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On the Power Series Coefficients of the Riemann Zeta Function

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§1. Introduction and the main result.

The Laurent expansion of the Riemann zeta function $\zeta(s)$ about the pole can be written in the form, in [2],

$$\zeta(s)=\frac{1}{s-1}+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\gamma_n(s-1)^n \tag{1}$$

with

$$\gamma_n=\lim_{N\to\infty}\left(\sum_{k=1}^{N}\frac{\log^n k}{k}-\frac{\log^{n+1}N}{n+1}\right).$$

Here $\log^0 k$ mean 1 for all k including $k=1$. γ_0 is the well known Euler constant, and, for $n\geqq 1$, γ_n, sometimes called generalized Euler constants, have been studied by many authors ([1], Entry 13; or [3], p. 51). In this paper we shall give an asymptotic expansion of γ_n for arbitrary large n, which yields some interesting results on γ_n. They can be found in [4].

We begin by defining some notations. Let N be a nonnegative integer, and let n be a positive integer. In order to write our theorem, we need two functions $a=a(n)$ and $b=b(n)$ which are given by the following lemma.

LEMMA 1. *If $n>c_1$, where c_1 is a sufficiently large constant, then the system of the equations*

$$-(n+1)\frac{y}{x^2+y^2}+\frac{1}{2}\pi-\operatorname{Im}\psi(x+iy)=0\ , \tag{2}$$

$$-(n+1)\frac{x}{x^2+y^2}-\log 2\pi+\operatorname{Re}\psi(x+iy)=0\ , \tag{3}$$

with unknown x and y, satisfying $0<y<x$ and $n^{1/2}<x<n$, has a unique

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solution $x=a$, $y=b$, where $\psi(y)$ is the logarithmic derivative of the gamma function, i.e., $\psi(z)=\Gamma'(z)/\Gamma(z)$, and $\operatorname{Re} z$ *and* $\operatorname{Im} z$ *mean the real and imaginary parts of z, respectively.*

For given n with $n>c_1$, the pair $x=a$, $y=b$ is uniquely determined by the lemma. Hence these a and b can be considered as the functions of n, so that we denote these new functions by $a=a(n)$ and $b=b(n)$. We next define the functions $\phi(z)$, $g(y)$, $f(y)$ by

$$\phi(z)=-(n+1)\log z-z\log(2\pi i)+\log\Gamma(z)\ ,$$
$$g(y)=\operatorname{Re}\phi(a+iy) \quad \text{and} \quad f(y)=\operatorname{Im}\phi(a+iy)$$

with a real variable y, namely,

$$g(y)=-\frac{1}{2}(n+1)\log(a^2+y^2)-a\log 2\pi+\frac{1}{2}\pi y+\operatorname{Re}\log\Gamma(a+iy)\ ,$$
$$f(y)=-(n+1)\arctan\left(\frac{y}{a}\right)-y\log 2\pi-\frac{1}{2}\pi a+\operatorname{Im}\log\Gamma(a+iy)\ .$$

Moreover, define the sequences h_k, u_k, v_k ($k=0, 1, 2, 3, \cdots$) by

$$\sum_{k=0}^{\infty} h_k(y-b)^k=\exp\left\{\phi(a+iy)-\phi(a+ib)+\frac{1}{2}\phi''(a+ib)(y-b)^2\right\}\ , \tag{4}$$
$$u_k=\operatorname{Re} h_k \quad \text{and} \quad v_k=\operatorname{Im} h_k\ ,$$

then we have the following theorem which states the asymptotic expansion of γ_n.

THEOREM. *Let N be a nonnegative integer. If $n>c_2\Gamma(N/3+17/6)$, where c_2 is a sufficiently large constant, then*

$$\gamma_n=\frac{1}{\pi}n!\ e^{g(b)}\sum_{k=0}^{N}|h_{2k}|2^{k+1/2}\Gamma\left(k+\frac{1}{2}\right)\{g''(b)^2+f''(b)^2\}^{-k/2-1/4}$$
$$\times\cos\left\{f(b)-\left(k+\frac{1}{2}\right)\arctan\left(\frac{f''(b)}{g''(b)}\right)+\arctan\left(\frac{v_{2k}}{u_{2k}}\right)\right\}$$
$$+O\left\{\Gamma\left(N+\frac{3}{2}\right)2^N n!\ e^{g(b)}n^{-N/3+1/6}\log^{2N/3-1/3}n\right\}\ .$$

§2. Proof of Lemma 1 and some other lemmas.

PROOF OF LEMMA 1. The equations (2) and (3) are the imaginary and real parts of the equation

$$\phi'(x+iy) = -\frac{n+1}{x+iy} - \log(2\pi i) + \psi(x+iy) = 0 \ ,$$

respectively. Hence we will show that the equation $z\phi'(z)=0$ with $z=x+iy$ has a unique solution under the assumption of the lemma. Now, we put

$$(5) \qquad h(y) = \operatorname{Im} z\phi'(z) = -y\log 2\pi - \frac{1}{2}\pi x + \operatorname{Im}(x+iy)\psi(x+iy) \ .$$

Then, if we use the asymptotic expansion of $\psi(z)$ which is derived from the asymptotic expansion of $\log\Gamma(z)$ ([6], p. 251), we can prove that, for sufficiently large x, $h(y)$ is steadily increasing in $0 \leqq y \leqq x$, having the values $h(0)<0$ and $h(x)>0$. Therefore the equation $h(y)=0$ has a unique solution, say, y_x, in $0<y<x$. We further set $z_x = x+iy_x$ and define

$$(6) \qquad u(x) = \operatorname{Re} z_x\phi'(z_x) = -(n+1) + x\log 2\pi + \frac{1}{2}\pi y_x + \operatorname{Re}(x+iy_x)\psi(x+iy_x)$$

$$= -n-1+\left(x+\frac{y_x}{x}\right)(-\log 2\pi + \operatorname{Re}\psi(x+iy_x)) \ .$$

Then, using the asymptotic expansion of $\psi(z)$, we can prove that, for sufficiently large n, $u(x)$ is steadily increasing in $n^{1/2} \leqq x \leqq n$, having the values $u(n^{1/2})<0$ and $u(n)>0$. Hence the equation $u(x)=0$ has a unique solution, say a, in $n^{1/2}<x<n$. It follows that $z=a+iy_a$ is the unique solution of $z\phi'(z)=0$. This completes the proof.

LEMMA 2. *For sufficiently large* n,

$$(7) \qquad n\log^{-1}n < a < n\log^{-1}n + 2n\log^{-2}n\log\log n \ ,$$

and

$$(8) \qquad b = \frac{1}{2}\pi n\log^{-2}n + O(n\log^{-3}n\log\log n) \ .$$

PROOF. We have from (5)

$$(9) \qquad -b\log 2\pi - \frac{1}{2}\pi a + \operatorname{Im}(a+ib)\psi(a+ib) = 0 \ .$$

Using (9), and $\psi(z) = \log z - z^{-1}/2 + O(|z|^{-2})$, we have $b/a < 2\log^{-1}a$, and further

$$(10) \qquad \frac{b}{a} = \frac{1}{2}\pi\log^{-1}a + O(\log^{-2}a) \ .$$

It follows from (6) that

$$n=-1+\left(a+\frac{b}{a}\right)(-\log 2\pi+\operatorname{Re}\psi(a+ib))$$

$$=a\log a-a\log 2\pi+O(a\log^{-1}a)\,. \tag{11}$$

Here we suppose that $a\leqq n\log^{-1}n$. Then we have from (11) $n< a\log a\leqq n\log^{-1}n\log(n\log^{-1}n)<n$ for sufficiently large n. This is a contradiction, and hence $a>n\log^{-1}n$. We next suppose that $a\geqq n\log^{-1}n+2a\log^{-2}n\log\log n$. Then, by (11), we have $n>a\log a-2a\geqq n\log^{-1}n(1+2\log^{-1}n\log\log n)\log\{n\log^{-1}n(1+2\log^{-1}n\log\log n)\}-2n\log^{-1}n(1+2\log^{-1}n\times\log\log n)>n$. This is a contradiction, and hence $a<n\log^{-1}n+2\log^{-2}n\times\log\log n$, which completes (7). Moreover, we can obtain (8) from (7) and (10).

LEMMA 3. *$g(y)$ is steadily increasing in $0\leqq y\leqq b$, has the maximum at $y=b$, and is steadily decreasing in $b\leqq y\leqq a$, having the properties $g(b)-g(b-\Delta)>(1/3)\log^3 a$, $g(b)-g(b+\Delta)>(1/3)\log^3 a$, where $\Delta=a^{1/2}\log a$; further, $g''(b)=-a^{-1}\log a+(\log 2\pi-1)a^{-1}+O(a^{-1}\log^{-1}a)$, and $f''(b)=a^{-1}\pi+O(a^{-1}\log^{-1}a)$.*

PROOF. Using the asymptotic expansion of $\log\Gamma(z)$, we have, for $0\leqq y\leqq a$,

$$g''(y)=-\left(n+\frac{3}{2}\right)\frac{a^2-y^2}{(a^2+y^2)^2}-\frac{a}{a^2+y^2}+O(a^{-3})<0\,.$$

It follows that $g'(y)$ is steadily decreasing in $0\leqq y\leqq a$. Moreover, $g'(0)>0$ and $g'(a)<0$ for sufficiently large n, which leads to that $g(y)$ is steadily increasing in $0\leqq y\leqq b$, has the maximum at $y=b$, and is steadily decresing in $b\leqq y\leqq a$, since $g'(b)=0$.

Let $\delta=(\pi/2)\log^{-1}a$. Then $a+ib=a(1+i\delta)(1+O(\delta))$. It follows that

$$\phi''(a+ib)=\left(n+\frac{3}{2}\right)a^{-2}(1+i\delta)^{-2}+a^{-1}(1+i\delta)^{-1}+O(a^{-1}\delta)$$

$$=a^{-1}(\log a-\log 2\pi)(1-2i\delta)+a^{-1}+O(a^{-1}\delta)$$

Hence, we obtain $g''(b)=-a^{-1}\log a+(\log 2\pi-1)a^{-1}+O(a^{-1}\delta)$ and $f''(b)=a^{-1}\pi+O(a^{-1}\delta)$.

Now, if we use Taylor's theorem ([6], p. 96), then we have

$$g(b+\Delta)=g(b)+\frac{1}{2}g''(b)\Delta^2+\frac{1}{6}g'''(\xi)\Delta^3 \tag{12}$$

with $b<\xi<b+\Delta$. Using (12), and

$$g'''(\xi)=-\left(n+\frac{3}{2}\right)\frac{2\xi^5-4a^2\xi^3-6a^4\xi}{(a^2+\xi^2)^4}+\frac{2a\xi}{(a^2+\xi^2)^2}+O(a^{-4}) ,$$

we have

$$g'''(\xi)<12\left(n+\frac{3}{2}\right)a^{-3}+2a^{-2}+O(a^{-4})<12a^{-2}\log a .$$

It follows that

$$g(b+\Delta)-g(b)<-\frac{3}{8}\Delta^2a^{-1}\log a+2\Delta^3a^{-2}\log a<-\frac{1}{3}\log^3 a ,$$

since $g''(b)<-(3/4)a^{-1}\log a$. In case $g(b)-g(b-\Delta)$, we have the same result as well, which completes the proof.

LEMMA 4. *For* $j\geqq 0$, $|h_j|\leqq(a^{-1}\log a)^{2j/3}$.

PROOF. We first prove that, if p_j $(j=0, 1, 2, 3, \cdots)$ is the power series coefficients defined by $\phi(a+iy)=\sum_{j=0}^{\infty} p_j(y-b)^j$, then, for $j\geqq 2$,

$$|p_j|\leqq\frac{1}{j}(a^{-1}\log a)^{j-1} . \tag{13}$$

In case $j\geqq 3$, differentiating $\phi(a+iy)$ two times by y, we have

$$j(j-1)p_j=-\frac{1}{2\pi i}\int_C\phi''(a+iw)(w-b)^{-j+1}dw ,$$

where C is the circle with center b and radius $a/2$ in the positive direction. We have, on C, $\min|a+iw|=a/2+O(a\log^{-2}a)$, and $|\phi''(a+iw)|<5a^{-1}\log a$. Therefore, for $j\geqq 3$,

$$|p_j|<\frac{1}{j}(a^{-1}\log a)^{j-1}\frac{5}{j-1}(2\log^{-1}a)^{j-2}<\frac{1}{j}(a^{-1}\log a)^{j-1} .$$

In case $j=2$, we have, by Lemma 3, $|p_2|=(1/2)|g''(b)+if''(b)|<(1/2)a^{-1}\log a$, which implies (13) for all $j\geqq 2$.

We next prove the inequality in the lemma. If we differentiate (4), write the power series expansion with the variable $y-b$, and equate the terms of order 0, 1, 2, 3, $\cdots$ of both sides, then we have the system of the following infinite set of equations;

$$h_j=0 \qquad (j=1, 2) ,$$

$$jh_j=\sum_{k=2}^{j-1}(k+1)p_{k+1}h_{j-1-k} \qquad (j=3, 4, 5, 6, \cdots) .$$

In order to prove the lemma, we use the mathematical induction with respect to j. For $j=0, 1, 2$, the inequality holds, since $h_0=1$, $h_1=0$, and $h_2=0$. We thus suppose that the inequality holds for $j=0, 1, 2, 3, \cdots, J-1$. Then we have from (13)

$$|Jh_J| = \left|\sum_{k=2}^{J-1}(k+1)p_{k+1}h_{J-1-k}\right| \leqq \sum_{k=2}^{J-1}(a^{-1}\log a)^k(a^{-1}\log a)^{2(J-1-k)/3}$$
$$<\{1-(a^{-1}\log a)^{1/3}\}^{-1}(a^{-1}\log a)^{2J/3} < J(a^{-1}\log a)^{2J/3} ,$$

since $\{1-(a^{-1}\log a)^{1/3}\}^{-1}<3$ for sufficiently large a, which implies that the inequality holds for $j=J$. This completes the proof.

§ 3. Proof of Theorem.

In this section we use Vinogradov's symbol $\ll$ ([3], p. XVI). A key idea in our proof is the fact that we can apply the saddle point method to our integral expression of γ_n. Here, we shall prove that $\gamma_n=I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8+J_9$, I's are the error terms, and J_9 is the main term.

We first show

$$\text{(14)}\qquad \gamma_n=\frac{1}{\pi}n!\ \mathrm{Re}\int_0^\infty (a+iy)^{-n-1}(2\pi)^{-a-iy}e^{-\pi y/2+\pi ia/2}\Gamma(a+iy)\zeta(a+iy)dy$$
$$+\frac{1}{\pi}n!\ \mathrm{Re}\int_0^\infty (a+iy)^{-n-1}(2\pi)^{-a-iy}e^{\pi y/2-\pi ia/2}\Gamma(a+iy)\zeta(a+iy)dy$$
$$=I_1+J_2 ,$$

say. We have from (1)

$$\gamma_n=\frac{n!}{2\pi i}\int_C z^{-n-1}\zeta(1-z)dz ,$$

where C is a contour enclosing 0. Using the order of $\zeta(s)$ ([5], pp. 81-82), we can deform C into the line from $a-i\infty$ to $a+i\infty$. Moreover, using the functional equation of $\zeta(s)$ ([5], p. 13), and changing the variable z to y with $z=a+iy$, we have

$$\gamma_n=\frac{n!}{2\pi}\int_{-\infty}^\infty (a+iy)^{-n-1}2(2\pi)^{-a-iy}\cos\left\{\frac{1}{2}\pi(a+iy)\right\}\Gamma(a+iy)\zeta(a+iy)dy .$$

We now divide the integral into two parts; $\int_{-\infty}^0+\int_0^\infty$. If we notice that the first intergal is the complex conjugate of the second, we obtain (14), using $2\cos\{\pi(a+iy)/2\}=\exp(-\pi y/2+\pi ia/2)+\exp(\pi y/2-\pi ia/2)$.

We next prove

(15) $$I_1 \ll n!\, e^{g(b)} n^{-N/3-1/3} \log^{N/3+1/3} n\ .$$

Using asymptotic expansion of $\log \Gamma(a+iy)$, we have

$$I_1 \ll n! \int_0^\infty (a^2+y^2)^{-(n-a+3/2)/2}(2\pi e)^{-a} e^{-\pi y/2 - y\arctan(y/a)}dy$$

$$\ll n!\, \exp\{-(n-a+1)\log a - a(\log 2\pi+1)\} \ll n!\, \exp\{g(0)\}\ ,$$

since $g(0) = -(n-a+3/2)\log a - a(\log 2\pi+1)+O(1)$. It follows from Lemma 3 that

$$I_1 \ll n!\, \exp\{g(b-\Delta)\} \ll n!\, \exp\left\{g(b) - \frac{1}{3}\log^3 a\right\}\ ,$$

where $\Delta = a^{1/2}\log a$. By Lemma 1 and Lemma 2, we can prove $\log^2 a < N+4$, and hence we obtain (15).

We next divide J_2 into two parts;

$$J_2 = \frac{1}{\pi} n!\, \mathrm{Re}\left(\int_0^a + \int_a^\infty\right) = J_3 + I_2\ ,$$

say. We have by Lemma 2 and Lemma 3

$$I_2 \ll n!\, (2\pi e)^{-a} \int_a^\infty (a^2+y^2)^{-(n-a+3/2)/2} e^{\pi y/2 - y\arctan(y/a)}dy$$

(16) $$\ll n!\, e^{g(b)} n^{-N/3-4/3} \log^{N/3+4/3} n\ .$$

We next divide J_3 into two parts;

$$J_3 = \frac{1}{\pi} n!\, \mathrm{Re}\left\{\int_0^a e^{\phi(a+iy)}dy + \int_0^a e^{\phi(a+iy)}(\zeta(a+iy)-1)dy\right\} = J_4 + I_3\ ,$$

say. If we use the estimate $\zeta(a+iy)-1 = O(2^{-a})$, then we obtain

(17) $$I_3 \ll n!\, e^{g(b)} n \log^{-1} n \exp(-n \log^{-1} n \log 2)\ .$$

We further divide J_4 into three parts;

$$J_4 = \frac{1}{\pi} n!\, \mathrm{Re}\left(\int_0^{b-\Delta} + \int_{b-\Delta}^{b+\Delta} + \int_{b+\Delta}^a\right) = I_4 + J_6 + I_5\ ,$$

say. We then have by Lemma 3

$$\int_0^{b-\Delta} e^{\phi(a+iy)}dy \leq \int_0^{b-\Delta} e^{g(y)}dy < a e^{g(b-\Delta)}\ ,$$

so that

$$I_4 \ll n!\, e^{g(b)} n^{-N/3-1/3} \log^{N/3-1/3} n \,. \tag{18}$$

We have also

$$I_5 \ll n!\, e^{g(b)} n^{-N/3-1/3} \log^{N/3-1/3} n \,. \tag{19}$$

We next consider J_6. We can write

$$J_6 = \frac{1}{\pi} n!\, \mathrm{Re} \int_{b-\Delta}^{b+\Delta} \exp\left\{\phi(a+ib) - \frac{1}{2}\phi''(a+ib)(y-b)^2\right\}$$
$$\times \exp\left\{\phi(a+iy) - \phi(a+ib) + \frac{1}{2}\phi''(a+ib)(y-b)^2\right\} dy \,.$$

We know that $\phi(a+iy)$ has the nearest singularity at $y=ia$, so that the power series $\sum_{k=0}^{\infty} h_k(y-b)^k$ defined by (4) converges in $b-\Delta \leqq y \leqq b+\Delta$. We now divide the power series into two parts;

$$\sum_{k=0}^{\infty} h_k(y-b)^k = \sum_{k=0}^{2N+1} h_k(y-b)^k + U_N$$

with

$$U_N = \sum_{k=2N+2}^{\infty} h_k(y-b)^k \,.$$

We thus have

$$J_6 = \frac{1}{\pi} n!\, \mathrm{Re} \int_{b-\Delta}^{b+\Delta} \exp\left\{\phi(a+ib) - \frac{1}{2}\phi''(a+ib)(y-b)^2\right\} \sum_{k=0}^{2N+1} h_k(y-b)^k dy$$
$$+ \frac{1}{\pi} n!\, \mathrm{Re} \int_{b-\Delta}^{b+\Delta} \exp\left\{\phi(a+ib) - \frac{1}{2}\phi''(a+ib)(y-b)^2\right\} U_N dy = J_7 + I_6 \,,$$

say. We are going to estimate I_6. We have by Lemma 4

$$|U_N| \leqq \sum_{k=2N+2}^{\infty} |h_k|\, |y-b|^k < \sum_{k=2N+2}^{\infty} \{(a^{-1}\log a)^{2/3}\Delta\}^k < 2\{(a^{-1}\log a)^{2/3}\Delta\}^{2N+2} \,,$$

since $(a^{-1}\log a)^{2/3}\Delta < 1/2$. We have further, for $b-\Delta \leqq y \leqq b+\Delta$, $(1/2)g''(b) \times (y-b)^2 < 0$, so that

$$I_6 \ll n!\, \Delta e^{g(b)} \{(a^{-1}\log a)^{2/3}\Delta\}^{2N+2} \,.$$

Hence we get

$$I_6 \ll n!\, e^{g(b)} n^{-N/3+1/6} \log^{11N/3+25/6} n \,. \tag{20}$$

We next divide J_7 into three parts;

$$J_7=\frac{1}{\pi}n!\ \mathrm{Re}\sum_{k=0}^{2N+1}h_k\exp\{\phi(a+ib)\}\Big(\int_{-\infty}^{\infty}-\int_{b+\varDelta}^{\infty}-\int_{-\infty}^{b-\varDelta}\Big)=J_9+I_7+I_8\ ,$$

say. We are going to estimate I_7. For all k, the integrals in I_7 are estimated by

$$\ll\varDelta^{k+1}\int_1^{\infty}\exp\Big(-\frac{1}{3}\log^3 ay^2\Big)y^k dy\ ,$$

since $g''(b)<-(1/3)a^{-1}\log a$. We can now write the integrand as

$$\exp\Big(-\frac{1}{6}\log^3 ay^2\Big)\times\exp\Big(-\frac{1}{6}\log^3 ay^2\Big)y^k\ .$$

We see that the second term has the maximum at $y=(3k\log^{-1}a)^{1/2}<1$. Hence the integrals are estimated by

$$\begin{aligned}&\ll\varDelta^{k+1}\exp\Big(-\frac{1}{6}\log^3 a\Big)\int_1^{\infty}\exp\Big(-\frac{1}{6}\log^3 ay^2\Big)dy\\&<\varDelta^{k+1}\exp\Big(-\frac{1}{6}\log^3 a\Big)\int_1^{\infty}y\exp\Big(-\frac{1}{6}\log^3 ay^2\Big)dy\\&=3\varDelta^{k+1}\exp\Big(-\frac{1}{6}\log^3 a\Big)\log^{-3}a\ .\end{aligned}$$

It follows that

(21) $$I_7\ll n!\ e^{g(b)}n^{-N/3-5/6}\log^{N/3-7/6}n\ .$$

We have also

(22) $$I_8\ll n!\ e^{g(b)}n^{-N/3-5/6}\log^{N/3-7/6}n\ .$$

We finally consider J_9. If we notice that the integrals in J_9 are zeros for all odd k, we get

$$J_9=\frac{1}{\pi}n!\ \mathrm{Re}\sum_{k=0}^{N}h_{2k}e^{\phi(a+ib)}\int_{-\infty}^{\infty}\exp\Big\{-\frac{1}{2}\phi''(a+ib)(y-b)^2\Big\}(y-b)^{2k}dy\ .$$

Here we change the variable y to x with $x=(y-b)^2$ in the integral, and recall the Euler's expression of $\Gamma(z)$ ([6], p. 241). Then we see that the integral is expressed by

$$\Gamma\Big(k+\frac{1}{2}\Big)2^{k+1/2}\{g''(b)^2+f''(b)^2\}^{-k/2-1/4}\exp\Big\{-i\Big(k+\frac{1}{2}\Big)\arctan\Big(\frac{f''(b)}{g''(b)}\Big)\Big\}\ .$$

Hence we finally obtain

$$(23)\qquad J_9=\frac{1}{\pi}n!\,e^{g(b)}\sum_{k=0}^{N}|h_{2k}|2^{k+1/2}\Gamma\Big(k+\frac{1}{2}\Big)\{g''(b)^2+f''(b)^2\}^{-k/2-1/4}$$
$$\times\cos\Big\{f(b)-\Big(k+\frac{1}{2}\Big)\arctan\Big(\frac{f''(b)}{g''(b)}\Big)+\arctan\Big(\frac{v_{2k}}{u_{2k}}\Big)\Big\}\,.$$

If we take account of the fact that $\gamma_n=I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8+J_9$, we get the formula from (15), (16), (17), (18), (19), (20), (21), (22), and (23). Although the O-term in the theorem is sharper than what we get, we can obtain the theorem, taking $N+1$ instead of N, and estimating the last term in the sum, and the O-term. This completes the proof.

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