in Theorem 1], we obtain that $(\alpha, \infty)\subset\rho(A)$ and

$$R(\lambda: A)x=\int_0^\infty \lambda^{n+1}e^{-\lambda t}U(t)x\,dt \qquad \text{for} \quad x\in X \text{ and } \lambda>\alpha ,$$

where $\|S(t)\|\leqq M'e^{\alpha t}$ for $t\geqq 0$ and $\alpha>0$. It follows from [1, Theorem 3.1] that $U(t)$, $t\geqq 0$, satisfy (0.6) with n replaced by $n+1$. Thus $\{U(t): t\geqq 0\}$ is an $(n+1)$-times integrated semigroup on X with generator A. (We note here that

$$U(t)x=\int_0^t\int_0^{t_1}\cdots\int_0^{t_n}S(t_{n+1})(c-A)^{n+1}x\,dt_{n+1}\cdots dt_2dt_1$$

for $x\in D(A^{n+1})$ by (1.6).)

Finally, we show "(i)$\Rightarrow$(iii)". Let A be the generator of an $(n+1)$-times integrated semigroup $\{U(t): t\geqq 0\}$ on X satisfying $\|U(t+h)-U(t)\|\leqq K'e^{\omega'(t+h)}h$ for t, $h\geqq 0$. We first note that for $x\in X$ and $x^*\in X^*$, $\langle U(t)x, x^*\rangle$

is differentiable a.e. $t \in [0, \infty)$ and

$$(3.14) \qquad \left|\frac{d}{dt}\langle U(t)x, x^*\rangle\right| \leqq K' e^{\omega' t}\|x\|\,\|x^*\| \qquad \text{for a.e. } t \in [0, \infty)\,.$$

Now, by the definition of the generator, $(\omega', \infty) \subset \rho(A)$ and

$$R(\lambda: A)x = \int_0^\infty \lambda^{n+1} e^{-\lambda t} U(t)x\,dt \qquad \text{for} \quad x \in X \text{ and } \lambda > \omega'\,.$$

Since

$$(\lambda - A)\left(\int_0^\infty e^{-\lambda t} \sum_{k=0}^{n-1} \frac{t^k}{k!} A^k x\,dt + \int_0^\infty \lambda e^{-\lambda t} U(t) A^n x\,dt\right) = x$$

for $x \in D(A^n)$ and $\lambda > |\omega'|$, we obtain

$$R(\lambda: A)x = \int_0^\infty e^{-\lambda t} \sum_{k=0}^{n-1} \frac{t^k}{k!} A^k x\,dt + \int_0^\infty \lambda e^{-\lambda t} U(t) A^n x\,dt$$

for $x \in D(A^n)$ and $\lambda > |\omega'|$. Therefore for $x \in D(A^n)$, $x^* \in X^*$ and $\lambda > |\omega'|$

$$\langle R(\lambda: A)x, x^*\rangle = \int_0^\infty e^{-\lambda t} \sum_{k=0}^{n-1} \frac{t^k}{k!} \langle A^k x, x^*\rangle dt + \int_0^\infty e^{-\lambda t} \frac{d}{dt} \langle U(t) A^n x, x^*\rangle dt\,.$$

Differentiating this $m-1$ times with respect to λ and using (3.14),

$$\begin{aligned}
&(m-1)!\,|\langle R(\lambda: A)^m R(c: A)^n x, x^*\rangle| \\
&\leqq \int_0^\infty t^{m-1} e^{-\lambda t} \left\{\sum_{k=0}^{n-1} \frac{t^k}{k!} \|A^k R(c: A)^n\| + K' e^{\omega' t} \|A^n R(c: A)^n\|\right\} dt \|x\|\,\|x^*\| \\
&\leqq \frac{(m-1)!\,M\|x\|\,\|x^*\|}{(\lambda - a)^m} \quad \text{for } x \in X,\ x^* \in X^*,\ \lambda > a \text{ and } m \geqq 1\,,
\end{aligned}$$

where $M = 2\max\{\|A^k R(c: A)^n\|,\ k = 0, 1, \cdots, (n-1),\ K'\|A^n R(c: A)^n\|\}$ and $a = \max\{1, |\omega'|\}$. Thus (iii) holds good.

Now, we shall prove (c_1)-(c_3). We first note that A_1 is a closed linear operator in the Banach space $\overline{D(A)}$ and

$$(3.15) \qquad \begin{cases} (a, \infty) \subset \rho(A_1) \equiv \{\lambda: (\lambda - A_1)^{-1} \in B(\overline{D(A)})\} \\ (\lambda - A_1)^{-1} = R(\lambda: A)|_{\overline{D(A)}} \qquad \text{for } \lambda > a\,. \end{cases}$$

Let $\{S_1(t): t \geqq 0\}$ be the C_1-semigroup on $\overline{D(A)}$ defined by (3.3). Since $C_1^{-1} A_1 C_1$ is the generator of $\{S_1(t): t \geqq 0\}$ by Theorem 2.1 and $A_1 \subset C_1^{-1} A_1 C_1$, (3.15) implies $A_1 = C_1^{-1} A_1 C_1$. This proves (c_1). (c_2) and (c_3) can be proved by the same way as in the proof of [10, (ii)$\Rightarrow$(i) in Theorem 1]. Q.E.D.

Addendum.

After this paper was submitted for publication, the authors received the following due to R. deLaubenfels:

[d_1] *C*-semigroups and the Cauchy problem, J. Funct. Anal., to appear.

[d_2] Integrated semigroups, *C*-semigroups and the abstract Cauchy problem, preprint.

Theorem 2.4 (b) and Lemma 2.8 in [d_1] show that (1.6) and (1.7) hold true even if $\{S(t): t \geqq 0\}$ does not satisfy (0.3). Proposition 1.1, (1.5) and Proposition 1.4 (a_1) are also obtained in [d_1]. It should be noted that Proposition 1.4 (a_1) does not hold if (0.3) is not assumed. (See [d_1, Example 6.1].)

Let A, c and n be as in Theorem 3.1. It is shown in [d_2, Theorem 2.4] that A is the generator of an $(n+1)$-times integrated semigroup on X if and only if A is the generator of a C-semigroup on X with $C=R(c; A)^{n+1}$.

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