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Automorphisms, Diffeomorphisms and Strong Morita Equivalence of Irrational Rotation C^*-Algebras

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Abstract. Let A_θ be an irrational rotation C^*-algebra by θ and $\mathrm{Aut}(A_\theta)$ (resp. $\mathrm{Diff}(A_\theta)$) be the group of all automorphisms (resp. diffeomorphisms) of A_θ. Let $\mathrm{Int}(A_\theta)$ be the normal subgroup of $\mathrm{Aut}(A_\theta)$ of inner automorphisms of A_θ and let $\mathrm{Int}^\infty(A_\theta)=\mathrm{Int}(A_\theta)\cap\mathrm{Diff}(A_\theta)$. Let A_η be an irrational rotation C^*-algebra by η which is strongly Morita equivalent to A_θ. In the present paper we will show that $\mathrm{Aut}(A_\theta)/\mathrm{Int}(A_\theta)$ (resp. $\mathrm{Diff}(A_\theta)/\mathrm{Int}^\infty(A_\theta)$) is isomorphic to $\mathrm{Aut}(A_\eta)/\mathrm{Int}(A_\eta)$ (resp. $\mathrm{Diff}(A_\eta)/\mathrm{Int}^\infty(A_\eta)$) and that if A_η has a diffeomorphism of non Elliott type, so does A_θ.

§1. Preliminaries.

Let A_θ be an irrational rotation C^*-algebra by a rotation θ and $\mathrm{Aut}(A_\theta)$ be the group of all automorphisms of A_θ. Let $\mathrm{Int}(A_\theta)$ be the normal subgroup of $\mathrm{Aut}(A_\theta)$ of all inner automorphisms of A_θ. Let A_θ^∞ be the dense $*$-subalgebra of smooth elements of A_θ with respect to the canonical action of the two dimensional torus.

DEFINITION. Let $\alpha\in\mathrm{Aut}(A_\theta)$. We say that *$\alpha$ is a diffeomorphism of A_θ* if $\alpha(A_\theta^\infty)=A_\theta^\infty$. We denote by $\mathrm{Diff}(A_\theta)$ the group of all diffeomorphisms of A_θ. Let $\mathrm{Int}^\infty(A_\theta)=\mathrm{Int}(A_\theta)\cap\mathrm{Diff}(A_\theta)$.

Let A_η be an irrational rotation C^*-algebra by a rotation η. Rieffel [6] showed that A_θ and A_η are strongly Morita equivalent if and only if there is a $g=\begin{bmatrix} a & b \\ c & d\end{bmatrix}\in GL(2, \boldsymbol{Z})$ such that $\eta=\dfrac{a\theta+b}{c\theta+d}$ where $GL(2, \boldsymbol{Z})$ is the group of 2×2 matrices over $\boldsymbol{Z}$ with determinant 1 or -1. Throughout the present paper we suppose that $\eta=\dfrac{a\theta+b}{c\theta+d}$ where $g=\begin{bmatrix} a & b \\ c & d\end{bmatrix}\in GL(2, \boldsymbol{Z})$. For any $n\in\boldsymbol{N}$ let M_n be the $n\times n$ matrix algebra over $\boldsymbol{C}$ and $M_n(A_\theta)$ (resp. $M_n(A_\theta^\infty)$) be the $n\times n$ matrix algebra over A_θ (resp. A_θ^∞). We identify $M_n(A_\theta)$ with $A_\theta\otimes M_n$.

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§ 2. Homomorphisms of $\mathrm{Aut}(A_\eta)/\mathrm{Int}(A_\eta)$ to $\mathrm{Aut}(A_\theta)/\mathrm{Int}(A_\theta)$.

By Rieffel [6, Proposition 2.1] there are a positive integer n and a projection $p \in M_n(A_\eta^\infty)$ such that A_θ is isomorphic to $pM_n(A_\eta)p$. Let Ψ be an isomorphism of A_θ onto $pM_n(A_\eta)p$ with $\Psi(A_\theta^\infty)=pM_n(A_\eta^\infty)p$. Such an isomorphism is constructed in Rieffel [6, Proposition 2.1]. For any $\alpha \in \mathrm{Aut}(A_\eta)$, $[(\alpha\otimes\mathrm{id}_{M_n})(p)]=[p]$ in $K_0(A_\eta)$. Then we can find a unitary element $w \in M_n(A_\eta)$ such that $(\alpha\otimes\mathrm{id}_{M_n})(p)=wpw^*$ by Rieffel [7, 2.5. Corollary]. Hence $(\mathrm{Ad}(w^*)\circ\alpha\otimes\mathrm{id}_{M_n})|_{pM_n(A_\eta)p}$ is in $\mathrm{Aut}(pM_n(A_\eta)p)$. Therefore we obtain an automorphism $\Psi^{-1}\circ\mathrm{Ad}(w^*)\circ\alpha\otimes\mathrm{id}_{M_n}\circ\Psi$ of A_θ. And if $\alpha \in \mathrm{Diff}(A_\eta)$, we can find a unitary element $w \in M_n(A_\eta^\infty)$ such that $(\alpha\otimes\mathrm{id}_{M_n})(p)=wpw^*$. Thus we obtain a diffeomorphism $\Psi^{-1}\circ\mathrm{Ad}(w^*)\circ\alpha\otimes\mathrm{id}_{M_n}\circ\Psi$ of A_θ.

LEMMA 1. *Let p and Ψ be as above. Let $\alpha \in \mathrm{Aut}(A_\eta)$. Let w_j $(j=1, 2)$ be unitary elements in $M_n(A_\eta)$ such that $\beta_j=\Psi^{-1}\circ\mathrm{Ad}(w_j^*)\circ\alpha\otimes\mathrm{id}_{M_n}\circ\Psi$ $(j=1, 2)$ are automorphisms of A_θ. Then $\beta_1^{-1}\circ\beta_2 \in \mathrm{Int}(A_\theta)$.*

PROOF.

$$\begin{aligned}\beta_1^{-1}\circ\beta_2&=\Psi^{-1}\circ(\alpha\otimes\mathrm{id}_{M_n})^{-1}\circ\mathrm{Ad}(w_1w_2^*)\circ\alpha\otimes\mathrm{id}_{M_n}\circ\Psi\\&=\Psi^{-1}\circ\mathrm{Ad}((\alpha\otimes\mathrm{id}_{M_n})^{-1}(w_1w_2^*))\circ\Psi\ .\end{aligned}$$

On the other hand clearly $(\mathrm{Ad}(w_j^*)\circ(\alpha\otimes\mathrm{id}_{M_n}))|_{pM_n(A_\eta)p}$ is an automorphism of $pM_n(A_\eta)p$. Thus $(\alpha\otimes\mathrm{id}_{M_n})(p)=w_jpw_j^*$. Hence

$$p=(\alpha\otimes\mathrm{id}_{M_n})^{-1}(w_j)(\alpha\otimes\mathrm{id}_{M_n})^{-1}(p)(\alpha\otimes\mathrm{id}_{M_n})^{-1}(w_j^*)\ .$$

Therefore we get

$$(\alpha\otimes\mathrm{id}_{M_n})^{-1}(p)=(\alpha\otimes\mathrm{id}_{M_n})^{-1}(w_j^*)p(\alpha\otimes\mathrm{id}_{M_n})^{-1}(w_j)$$

for $j=1, 2$. By the above equations we obtain that

$$(\alpha\otimes\mathrm{id}_{M_n})^{-1}(w_1^*)p(\alpha\otimes\mathrm{id}_{M_n})^{-1}(w_1)=(\alpha\otimes\mathrm{id}_{M_n})^{-1}(w_2^*)p(\alpha\otimes\mathrm{id}_{M_n})^{-1}(w_2)\ .$$

Thus

$$(\alpha\otimes\mathrm{id}_{M_n})^{-1}(w_1w_2^*)p=p(\alpha\otimes\mathrm{id}_{M_n})(w_1w_2^*)\ .$$

Therefore

$$p(\alpha\otimes\mathrm{id}_{M_n})^{-1}(w_1w_2^*)p=(\alpha\otimes\mathrm{id}_{M_n})^{-1}(w_1w_2^*)p \in pM_n(A_\eta)p$$

and

$$p(\alpha\otimes\mathrm{id}_{M_n})^{-1}(w_2w_1^*) \in pM_n(A_\eta)p\ .$$

Hence for any $x \in A_\theta$

$$(\beta_1^{-1}\circ\beta_2)(x) = \Psi^{-1}((\alpha\otimes \mathrm{id}_{M_n})^{-1}(w_1w_2^*)\Psi(x)(\alpha\otimes \mathrm{id}_{M_n})^{-1}(w_2w_1^*))$$
$$= \Psi^{-1}((\alpha\otimes \mathrm{id}_{M_n})^{-1}(w_1w_2^*)p\Psi(x)p(\alpha\otimes \mathrm{id}_{M_n})^{-1}(w_2w_1^*))$$

since $\Psi(x)\in pM_n(A_\eta)p$. Since $(\alpha\otimes \mathrm{id}_{M_n})^{-1}(w_1w_2^*)p$ and $p(\alpha\otimes \mathrm{id}_{M_n})^{-1}(w_2w_1^*)$ are in $pM_n(A_\eta)p$, $\Psi^{-1}((\alpha\otimes \mathrm{id}_{M_n})^{-1}(w_1w_2^*)p)$ and $\Psi^{-1}(p(\alpha\otimes \mathrm{id}_{M_n})^{-1}(w_2w_1^*))$ are in A_θ. Hence

$$(\beta_1^{-1}\circ\beta_2)(x) = \Psi^{-1}((\alpha\otimes \mathrm{id}_{M_n})^{-1}(w_1w_2^*)p)x\Psi^{-1}(p(\alpha\otimes \mathrm{id}_{M_n})^{-1}(w_2w_1^*))$$

for any $x\in A_\theta$. Furthermore

$$\Psi^{-1}((\alpha\otimes \mathrm{id}_{M_n})^{-1}(w_1w_2^*)p)\Psi^{-1}((\alpha\otimes \mathrm{id}_{M_n})^{-1}(w_1w_2^*)p)^*$$
$$= \Psi^{-1}((\alpha\otimes \mathrm{id}_{M_n})^{-1}(w_1w_2^*)p(\alpha\otimes \mathrm{id}_{M_n})^{-1}(w_2w_1^*))$$
$$= \Psi^{-1}(p(\alpha\otimes \mathrm{id}_{M_n})^{-1}(w_1w_2^*w_2w_1^*))$$
$$= \Psi^{-1}(p)$$
$$= 1\,.$$

Similarly we obtain that $\Psi^{-1}((\alpha\otimes \mathrm{id}_{M_n})^{-1}(w_1w_2^*)p)^*\Psi^{-1}((\alpha\otimes \mathrm{id}_{M_n})^{-1}(w_1w_2^*)p) = 1$. Thus $\Psi^{-1}((\alpha\otimes \mathrm{id}_{M_n})^{-1}(w_1w_2^*)p)$ is a unitary element in A_θ. Hence $\beta_1^{-1}\circ\beta_2\in \mathrm{Int}(A_\theta)$. Q.E.D.

We will define the homomorphism $T_{\eta,\theta}(\Psi)$ of $\mathrm{Aut}(A_\eta)/\mathrm{Int}(A_\eta)$ to $\mathrm{Aut}(A_\theta)/\mathrm{Int}(A_\theta)$ as follows: For any $\alpha\in \mathrm{Aut}(A_\eta)$

$$T_{\eta,\theta}(\Psi)([\alpha]) = [\Psi^{-1}\circ \mathrm{Ad}(w^*)\circ(\alpha\otimes \mathrm{id}_{M_n})\circ\Psi]$$

where $[\alpha]$ denotes the class of α in $\mathrm{Aut}(A_\eta)/\mathrm{Int}(A_\eta)$ and w is a unitary element in $M_n(A_\eta)$ such that $(\alpha\otimes \mathrm{id}_{M_n})(p) = wpw^*$. By Lemma 1 we can see easily that $T_{\eta,\theta}(\Psi)$ is well defined. And if $\alpha\in \mathrm{Diff}(A_\eta)$, $\Psi^{-1}\circ \mathrm{Ad}(w^*)\circ \alpha\otimes \mathrm{id}_{M_n}\circ\Psi\in \mathrm{Diff}(A_\theta)$ where w is a unitary element in $M_n(A_\eta^\infty)$. Hence we can define in the same way the homomorphism $T_{\eta,\theta}(\Psi)$ of $\mathrm{Diff}(A_\eta)/\mathrm{Int}^\infty(A_\eta)$ to $\mathrm{Diff}(A_\theta)/\mathrm{Int}^\infty(A_\theta)$.

§3. An isomorphism of $\mathrm{Aut}(A_\eta)/\mathrm{Int}(A_\eta)$ onto $\mathrm{Aut}(A_\theta)/\mathrm{Int}(A_\theta)$.

Let m be a positive integer and q be a projection in $M_m(A_\theta^\infty)$ such that A_η is isomorphic to $qM_m(A_\theta)q$. Let Φ be an isomorphism of A_η onto $qM_m(A_\theta)q$ with $\Phi(A_\eta^\infty) = qM_m(A_\theta^\infty)q$. Hence we obtain an isomorphism $\Phi\otimes \mathrm{id}_{M_n}\circ\Psi$ of A_θ onto $(\Phi\otimes \mathrm{id}_{M_n})(p)M_{mn}(A_\theta)(\Phi\otimes \mathrm{id}_{M_n})(p)$ with $(\Phi\otimes \mathrm{id}_{M_n}\circ\Psi)(A_\theta^\infty) = (\Phi\otimes \mathrm{id}_{M_n})(p)M_{mn}(A_\theta^\infty)(\Phi\otimes \mathrm{id}_{M_n})(p)$ where we identify $M_m(A_\theta)\otimes M_n$ with $M_{mn}(A_\theta)$. Let τ_θ (resp. τ_η) be the unique tracial state on A_θ (resp. A_η)

or the non normalized trace on $M_m(A_\theta)$ (resp. $M_n(A_\eta)$) induced by the unique tracial state on A_θ (resp. A_η).

LEMMA 2. *With the above notations*

$$\tau_\theta((\Phi\otimes\mathrm{id}_{M_n}\circ\Psi)(1))=\tau_\theta((\Phi\otimes\mathrm{id}_{M_n})(p))=1\ .$$

PROOF. Since A_θ and A_η are strongly Morita equivalent, there is an $A_\eta-A_\theta$-equivalence bimodule (i.e., imprimitivity bimodule) X. Let $\langle\ ,\ \rangle_{A_\theta}$ and $\langle\ ,\ \rangle_{A_\eta}$ be the A_θ and A_η-valued inner products on X respectively. By Rieffel [6, Proposition 2.1] there are $2n$ elements $\xi_1,\cdots,\xi_n,\zeta_1,\cdots,\zeta_n\in X$ such that

$$\sum_{k=1}^n\langle\xi_k,\zeta_k\rangle_{A_\theta}=1\ .$$

We consider X^n as an $M_n(A_\eta)-A_\theta$-equivalence bimodule in the trivial way. Let $\xi=\{\xi_k\}_{k=1}^n$ and $\zeta=\{\zeta_k\}_{k=1}^n$ in X^n. Then $\langle\xi,\zeta\rangle_{A_\theta}=1$. Let $\bar{\xi}=\langle\zeta,\zeta\rangle_{M_n(A_\eta)}^{1/2}\xi$ where $\langle\ ,\ \rangle_{M_n(A_\eta)}$ is the $M_n(A_\eta)$-valued inner product on X^n. By Rieffel [6, Proposition 2.1] there is the automorphism $\alpha_\theta\in\mathrm{Aut}(A_\theta)$ such that

$$\Psi(x)=\langle\bar{\xi}\alpha_\theta(x),\bar{\xi}\rangle_{M_n(A_\eta)}$$

for any $x\in A_\theta$. Hence $p=\Psi(1)=\langle\bar{\xi},\bar{\xi}\rangle_{M_n(A_\eta)}$. By Rieffel [6, Proposition 2.1] there are $2m$ elements $\mu_1,\cdots,\mu_m,\nu_1,\cdots,\nu_m\in X$ such that

$$\sum_{j=1}^m\langle\mu_j,\nu_j\rangle_{A_\eta}=1\ .$$

We consider X^m as an $A_\eta-M_m(A_\theta)$-equivalence bimodule in the trivial way. Let $\mu=\{\mu_j\}_{j=1}^m$ and $\nu=\{\nu_j\}_{j=1}^m$ in X^m. Then $\langle\mu,\nu\rangle_{A_\eta}=1$. Let $\bar{\mu}=\mu\langle\nu,\nu\rangle_{M_m(A_\theta)}^{1/2}$ where $\langle\ ,\ \rangle_{M_m(A_\theta)}$ is the $M_m(A_\theta)$-valued inner product on X^m. By Rieffel [6, Proposition 2.1] there is the automorphism $\alpha_\eta\in\mathrm{Aut}(A_\eta)$ such that

$$\Phi(x)=\langle\alpha_\eta(x)\bar{\mu},\bar{\mu}\rangle_{M_m(A_\theta)}$$

for any $x\in A_\eta$. Let $p=\sum_{k,l=1}^n p_{kl}\otimes e_{kl}$ where $\{e_{kl}\}_{k,l=1}^n$ are matrix units of M_n and $p_{kl}\in A_\eta$. Then

$$(\Phi\otimes\mathrm{id}_{M_n})(p)=\sum_{k,l=1}^n\langle\alpha_\eta(p_{kl})\bar{\mu},\bar{\mu}\rangle_{M_m(A_\theta)}\otimes e_{kl}\ .$$

Hence

$$\begin{aligned}\tau_\theta((\Phi\otimes\mathrm{id}_{M_n})(p))&=\sum_{k=1}^n\tau_\theta(\langle\alpha_\eta(p_{kk})\bar{\mu},\bar{\mu}\rangle_{M_m(A_\theta)})\\&=\sum_{k=1}^n\tau_\theta(\langle\alpha_\eta(p_{kk})\mu\langle\nu,\nu\rangle_{M_m(A_\theta)}^{1/2},\mu\langle\nu,\nu\rangle_{M_m(A_\theta)}^{1/2}\rangle_{M_m(A_\theta)})\end{aligned}$$

$$=\sum_{k=1}^{n}\tau_\theta(\langle\nu,\nu\rangle^{1/2}_{M_m(A_\theta)}\langle\alpha_\eta(p_{kk})\mu,\mu\rangle_{M_m(A_\theta)}\langle\nu,\nu\rangle^{1/2}_{M_m(A_\theta)})$$
$$=\sum_{k=1}^{n}\tau_\theta(\langle\alpha_\eta(p_{kk})\mu,\mu\rangle_{M_m(A_\theta)}\langle\nu,\nu\rangle_{M_m(A_\theta)})$$
$$=\sum_{k=1}^{n}\tau_\theta(\langle\alpha_\eta(p_{kk})\mu,\mu\langle\nu,\nu\rangle_{M_m(A_\theta)}\rangle_{M_m(A_\theta)})$$
$$=\sum_{k=1}^{n}\tau_\theta(\langle\alpha_\eta(p_{kk})\mu,\langle\mu,\nu\rangle_{A_\eta}\nu\rangle_{M_m(A_\theta)})$$
$$=\sum_{k=1}^{n}\tau_\theta(\langle\alpha_\eta\langle p_{kk})\mu,\nu\rangle_{M_m(A_\theta)})$$
$$=\sum_{k=1}^{n}\sum_{j=1}^{m}\tau_\theta(\langle\alpha_\eta(p_{kk})\mu_j,\nu_j\rangle_{A_\theta}) .$$

Furthermore by Rieffel [7, Proof of Theorem 1.4] we have the following equation;

$$\tau_\theta(\langle\xi,\zeta\rangle_{A_\theta})=|c\theta+d|\tau_\eta(\langle\zeta,\xi\rangle_{A_\eta})$$

for any $\xi,\zeta\in X$ since $\eta=\dfrac{a\theta+b}{c\theta+d}$, $\begin{bmatrix}a & b\\ c & d\end{bmatrix}\in GL(2,\boldsymbol{Z})$. Thus we obtain that

$$\begin{aligned}\tau_\theta((\Phi\otimes\mathrm{id}_{M_n})(p))&=\sum_{k=1}^{n}\sum_{j=1}^{m}|c\theta+d|\tau_\eta(\langle\nu_j,\alpha_\eta(p_{kk})\mu_j\rangle_{A_\eta})\\
&=\sum_{k=1}^{n}\sum_{j=1}^{m}|c\theta+d|\tau_\eta(\langle\nu_j,\mu_j\rangle_{A_\eta}\alpha_\eta(p_{kk}))\\
&=\sum_{k=1}^{n}|c\theta+d|\tau_\eta\Big(\sum_{j=1}^{m}\langle\nu_j,\mu_j\rangle_{A_\eta}\alpha_\eta(p_{kk})\Big)\\
&=\sum_{k=1}^{n}|c\theta+d|\tau_\eta(\alpha_\eta(p_{kk}))\\
&=|c\theta+d|\tau_\eta(p) .\end{aligned}$$

On the other hand

$$\begin{aligned}\tau_\eta(p)&=\tau_\eta(\langle\bar\xi,\bar\xi\rangle_{M_n(A_\eta)})\\
&=\tau_\eta(\langle\langle\zeta,\zeta\rangle^{1/2}_{M_n(A_\eta)}\xi,\langle\zeta,\zeta\rangle^{1/2}_{M_n(A_\eta)}\xi\rangle_{M_n(A_\eta)})\\
&=\tau_\eta(\langle\zeta,\zeta\rangle^{1/2}_{M_n(A_\eta)}\langle\xi,\xi\rangle_{M_n(A_\eta)}\langle\zeta,\zeta\rangle^{1/2}_{M_n(A_\eta)})\\
&=\tau_\eta(\langle\zeta,\zeta\rangle_{M_n(A_\eta)}\langle\xi,\xi\rangle_{M_n(A_\eta)})\\
&=\tau_\eta(\langle\langle\zeta,\zeta\rangle_{M_n(A_\eta)}\xi,\xi\rangle_{M_n(A_\eta)})\\
&=\tau_\eta(\langle\zeta\langle\zeta,\xi\rangle_{A_\theta},\xi\rangle_{M_n(A_\eta)})\\
&=\tau_\eta(\langle\zeta,\xi\rangle_{M_n(A_\eta)})\\
&=\tau_\eta\Big(\sum_{k=1}^{n}\langle\zeta_k,\xi_k\rangle_{A_\eta}\Big) .\end{aligned}$$

Since we have that $\tau_\theta(\langle\xi, \zeta\rangle_{A_\theta}) = |c\theta+d|\tau_\eta(\langle\zeta, \xi\rangle_{A_\eta})$ for any $\xi, \zeta \in X$, we obtain that

$$\begin{aligned}\tau_\eta(p) &= \sum_{k=1}^{n} |c\theta+d|^{-1}\tau_\theta(\langle\xi_k, \zeta_k\rangle_{A_\theta}) \\ &= |c\theta+d|^{-1}\tau_\theta\Big(\sum_{k=1}^{n} \langle\xi_k, \zeta_k\rangle_{A_\theta}\Big) \\ &= |c\theta+d|^{-1} .\end{aligned}$$

Therefore we obtain that $\tau_\theta((\Phi\otimes \mathrm{id}_{M_n})(p))=1$. Q.E.D.

LEMMA 3. *Let ϕ_0 be the monomorphism of A_θ into $M_k(A_\theta)$ defined by $\phi_0(x)=x\otimes e_{11}$ for any $x\in A_\theta$ where $\{e_{ij}\}_{i,j=1}^{k}$ are matrix units of $M_k(A_\theta)$. Let ϕ be a monomorphism of A_θ into $M_k(A_\theta)$ with $\phi(A_\theta)=fM_k(A_\theta)f$ where f is a projection in $M_k(A_\theta)$ with $\tau_\theta(f)=1$. Then there are an automorphism β of A_θ and a unitary element $z\in M_k(A_\theta)$ such that $\phi=\mathrm{Ad}(z^*)\circ\phi_0\circ\beta$.*

PROOF. Since $\tau_\theta(f)=\tau_\theta(1\otimes e_{11})=1$, there is a unitary element $z\in M_k(A_\theta)$ such that $zfz^*=1\otimes e_{11}$ by Rieffel [7, 2.5. Corollary]. For any $x\in M_k(A_\theta)$, $zfxfz^* = (1\otimes e_{11})zxz^*(1\otimes e_{11}) \in (1\otimes e_{11})M_k(A_\theta)(1\otimes e_{11})$. Hence $\mathrm{Ad}(z)$ is an isomorphism of $fM_k(A_\theta)f$ onto $(1\otimes e_{11})M_k(A_\theta)(1\otimes e_{11})$. Since $\phi_0(A_\theta)= (1\otimes e_{11})M_k(A_\theta)(1\otimes e_{11})$ and $\phi(A_\theta)=fM_k(A_\theta)f$, if $\beta\in \mathrm{Aut}(A_\theta)$ is defined by $\beta=\phi_0^{-1}\circ\mathrm{Ad}(z)\circ\phi$, we obtain the conclusion. Q.E.D.

COROLLARY 4. *Let ϕ_0 be as above. Let ϕ be a monomorphism of A_θ into $M_k(A_\theta)$ with $\phi(A_\theta)=fM_k(A_\theta)f$ and $\phi(A_\theta^\infty)=fM_k(A_\theta^\infty)f$ where f is a projection in $M_k(A_\theta^\infty)$ with $\tau_\theta(f)=1$. Then there are a diffeomorphism β of A_θ and a unitary element $z\in M_k(A_\theta^\infty)$ such that $\phi=\mathrm{Ad}(z^*)\circ\phi_0\circ\beta$.*

PROOF. We have the same result as Rieffel [7, 2.5. Corollary] for A_θ^∞. Hence there is a unitary element $z\in M_k(A_\theta^\infty)$ such that $zfz^*=1\otimes e_{11}$. If we repeat the same discussion as Lemma 3, we obtain the automorphism β of A_θ such that

$$\beta=\phi_0^{-1}\circ\mathrm{Ad}(z)\circ\phi .$$

Since $\phi_0(A_\theta^\infty)=(1\otimes e_{11})M_k(A_\theta^\infty)(1\otimes e_{11})$ and $\phi(A_\theta^\infty)=fM_k(A_\theta^\infty)f$, $\beta(A_\theta^\infty)=A_\theta^\infty$.
Q.E.D.

LEMMA 5. *Let ϕ, ϕ_0, β and z be as in Lemma 3. Let $\alpha\in\mathrm{Aut}(A_\theta)$ and w be a unitary element in $M_k(A_\theta)$ such that $\mathrm{Ad}(w^*)\circ\alpha\otimes\mathrm{id}_{M_k}\in \mathrm{Aut}(fM_k(A_\theta)f)$ with $(\alpha\otimes\mathrm{id}_{M_k})(f)=wfw^*$. Then there is a unitary element $a\in A_\theta$ such that*

$$\phi^{-1}\circ\mathrm{Ad}(w^*)\circ\alpha\otimes\mathrm{id}_{M_k}\circ\phi=\beta^{-1}\circ\mathrm{Ad}(a)\circ\alpha\circ\beta\ .$$

PROOF. By Lemma 3 we have $\phi=\mathrm{Ad}(z^*)\circ\phi_0\circ\beta$. Thus

$$\begin{aligned}\phi^{-1}\circ\mathrm{Ad}(w^*)\circ\alpha\otimes\mathrm{id}_{M_k}\circ\phi&=\beta^{-1}\circ\phi_0^{-1}\circ\mathrm{Ad}(z)\circ\mathrm{Ad}(w^*)\circ\alpha\otimes\mathrm{id}_{M_k}\circ\mathrm{Ad}(z^*)\circ\phi_0\circ\beta\\&=\beta^{-1}\circ\phi_0^{-1}\circ\mathrm{Ad}(zw^*(\alpha\otimes\mathrm{id}_{M_k})(z^*))\circ\alpha\otimes\mathrm{id}_{M_k}\circ\phi_0\circ\beta\ .\end{aligned}$$

Since $(\phi^{-1}\circ\mathrm{Ad}(w^*)\circ\alpha\otimes\mathrm{id}_{M_k}\circ\phi)(1)=1$, $\mathrm{Ad}(zw^*(\alpha\otimes\mathrm{id}_{M_k})(z^*))(1\otimes e_{11})=1\otimes e_{11}$. Hence by an easy computation we can see that there are unitary elements $a\in A_\theta$ and $b\in M_{k-1}(A_\theta)$ such that $zw^*(\alpha\otimes\mathrm{id}_{M_k})(z^*)=a\oplus b$. Therefore for any $x\in A_\theta$

$$\begin{aligned}(\phi^{-1}\circ\mathrm{Ad}(w^*)\circ\alpha\otimes\mathrm{id}_{M_k}\circ\phi)(x)&=(\beta^{-1}\circ\phi_0^{-1}\circ\mathrm{Ad}(a\oplus b)\circ\alpha\otimes\mathrm{id}_{M_k}\circ\phi_0\circ\beta)(x)\\&=(\beta^{-1}\circ\phi_0^{-1}\circ\mathrm{Ad}(a\oplus b)\circ\alpha\otimes\mathrm{id}_{M_k})(\beta(x)\otimes e_{11})\\&=(\beta\circ\phi_0^{-1})((a\oplus b)((\alpha\circ\beta)(x)\otimes e_{11})(a^*\oplus b^*))\\&=(\beta\circ\phi_0^{-1})(a(\alpha\circ\beta)(x)a^*\otimes e_{11})\\&=(\beta^{-1}\circ\mathrm{Ad}(a)\circ\alpha\circ\beta)(x)\ .\end{aligned}$$

Thus we obtain the conclusion. Q.E.D.

COROLLARY 6. *Let ϕ, ϕ_0, β and z be as in Corollary 4. Let $\alpha\in\mathrm{Diff}(A_\theta)$ and w be a unitary element in $M_k(A_\theta^\infty)$ such that $\mathrm{Ad}(w^*)\circ\alpha\otimes\mathrm{id}_{M_k}\in\mathrm{Aut}(f(M_k(A_\theta)f))$ with $(\alpha\otimes\mathrm{id}_{M_k})(f)=wfw^*$ and $(\mathrm{Ad}(w^*)\circ\alpha\otimes\mathrm{id}_{M_k})(fM_k(A_\theta^\infty)f)=fM_k(A_\theta^\infty)f$. Then there is a unitary element $a\in A_\theta^\infty$ such that*

$$\phi^{-1}\circ\mathrm{Ad}(w^*)\circ\alpha\otimes\mathrm{id}_{M_k}\circ\phi=\beta^{-1}\circ\mathrm{Ad}(a)\circ\alpha\circ\beta\ .$$

PROOF. By Corollary 4 we have $\phi=\mathrm{Ad}(z^*)\circ\phi_0\circ\beta$. Thus

$$\phi^{-1}\circ\mathrm{Ad}(w^*)\circ\alpha\otimes\mathrm{id}_{M_k}\circ\phi=\beta^{-1}\circ\phi_0^{-1}\circ\mathrm{Ad}(zw^*(\alpha\otimes\mathrm{id}_{M_k})(z^*))\circ\alpha\otimes\mathrm{id}_{M_k}\circ\phi_0\circ\beta\ .$$

By the assumptions z, $w\in M_k(A_\theta^\infty)$ and $\alpha\in\mathrm{Diff}(A_\theta)$. Thus $zw^*(\alpha\otimes\mathrm{id}_{M_k})(z^*)\in M_k(A_\theta^\infty)$. By the proof of Lemma 5 $zw^*(\alpha\otimes\mathrm{id}_{M_k})(z^*)=a\oplus b$. Hence in this case $a\in A_\theta^\infty$ and $b\in M_{k-1}(A_\theta^\infty)$. Therefore we can obtain the conclusion.
Q.E.D.

Now recall that Ψ is an isomorphism of A_θ onto $pM_n(A_\eta)p$ with $\Psi(A_\theta^\infty)=pM_n(A_\eta^\infty)p$ where p is a projection in $M_n(A_\eta^\infty)$ and that Φ is an isomorphism of A_θ onto $qM_m(A_\theta)q$ with $\Phi(A_\eta^\infty)=qM_m(A_\theta^\infty)q$ where q is a projection in $M_m(A_\theta^\infty)$. Furthermore $\Phi\otimes\mathrm{id}_{M_n}\circ\Psi$ is an isomorphism of A_θ onto $(\Phi\otimes\mathrm{id}_{M_n})(p)M_{mn}(A_\theta)(\Phi\otimes\mathrm{id}_{M_n})(p)$ with $(\Phi\otimes\mathrm{id}_{M_n}\circ\Psi)(A_\theta^\infty)=(\Phi\otimes\mathrm{id}_{M_n})(p)\times M_{mn}(A_\theta^\infty)(\Phi\otimes\mathrm{id}_{M_n})(p)$.

Let Φ_0 be the monomorphism of A_θ into $M_{mn}(A_\theta)$ defined by $\Phi_0(x)=x\otimes e_{11}$ for any $x\in A_\theta$ where $\{e_{ij}\}_{i,j=1}^{mn}$ are matrix units of M_{mn}.

LEMMA 7. *With the above notations there are a diffeomorphism* β_θ *of* A_θ *and a unitary element* $z_\theta \in M_{mn}(A_\theta^\infty)$ *such that*

$$\Phi \otimes \mathrm{id}_{M_n} \circ \Psi = \mathrm{Ad}(z_\theta{}^*) \circ \Phi_0 \circ \beta_\theta .$$

PROOF. This is clear by Lemma 2 and Corollary 4. Q.E.D.

LEMMA 8. *Let* Φ, Ψ *and* β_θ *be as above. Let* $\alpha \in \mathrm{Aut}(A_\theta)$ *and* w *be a unitary element in* $M_{mn}(A_\theta)$ *such that* $\mathrm{Ad}(w^*) \circ \alpha \otimes \mathrm{id}_{M_{mn}} \in \mathrm{Aut}((\Phi \otimes \mathrm{id}_{M_n})(p) \times M_{mn}(A_\theta)(\Phi \otimes \mathrm{id}_{M_n})(p))$ *with* $(\alpha \otimes \mathrm{id}_{M_{mn}})((\Phi \otimes \mathrm{id}_{M_n})(p)) = w(\Phi \otimes \mathrm{id}_{M_n})(p)w^*$. *Then there is a unitary element* $a \in A_\theta$ *such that*

$$\Psi^{-1} \circ (\Phi \otimes \mathrm{id}_{M_n})^{-1} \circ \mathrm{Ad}(w^*) \circ \alpha \otimes \mathrm{id}_{M_{mn}} \circ (\Phi \otimes \mathrm{id}_{M_n}) \circ \Psi = \beta_\theta^{-1} \circ \mathrm{Ad}(a) \circ \alpha \circ \beta_\theta .$$

In particular if $\alpha \in \mathrm{Diff}(A_\theta)$ *and* w *is a unitary element in* $M_{mn}(A_\theta^\infty)$ *such that* $\mathrm{Ad}(w^*) \circ \alpha \otimes \mathrm{id}_{M_{mn}} \in \mathrm{Aut}((\Phi \otimes \mathrm{id}_{M_n})(p) M_{mn}(A_\theta)(\Phi \otimes \mathrm{id}_{M_n})(p))$ *with* $(\alpha \otimes \mathrm{id}_{M_{mn}})((\Phi \otimes \mathrm{id}_{M_n})(p)) = w(\Phi \otimes \mathrm{id}_{M_n})(p)w^*$, *then there is a unitary element* $a \in A_\theta^\infty$ *such that*

$$\Psi^{-1} \circ (\Phi \otimes \mathrm{id}_{M_n})^{-1} \circ \mathrm{Ad}(w^*) \circ \alpha \otimes \mathrm{id}_{M_{mn}} \circ (\Phi \otimes \mathrm{id}_{M_n}) \circ \Psi = \beta_\theta^{-1} \circ \mathrm{Ad}(a) \circ \alpha \circ \beta_\theta .$$

PROOF. This is clear by Lemma 5 and Corollary 6. Q.E.D.

For any isomorphism Φ of A_η onto $qM_m(A_\theta)q$ with $\Phi(A_\eta^\infty) = qM_m(A_\theta^\infty)q$ we define the homomorphism $T_{\theta,\eta}(\Phi)$ of $\mathrm{Aut}(A_\theta)/\mathrm{Int}(A_\theta)$ to $\mathrm{Aut}(A_\eta)/\mathrm{Int}(A_\eta)$ as follows;

$$T_{\theta,\eta}(\Phi)([\alpha]) = [\Phi^{-1} \circ \mathrm{Ad}(w_\theta{}^*) \circ \alpha \otimes \mathrm{id}_{M_m} \circ \Phi]$$

where $\alpha \in \mathrm{Aut}(A_\theta)$ and w_θ is a unitary element in $M_m(A_\theta)$ such that $(\alpha \otimes \mathrm{id}_{M_m})(q) = w_\theta q w_\theta{}^*$. Similarly for any isomorphism Ψ of A_θ onto $pM_n(A_\eta)p$ with $\Psi(A_\theta^\infty) = pM_n(A_\eta^\infty)p$ we define the homomorphism $T_{\eta,\theta}(\Psi)$ of $\mathrm{Aut}(A_\eta)/\mathrm{Int}(A_\eta)$ to $\mathrm{Aut}(A_\theta)/\mathrm{Int}(A_\theta)$. And since $\Phi(A_\eta^\infty) = qM_m(A_\theta^\infty)q$ and $\Psi(A_\theta^\infty) = pM_n(A_\eta^\infty)p$, $T_{\theta,\eta}(\Phi)$ and $T_{\eta,\theta}(\Psi)$ can be also considered as a homomorphism of $\mathrm{Diff}(A_\theta)/\mathrm{Int}^\infty(A_\theta)$ to $\mathrm{Diff}(A_\eta)/\mathrm{Int}^\infty(A_\eta)$ and a homomorphism of $\mathrm{Diff}(A_\eta)/\mathrm{Int}^\infty(A_\eta)$ to $\mathrm{Diff}(A_\theta)/\mathrm{Int}^\infty(A_\theta)$ respectively.

LEMMA 9. *Let* Φ, Ψ *and* β_θ *be as in Lemma 7. Let* $T_{\eta,\theta}$ *and* $T_{\theta,\eta}$ *be as above. Then*

$$(T_{\eta,\theta}(\Psi \circ \beta_\theta^{-1}) \circ T_{\theta,\eta}(\Phi))([\alpha]) = [\alpha]$$

for any $\alpha \in \mathrm{Aut}(A_\theta)$ *(resp.* $\alpha \in \mathrm{Diff}(A_\theta)$*).*

PROOF. For any $\alpha \in \mathrm{Aut}(A_\theta)$

$$\begin{aligned}(T_{\eta,\theta}(\Psi\circ\beta_\theta^{-1})\circ T_{\theta,\eta}(\Phi))([\alpha]) &= [\beta_\theta\circ\Psi^{-1}\circ\mathrm{Ad}(w_\eta{}^*)\circ(\Phi^{-1}\circ\mathrm{Ad}(w_\theta{}^*)\circ\alpha\otimes\mathrm{id}_{M_m}\circ\Phi)\otimes\mathrm{id}_{M_n}\circ\Psi\circ\beta_\theta^{-1}] \\ &= [\beta_\theta\circ\Psi^{-1}\circ\mathrm{Ad}(w_\eta{}^*)\circ\Phi^{-1}\otimes\mathrm{id}_{M_n}\circ\mathrm{Ad}(w_\theta{}^*\otimes I_n)\circ\alpha\otimes\mathrm{id}_{M_{mn}}\circ\Phi\otimes\mathrm{id}_{M_n}\circ\Psi\circ\beta_\theta^{-1}] \\ &= [\beta_\theta\circ\Psi^{-1}\circ(\Phi\otimes\mathrm{id}_{M_n})^{-1}\circ\mathrm{Ad}((\Phi\otimes\mathrm{id}_{M_n})(w_\eta{}^*)(w_\theta{}^*\otimes I_n))\circ\alpha\otimes\mathrm{id}_{M_{mn}} \\ &\qquad \circ(\Phi\otimes\mathrm{id}_{M_n})\circ\Psi\circ\beta_\theta^{-1}]\,,\end{aligned}$$

where w_θ is a unitary element in $M_m(A_\theta^\infty)$ such that $(\alpha\otimes\mathrm{id}_{M_m})(q)=w_\theta q w_\theta{}^*$ and w_η is a unitary element in $M_n(A_\eta^\infty)$ such that $((\Phi^{-1}\circ\mathrm{Ad}(w_\theta{}^*)\circ\alpha\otimes\mathrm{id}_{M_m}\circ\Phi)\otimes\mathrm{id}_{M_n})(p)=w_\eta p w_\eta{}^*$. By Lemma 8 there is a unitary element $a_\theta\in A_\theta$ (or $a_\theta\in A_\theta^\infty$ if $\alpha\in\mathrm{Diff}(A_\theta)$) such that

$$\Psi^{-1}\circ(\Phi\otimes\mathrm{id}_{M_n})^{-1}\circ\mathrm{Ad}((\Phi\otimes\mathrm{id}_{M_n})(w_\eta{}^*)(w_\theta{}^*\otimes I_n))\circ\alpha\otimes\mathrm{id}_{M_{mn}}\circ(\Phi\otimes\mathrm{id}_{M_n})\circ\Psi = \beta_\theta^{-1}\circ\mathrm{Ad}(a_\theta)\circ\alpha\circ\beta_\theta\,.$$

Hence we obtain that

$$(T_{\eta,\theta}(\Psi\circ\beta_\theta^{-1})\circ T_{\theta,\eta}(\Phi))([\alpha])=[\mathrm{Ad}(a_\theta)\circ\alpha]=[\alpha]\,. \qquad \text{Q.E.D.}$$

Let Ψ_0 be the monomorphism of A_η into $M_{mn}(A_\eta)$ defined by $\Psi_0(x)=x\otimes e_{11}$ for any $x\in A_\eta$.

LEMMA 10. *Let Φ, Ψ and β_θ be as in Lemma 9 and let Ψ_0 be as above. Then there are a diffeomorphism β_η of A_η and a unitary element $z_\eta\in M_{mn}(A_\eta^\infty)$ such that*

$$(\Psi\circ\beta_\theta^{-1})\otimes\mathrm{id}_{M_m}\circ\Phi=\mathrm{Ad}(z_\eta{}^*)\circ\Psi_0\circ\beta_\eta\,.$$

PROOF. This is clear by Lemma 2 and Corollary 4. Q.E.D.

LEMMA 11. *Let Φ, Ψ, β_θ and β_η be as in Lemma 10. Let $\alpha\in\mathrm{Aut}(A_\eta)$ and w be a unitary element in $M_{mn}(A_\eta)$ such that $\mathrm{Ad}(w^*)\circ\alpha\otimes\mathrm{id}_{M_{mn}}\in\mathrm{Aut}(((\Psi\circ\beta_\theta^{-1})\otimes\mathrm{id}_{M_m})(q)M_{mn}(A_\eta)((\Psi\circ\beta_\theta^{-1})\otimes\mathrm{id}_{M_m})(q))$ with $(\alpha\otimes\mathrm{id}_{M_{mn}})(((\Psi\circ\beta_\theta^{-1})\otimes\mathrm{id}_{M_m})(q))=w((\Psi\circ\beta_\theta^{-1})\otimes\mathrm{id}_{M_m})(q)w^*$. Then there is a unitary element $a\in A_\eta$ such that*

$$(\Phi^{-1}\circ(\beta_\theta\circ\Psi^{-1})\otimes\mathrm{id}_{M_m})\circ\mathrm{Ad}(w^*)\circ\alpha\otimes\mathrm{id}_{M_{mn}}\circ((\Psi\circ\beta_\theta^{-1})\otimes\mathrm{id}_{M_m}\circ\Phi)=\beta_\eta^{-1}\circ\mathrm{Ad}(a)\circ\alpha\circ\beta_\eta\,.$$

In particular if $\alpha\in\mathrm{Diff}(A_\eta)$ and $w\in M_{mn}(A_\eta^\infty)$, $a\in A_\eta^\infty$.

PROOF. This is clear by Lemma 5 and Corollary 6. Q.E.D.

LEMMA 12. *Let $T_{\theta,\eta}$ and $T_{\eta,\theta}$ be as before. Let Φ, Ψ, β_θ and β_η be as in Lemma 11. Then*

$$(T_{\theta,\eta}(\Phi\circ\beta_\eta^{-1})\circ T_{\eta,\theta}(\Psi\circ\beta_\theta^{-1}))([\alpha])=[\alpha]$$

for any $\alpha \in \mathrm{Aut}(A_\eta)$ (*resp.* $\alpha \in \mathrm{Diff}(A_\eta)$).

PROOF. For any $\alpha \in \mathrm{Aut}(A_\eta)$

$$\begin{aligned}(T_{\theta,\eta}(\Phi\circ\beta_\eta^{-1})\circ T_{\eta,\theta}(\Psi\circ\beta_\theta^{-1}))([\alpha]) &= [\beta_\eta\circ\Phi^{-1}\circ\mathrm{Ad}(w_\theta{}^*)\circ(\beta_\theta\circ\Psi^{-1}\circ\mathrm{Ad}(w_\eta{}^*)\circ\alpha\otimes\mathrm{id}_{M_n}\circ\Psi\circ\beta_\theta^{-1})\otimes\mathrm{id}_{M_m}\circ\Phi\circ\beta_\eta^{-1}] \\ &= [\beta_\eta\circ\Phi^{-1}\circ(\beta_\theta\circ\Psi^{-1})\otimes\mathrm{id}_{M_m}\circ\mathrm{Ad}(((\Psi\circ\beta_\theta^{-1})\otimes\mathrm{id}_{M_m})(w_\theta{}^*)(w_\eta{}^*\otimes I_n)) \\ &\qquad \circ\alpha\otimes\mathrm{id}_{M_{mn}}\circ(\Psi\circ\beta_\theta^{-1})\otimes\mathrm{id}_{M_m}\circ\Phi\circ\beta_\eta^{-1}]\,.\end{aligned}$$

By Lemma 11 there is a unitary element $a_\eta \in A_\eta$ (or $a_\eta \in A_\eta^\infty$ if $\alpha \in \mathrm{Diff}(A_\eta)$) such that

$$\begin{aligned}&\Phi^{-1}\circ(\beta_\theta\circ\Psi^{-1})\otimes\mathrm{id}_{M_m}\circ\mathrm{Ad}(((\Psi\circ\beta_\theta^{-1})\otimes\mathrm{id}_{M_m})(w_\theta{}^*)(w_\eta{}^*\otimes I_n)) \\ &\qquad \circ\alpha\otimes\mathrm{id}_{M_{mn}}\circ((\Psi\circ\beta_\theta^{-1})\otimes\mathrm{id}_{M_m}\circ\Phi) = \beta_\eta^{-1}\circ\mathrm{Ad}(a_\eta)\circ\alpha\circ\beta_\eta\,.\end{aligned}$$

Hence we obtain that

$$(T_{\theta,\eta}(\Phi\circ\beta_\eta^{-1})\circ T_{\eta,\theta}(\Psi\circ\beta_\theta^{-1}))([\alpha]) = [\mathrm{Ad}(a_\eta)\circ\alpha] = [\alpha]\,. \qquad \text{Q.E.D.}$$

THEOREM 13. *If* A_θ *and* A_η *are strongly Morita equivalent,* $\mathrm{Aut}(A_\theta)/\mathrm{Int}(A_\theta)$ (*resp.* $\mathrm{Diff}(A_\theta)/\mathrm{Int}^\infty(A_\theta)$) *is isomorphic to* $\mathrm{Aut}(A_\eta)/\mathrm{Int}(A_\eta)$ (*resp.* $\mathrm{Diff}(A_\eta)/\mathrm{Int}^\infty(A_\eta)$).

PROOF. This follows from Lemmas 9 and 12. Q.E.D.

§4. Non generic numbers not satisfying the result of Elliott.

DEFINITION. Let θ be an irrational number. We say that θ *is generic* if there are $r>1$ and $C>0$ such that

$$|e^{2\pi i n\theta}-1| \geqq \frac{C}{n^r}$$

for any integer $n\neq 0$, that is, not a Liouville number.

For any $s,\ t\in \boldsymbol{R}$ let $\alpha_{(s,t)}$ be the diffeomorphism of A_θ defined by $\alpha_{(s,t)}(u)=e^{2\pi i s}u$ and $\alpha_{(s,t)}(v)=e^{2\pi i t}v$ where u and v are generators of A_θ with $uv=e^{2\pi i\theta}vu$. For any $g=\begin{bmatrix} a & b \\ c & d\end{bmatrix}\in SL(2, \boldsymbol{Z})$ let α_g be the diffeomorphism of A_θ defined by $\alpha_g(u)=u^a v^c$ and $\alpha_g(v)=u^b v^d$ where $SL(2, \boldsymbol{Z})$ is the group of all 2×2 matrices over $\boldsymbol{Z}$ with determinant 1.

Now we will state the result of Elliott.

THEOREM (Elliott [2]). *Let* θ *be a generic irrational number. For any* $\alpha\in\mathrm{Diff}(A_\theta)$ *there are a unitary element* $w\in A_\theta^\infty$, $g\in SL(2, \boldsymbol{Z})$ *and* s, $t\in\boldsymbol{R}$ *such that* $\alpha=\mathrm{Ad}(w)\circ\alpha_g\circ\alpha_{(s,t)}$.

In this section we will show that if A_η has a diffeomorphism not satisfying the above theorem, so does A_θ if $\theta \in GL(2, \boldsymbol{Z})\eta$. We use the notations as before. For $\alpha \in \mathrm{Diff}(A_\theta)$ let $\tilde{\tau}_\theta$ be the trace of $A_\theta \times_\alpha \boldsymbol{Z}$ induced by τ_θ and let $\tilde{\tau}_{\theta *}$ be the homomorphism of $K_0(A_\theta \times_\alpha \boldsymbol{Z})$ into $\boldsymbol{R}$ induced by $\tilde{\tau}_\theta$. And similarly we define $\tilde{\tau}_\eta$ and $\tilde{\tau}_{\eta *}$.

LEMMA 14. *Let $\alpha \in \mathrm{Diff}(A_\eta)$ with $\alpha_* = \mathrm{id}$ on $K_1(A_\eta)$ and $\tilde{\tau}_{\eta *}(K_0(A_\eta \times_\alpha \boldsymbol{Z})) = \boldsymbol{Z} + \boldsymbol{Z}\eta$. Let β be a diffeomorphism of A_θ such that*

$$\beta = \Psi^{-1} \circ \mathrm{Ad}(w_\eta^*) \circ \alpha \otimes \mathrm{id}_{M_n} \circ \Psi$$

where w_η is a unitary element in $M_n(A_\eta^\infty)$ with $(\alpha \otimes \mathrm{id}_{M_n})(p) = w_\eta p w_\eta^*$. Then $\beta_* = \mathrm{id}$ on $K_1(A_\theta)$ and $\tilde{\tau}_{\theta *}(K_0(A_\theta \times_\beta \boldsymbol{Z})) = \boldsymbol{Z} + \boldsymbol{Z}\theta$.

PROOF. By the definition of β, $\beta_* = \mathrm{id}$ on $K_1(A_\theta)$. Since $\tau_\eta \circ \Psi$ is a tracial state on A_θ and A_θ has the unique tracial state, $\tau_\theta = t(\tau_\eta \circ \Psi)$ where t is a positive number. However by the proof of Lemma 2 $\tau_\theta(1) = t(\tau_\eta \circ \Psi)(1) = t\tau_\eta(p) = t|c\theta + d|^{-1}$. Hence $t = |c\theta + d|$. Thus $\tau_\theta = |c\theta + d|(\tau_\eta \circ \Psi)$. Let u and v be unitary elements in A_θ with $uv = e^{2\pi i\theta} vu$. Since $\beta_* = \mathrm{id}$ on $K_1(A_\theta)$, there is a piecewise continuously differentiable path $h\colon [0, 1] \to U_k(A_\theta)$ such that $h(0) = 1 \otimes I_k$ and $h(1) = \beta(u)u^* \otimes I_k$ where $U_k(A_\theta)$ is the unitary group of $M_k(A_\theta)$. Hence $\Psi \otimes \mathrm{id}_k \circ h\colon [0, 1] \to U((p \otimes I_k) M_{kn}(A_\eta)(p \otimes I_k))$ is a piecewise continuously differentiable path from $p \otimes I_k$ to $(\mathrm{Ad}(w_\eta^*) \circ \alpha \otimes \mathrm{id}_{M_n})(\Psi(u))\Psi(u)^* \otimes I_k$ where $U((p \otimes I_k) M_{kn}(A_\eta)(p \otimes I_k))$ is the unitary group of $(p \otimes I_k) M_{kn}(A_\eta)(p \otimes I_k)$. Let $\tilde{h}\colon [0, 1] \to U_{kn}(A_\eta)$ be the piecewise continuously differentiable path from $1 \otimes I_{kn}$ to $(\mathrm{Ad}(w_\eta^*) \circ \alpha \otimes \mathrm{id}_{M_n})(\Psi(u))\Psi(u)^* \otimes I_k + 1 \otimes I_{kn} - p \otimes I_k$ defined by $\tilde{h}(t) = (\Psi \otimes \mathrm{id}_{M_k})(h(t)) + 1 \otimes I_{kn} - p \otimes I_k$. Then since $\tilde{\tau}_{\eta *}(K_0(A_\eta \times_\alpha \boldsymbol{Z})) = \boldsymbol{Z} + \boldsymbol{Z}\theta$,

$$\frac{1}{2\pi i}\int_0^1 \tau_\eta\Big(\tilde{h}(t)^* \frac{d}{dt}\tilde{h}(t)\Big)dt = l_1 + l_2\eta$$

by Pimsner [5] where l_1 and $l_2 \in \boldsymbol{Z}$. Therefore

$$\begin{aligned}
&\frac{1}{2\pi i}\int_0^1 \tau_\theta\Big(h(t)^* \frac{d}{dt}h(t)\Big)dt \\
&= \frac{1}{2\pi i}\int_0^1 |c\theta + d|(\tau_\eta \circ \Psi \otimes \mathrm{id}_{M_k})\Big(h(t)^* \frac{d}{dt}h(t)\Big)dt \\
&= \frac{1}{2\pi i}\int_0^1 |c\theta + d|\tau_\eta\Big(\tilde{h}(t)^* \frac{d}{dt}\tilde{h}(t)\Big)dt \\
&= |c\theta + d|(l_1 + l_2\eta)\ .
\end{aligned}$$

Since $\eta=\frac{a\theta+b}{c\theta+d}$, we obtain that

$$\frac{1}{2\pi i}\int_0^1\tau_\theta\Big(h(t)^*\frac{d}{dt}h(t)\Big)dt$$
$$=|c\theta+d|\Big(l_1+l_2\frac{a\theta+b}{c\theta+d}\Big)$$
$$=|c\theta+d|l_1+l_2\frac{|c\theta+d|}{c\theta+d}(a\theta+b)\in \boldsymbol{Z}+\boldsymbol{Z}\theta\ .$$

If we repeat the same discussion for $v\in A_\theta$, by Pimsner [5], $\tilde{\tau}_{\theta*}(K_0(A_\theta\times_\beta \boldsymbol{Z}))=\boldsymbol{Z}+\boldsymbol{Z}\theta$. Q.E.D.

We showed in [3] that there are a non generic irrational number η and $\alpha\in\mathrm{Diff}(A_\eta)$ satisfying the following conditions;

1) $\alpha_*=\mathrm{id}$ on $K_1(A_\eta)$,
2) $\tilde{\tau}_{\eta*}(K_0(A_\eta\times_\alpha \boldsymbol{Z}))=\boldsymbol{Z}+\boldsymbol{Z}\theta$,
3) $\Gamma(\alpha)=\boldsymbol{T}$,

where $\Gamma(\alpha)$ is its Connes spectrum. The above α does not satisfy the result of Elliott.

COROLLARY 15. *Let η be as above and $\eta=g\theta=\frac{a\theta+b}{c\theta+d}$ where $g=\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in GL(2, \boldsymbol{Z})$. Then there is a $\beta\in\mathrm{Diff}(A_\theta)$ satisfying the above conditions.*

PROOF. Let $\alpha\in\mathrm{Diff}(A_\eta)$ be as above. Let $\beta=\Psi^{-1}\circ\mathrm{Ad}(w_\eta)\circ\alpha\otimes\mathrm{id}_{M_n}\circ\Psi$ where w_η is a unitary element in $M_n(A_\eta^\infty)$ with $(\alpha\otimes\mathrm{id}_{M_n})(p)=w_\eta p w_\eta^*$. Then we can see by Theorem 13 that $\Gamma(\beta)=\boldsymbol{T}$ since $\Gamma(\alpha)=\boldsymbol{T}$. And we obtain by Lemma 14 that $\beta_*=\mathrm{id}$ on $K_1(A_\theta)$ and $\tilde{\tau}_{\theta*}(K_0(A_\theta\times_\beta \boldsymbol{Z}))=\boldsymbol{Z}+\boldsymbol{Z}\theta$. Q.E.D.

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