TOKYO J. MATH.
VOL. 12, No. 2, 1989

On Asymptotic Stability for the Yang-Mills Gradient Flow

Hideo KOZONO and Yoshiaki MAEDA

Nagoya University and Keio University

Dedicated to Professor Hiroshi Fujita on his sixtieth birthday

§1. Introduction and statement of results.

The purpose of this paper is to study the asymptotic stability in $W^{m,r}$-sense for the Yang-Mills gradient flow around stable Yang-Mills connections.

We first concern with a closed connected Riemannian n-manifold (M, h) and consider a G-vector bundle $E=P\times_{\rho}\boldsymbol{R}^N$ associated with a G-principal bundle P over M. Here, G is a compact connected Lie group and ρ is a faithful orthogonal representation $\rho: G\to O_N$ of G.

On the space C_E of connections on E preserving the inner product of E, we consider the *Yang-Mills functional* (Y-M functional)

$$\mathrm{YM}(\nabla)=\frac{1}{2}\int_M |R^{\nabla}|^2 d_h x\,. \tag{1.1}$$

Here R^{∇} and $d_h x$ denote the curvature tensor of connection ∇ and the Riemannian measure on (M, h), respectively and $|\ |$ is the norm determined by the inner product on E.

A critical point of the above functional (1.1) is called a *Yang-Mills connection* (a Y-M connection) and the corresponding curvature field is called the *Yang-Mills field* (the Y-M field), respectively. A Y-M connection is said to be *stable* if it minimizes (1.1) locally. Moreover, a Y-M connection ∇ is said to be *strictly stable* if the second variation of Y-M functional at ∇ is *strictly positive on a transversal orbit of the gauge group action* on C_E (see Definition 2.1). These notions are referred to Bourguignon-Lawson [3]. Typical examples of the stable Y-M connections are well-known self-dual connections on 4-sphere S^4. Moreover,

Received November 15, 1988
Revised June 1, 1989

Bourguignon-Lawson [3, Theorem 9.1] have given examples of the strictly stable connection on a certain vector bundle over S^n/Γ ($n\geqq 4$).

It is probable that in a suitable topology any connection near a stable Y-M connection should converge to the Y-M one through the integral curve of the gradient flow of Y-M functional:

$$\frac{d\nabla(t)}{dt} = -\mathrm{grad}\,\mathrm{YM}(\nabla(t)) \tag{1.2}$$

with the initial condition $\nabla(0)=\nabla_0\in C_E$, where $X=\mathrm{grad}\,\mathrm{YM}(\nabla)$ is the gradient vector field of Y-M functional.

In doing so, we have difficulties, however, because the vector field X is so *degenerate on orbits of the gauge group actions.* Introducing a *rigid coordinate* on C_E in accordance with the action of the gauge group, we observe the curve of gradient flow explicitly (cf. §2).

Leaving the precise definition in section 2, we state our first result as follows:

THEOREM A. *Let M be a closed connected Riemannian manifold and E is a G-vector bundle over M. A strictly stable Y-M connection $\nabla\in C_E$ is asymptotically stable in $W^{m,r}$-sense for $m\geqq 2$, $r>n$.*

Most of difficulties for obtaining Theorem A come from how we can recover the parabolicity. We shall reduce (1.2) to a system of semilinear evolution equations of parabolic type by using the *admissible coordinate* given in section 2. These equations seem to be similar to the Navier-Stokes ones for an incompressible fluid. We shall construct a global solution of (1.2) by making use of the fractional powers of a certain dissipative operator like Fujita-Kato [5].

Subsequently, we observe the *flat* connection ∇ on a smooth vector bundle E over (i) a bounded domain M in $\boldsymbol{R}^n$ with smooth boundary ∂M and (ii) the whole space $\boldsymbol{R}^n$. In these cases, we have $\mathrm{YM}(\nabla)\equiv 0$, so that ∇ attains the absolute minimum of the Y-M functional. Furthermore, both cases are strictly stable in a slightly modified sense of [3] (cf. Definition 2.2). In treating these, we may reformulate the situation and the definitions stated as above for these cases. Sections 2 and 3 will be devoted to a precise formulation included for these cases. Using a method similar to that in the proof of Theorem A, we shall show in section 4:

THEOREM B. *Let E be a smooth vector bundle over a bounded domain M in $\boldsymbol{R}^n$ with smooth boundary ∂M. Then the flat connection ∇*

is asymptotically stable in $W^{m,r}$*-sense for* $m \geqq 2$, $r > n$.

Finally in section 5, we shall give the following result whose proof needs some techniques different from the previous ones.

THEOREM C. *The flat connection* ∇ *on the smooth vector bundle* E *over* $\boldsymbol{R}^n$ *is asymptotically stable in* $W^{m,n}$*-sense for* $m \geqq 2$.

The reason why we offer Theorem C is that one seems to obtain the asymptotic stability by using the method of the proof similar to that of Theorem C *when* ∇ *is weakly stable.* In fact, in the proof of Theorem C, we shall not make use of the abstract theory of the *bounded* semi-group but the (L^r, L^q)-estimates for the solution of the heat equation on $\boldsymbol{R}^n$. The method of (L^r, L^q)-estimates seems to be useful in observing the asymptotic stability for the various self-dual connections, which will be presented in the forthcoming paper.

REMARK. Kono-Nagasawa [11] gave another treatment for the gradient flow of Y-M functional. They showed a global existence and a decay property of the solution for the Y-M gradient flow equation around the flat connection of a trivial vector bundle over $\boldsymbol{R}^n$. In these results, however, the gradient flows are restricted on the directions of the gauge orbit in the space of connections C_E. Therefore, they do not give any description around the whole neighborhood of the Y-M connections in C_E.

§2. Admissible coordinate on C_E.

We give our basic set-up and notations used throughout this paper, although these are mainly due to Bourguignon-Lawson [3].

Let M be a smooth, connected Riemannian n-manifold with the metric h. In what follows, we shall discuss such manifolds M as the following types (I), (II) and (III);

(I) compact Riemannian manifolds without boundary,

(II) bounded domains M in $\boldsymbol{R}^n$ with the smooth boundary ∂M,

(III) Euclidian n-space $\boldsymbol{R}^n$.

Take a coordinate neighborhood U of M with the coordinates $(x^1, \cdots, x^n)$, where we take the whole space M as U in the case of (II) or (III). Let E be the smooth vector bundle associated with the principal G-bundle P over M. We denote by k the inner product on E. On a coordinate neighbourhood U, we trivialize $E|_U = U \times \boldsymbol{R}^N$ and use the coordinate $(x^1, \cdots, x^n, u^1, \cdots, u^N) = (x, u)$ on it. Using this coordinate, we shall express some geometric quantities. For the Einstein's summation convention, we give a list of indices as follows:

$$i,\ j,\ l=1,\cdots,\ n=\dim M\ ,$$
$$a,\ b,\ c=1,\cdots,\ N=\dim E\ .$$

The Riemannian metric h on M and the inner product k on E can be written by $h=(h_{ij}(x))$ and $k=(k_{ab}(x))$ on U, respectively. Let $\mathfrak{g}_E$ be the smooth vector bundle whose fibre $\mathfrak{g}_{E,x}$ at $x\in M$ is the skew-symmetric endomorphisms of E_x with respect to k. We denote by $\Omega^p(\mathfrak{g}_E)$ the set of all $\mathfrak{g}_E$-valued smooth p-forms and by C_E the set of all smooth connections on E. For the connection $\nabla\in C_E$, we denote by $\omega_j(x)=(\omega_{jb}^{\ a}(x))$ the component of ∇ on U. For example, for $\phi\in\Omega^1(\mathfrak{g}_E)$ with the component $\phi(x)=(\phi_{jb}^{\ a}(x))$ in U, the covariant derivative of ϕ can be given by

$$\nabla_j\phi_{ib}^{\ a}(x)=\partial_j\phi_{ib}^{\ a}(x)-\Gamma_{ji}^{\ l}(x)\phi_{lb}^{\ a}(x)+\omega_{jc}^{\ a}(x)\phi_{ib}^{\ c}(x)-\omega_{jb}^{\ c}(x)\phi_{ic}^{\ a}(x) \tag{2.1}$$

on U, where $\Gamma_{ij}^{l}(x)$ is the Christoffel symbol of h. The connection ∇ acts on other $\mathfrak{g}_E$-valued tensor fields according to the derivation rules (cf. Bourguignon-Lawson [3, (2.1)]). In the same way, the exterior covariant differentiation d^∇ can be defined as usual (cf. [3, (2.8)]). We denote by $R^\nabla\in\Omega^2(\mathfrak{g}_E)$ the curvature tensor of $\nabla\in C_E$, which can be expressed as

$$R^{\nabla}{}_{ijb}^{\ a}(x)=\partial_i\omega_{jb}^{\ a}(x)-\partial_j\omega_{ib}^{\ a}(x)+\omega_{ic}^{\ a}(x)\omega_{jb}^{\ c}(x)-\omega_{jc}^{\ a}(x)\omega_{ib}^{\ c}(x)\ . \tag{2.2}$$

We choose an arbitrary connection $\nabla\in C_E$ in case (I). In case (II) or (III), we take ∇ as the flat connection, i. e. $\Gamma_{ji}^{\ l}(x)=0$ and $\omega_{jb}^{\ a}(x)=0$.

We introduce the Hilbert inner product on $\Omega^p(\mathfrak{g}_E)$ as

$$(A,\ B)=-\frac{1}{2}\int_M A^{i_1\cdots i_p}{}_b^{\ a}(x)B_{i_1\cdots i_p}{}_a^{\ b}(x)d_hx \tag{2.3}$$

for $A=(A_{i_1\cdots i_p}{}_b^{\ a}(x))$, $B=(B_{i_1\cdots i_p}{}_b^{\ a}(x))\in\Omega^p(\mathfrak{g}_E)$. Here $A^{i_1\cdots i_p}{}_b^{\ a}(x)=h^{i_1j_1}(x)\cdots\times h^{i_pj_p}(x)A_{j_1\cdots j_p}{}_b^{\ a}(x)$ $((h^{ij}(x))$; the inverse matrix of $(h_{ij}(x)))$. For $m=1,2,\cdots$ and $r>1$, we define the $W^{m,r}$-norm on $\Omega^p(\mathfrak{g}_E)$ by

$$\|A\|_{m,r}=-\left\{\sum_{s\leq m}\int_M[\nabla^{i_1\cdots i_s}A^{j_1\cdots j_p}{}_b^{\ a}(x)\nabla_{i_1\cdots i_s}A_{j_1\cdots j_p}{}_a^{\ b}(x)]^{r/2}d_hx\right\}^{1/r} \tag{2.4}$$

for $A\in\Omega^p(\mathfrak{g}_E)$. $W^{m,r}(\Omega^p(\mathfrak{g}_E))$ is the completion of the set $\{A\in\Omega^p(\mathfrak{g}_E);\ \|A\|_{m,r}<\infty\}$ with respect to the norm $\|\ \|_{m,r}$. In particular, we set $\Omega_0^1(\mathfrak{g}_E)=\Omega^1(\mathfrak{g}_E)$ if M is of type (I) or (III), $\Omega_0^1(\mathfrak{g}_E)=\{A\in\Omega^1(\mathfrak{g}_E);\ A$ is tangent to the boundary $\partial M\}$ if M is of type (II). We denote by $W^{m,r}(\Omega_0^1(\mathfrak{g}_E))$

the completion of the space $\{A\in\Omega_0^1(\mathfrak{g}_E);\ \|A\|_{m,r}<\infty\}$ with respect to the norm $\|\ \|_{m,r}$. For the definition and the properties of the differentials on $W^{m,r}$-space, see, for example, Aubin [1].

Let $C_{E,0}$ be the set $\nabla_A=\nabla+A$ of connections, where $A\in\Omega_0^1(\mathfrak{g}_E)$. Then, by the identification $C_{E,0}\cong\Omega_0^1(\mathfrak{g}_E)$, we can introduce $W^{m,r}$-norm on $C_{E,0}$. We denote by $W^{m,r}(C_{E,0})$ the completion of $C_{E,0}$ with respect to the $W^{m,r}$-norm. Clearly, $W^{m,r}(C_{E,0})=\{\nabla+A;\ A\in W^{m,r}(\Omega_0^1(\mathfrak{g}_E))\}$. In this stage, we can extend the Y-M functional YM on $C_{E,0}$ defined by (1.1) to

$$\mathrm{YM}^{m,r}:\ W^{m,r}(C_{E,0})\longrightarrow \boldsymbol{R} \tag{2.5}$$

for $m\geqq 1$ and $1/r\leqq \mathrm{Min.}\{1/2,\ 1/4+m/n\}$.

The following is easy to see (cf. [3, (2.11) Theorem]):

LEMMA 2.1. *If $\nabla\in C_{E,0}$ is a Y-M connection, then it holds*

$$(\delta^\nabla R^\nabla)_{ib}^{a}(x)=-\nabla^j R_{jib}^{a}(x)=0\ , \tag{2.6}$$

where δ^∇ is the formal adjoint of d^∇.

We denote by $\mathfrak{X}^{m,r}(C_E)$ the set of all $W^{m,r}$-vector fields on C_E. Associated with (1.1), we define the vector field $X^{m,r}$ on C_E which can be considered as an element of $\mathfrak{X}^{m,r}(C_E)^*$ (the dual space of $\mathfrak{X}^{m,r}(C_E)$):

$$(X^{m,r}(\nabla),\ W(\nabla))=d\mathrm{YM}^{m,r}(\nabla)(W(\nabla))\qquad \text{for}\quad W\in\mathfrak{X}^{m,r}(C_E)\ , \tag{2.7}$$

where $d\mathrm{YM}^{m,r}(\nabla)$ is the differential of (1.1) at $\nabla\in W^{m,r}(C_E)$ and $(\cdot,\cdot)$ denotes the duality between $\mathfrak{X}^{m,r}(C_E)^*$ and $\mathfrak{X}^{m,r}(C_E)$. We call $X^{m,r}$ determined by (2.7) the gradient vector field of the functional $\mathrm{YM}^{m,r}$ and denote it by $X^{m,r}=\mathrm{grad}\,\mathrm{YM}^{m,r}$. Obviously, $X^{m,r}$ is stationary at ∇ if ∇ is a Y-M connection.

Now, let us consider the integral curve of $\mathrm{grad}\,\mathrm{YM}^{m,r}$ in $W^{m,r}(C_E)$:

$$\frac{d\nabla(t)}{dt}=-\mathrm{grad}\,\mathrm{YM}^{m,r}(\nabla(t)) \tag{2.8}$$

with the initial condition $\nabla(0)=\nabla_0\in C_E$.

We denote by $\mathcal{G}$ the gauge group which is the set of all automorphisms g on E preserving the inner product of E. Expressing $g\in\mathcal{G}$ by $g=(g_b^a(x))$ in the coordinate U, we introduce the $W^{m,r}$-norm on $\mathcal{G}$ as

$$\|g\|_{m,r}=\left\{\sum_{s\leqq m}\int_M(\nabla^{i_1}\cdots\nabla^{i_s}g_b^a(x)\nabla_{i_1}\cdots\nabla_{i_s}g_a^b(x))^{r/2}d_hx\right\}^{1/r}\ . \tag{2.9}$$

Denote by $W^{m,r}(\mathcal{G})$ the completion of the space $\{g\in\mathcal{G};\ \|g\|_{m,r}<\infty\}$

with respect to the norm $\| \|_{m,r}$. Let $\mathfrak{g}$ be the set of all infinitesimal automorphisms s on E. Denoting $s=(s_b^a(x))$ on U, we get $s_{ab}(x)+s_{ba}(x)=0$, where $s_{ab}(x)=k_{ac}(x)s_b^c(x)$. It can be easily seen that $\mathfrak{g}\cong \Omega^0(\mathfrak{g}_E)$. Hence we may introduce the $W^{m,r}$-norm on $\mathfrak{g}$ by (2.4) and denote by $W^{m,r}(\mathfrak{g})$ the completion of the space $\{s\in \mathfrak{g}\cong \Omega^0(\mathfrak{g}_E);\ \|s\|_{m,r}<\infty\}$ with respect to this norm.

Let exp: $\boldsymbol{g}\to G$ be the exponential mapping. Then we get the mapping

$$\exp\colon \mathfrak{g}\longrightarrow \mathscr{G} \tag{2.10}$$

by $(\exp\phi)(x)=\exp\phi(x)$. Here for each $x\in M$, $\phi(x)$ can be considered as an element of $\mathfrak{g}$ by using the representation $\rho\colon G\to O_N$. The following proposition is proved by the successive approximation or the method similar to Omori [14]:

PROPOSITION 2.2. *Suppose that* $m\geqq 2$, $r\geqq n$ *and* $j=0, 1, \cdots$. *For* $\phi\in W^{m,r}(\Omega^0(\mathfrak{g}_E))$ *and* $s\in C^j([0, \infty); W^{m,r}(\mathfrak{g}))$, *there exists a unique* $g\in C^{j+1}([0, \infty); W^{m,r}(\mathscr{G}))$ *such that*

$$\frac{dg(t)}{dt}=g(t)\cdot s(t) \quad (t>0)\ , \qquad g(0)=\exp^{m,r}\phi\ . \tag{2.11}$$

Here $\exp^{m,r}$ *denotes the smooth extension of* (2.11) *as*

$$\exp^{m,r}\ :\ \ W^{m,r}(\mathfrak{g})\longrightarrow W^{m,r}(\mathscr{G})\ . \tag{2.12}$$

REMARK. For m and r as above, $W^{m,r}(\mathscr{G})$ is closed for the pointwise multiplication. Concerning (2.12), $W^{m,r}(\mathfrak{g})$ and $W^{m,r}(\mathscr{G})$ are considered not as the Lie algebra and the Lie group, respectively, but simply as the linear spaces.

Note that $\mathscr{G}$ acts naturally on C_E as

$$\nabla^g=g\circ\nabla\circ g^{-1} \qquad \text{for} \quad g\in\mathscr{G} \text{ and } \nabla\in C_E\ . \tag{2.13}$$

The Y-M functional (1.1) is left invariant under the action (2.13) above. Now, let us take a connection $\nabla\in C_E$ and fix it. Then we can get a natural splitting of the tangent space $T_\nabla C_E$ by

$$T_\nabla C_E\cong \Omega^1(\mathfrak{g}_E)=Z^1(\mathfrak{g}_E)\oplus\Omega^1_*(\mathfrak{g}_E) \quad \text{(direct sum)}\ , \tag{2.14}$$

where

$$Z^1(\mathfrak{g}_E)=\{d^\nabla\phi\in\Omega^1_0(\mathfrak{g}_E)\ ;\ \phi\in\Omega^0(\mathfrak{g}_E)\cap(\ker d^\nabla)^\perp\}\ ,$$
$$\Omega^1_{0,*}(\mathfrak{g}_E)=\{A\in\Omega^1_0(\mathfrak{g}_E)\ ;\ \delta^\nabla A=0\}\ .$$

Note that if ∇ is irreducible, then $(\ker d^\nabla)^\perp=\emptyset$. Let us consider a smooth mapping $\sigma\colon Z^1(\mathfrak{g}_E)\oplus\Omega^1_{0,*}(\mathfrak{g}_E)\to\Omega^1_0(\mathfrak{g}_E)$ defined by

$$\sigma(d^\nabla\phi, A)=\exp\phi(\nabla+A)\exp\phi^{-1}-\nabla\ . \tag{2.15}$$

Clearly, σ can be extended to the smooth map:

$$\sigma^{m,r}\ :\ W^{m,r}(Z^1(\mathfrak{g}_E))\oplus W^{m,r}(\Omega^1_{0,*}(\mathfrak{g}_E))\longrightarrow W^{m,r}(\Omega^1_0(\mathfrak{g}_E)) \tag{2.16}$$

($W^{m,r}(X)$; the completion of the space $\{u\in X;\ \|u\|_{m,r}<\infty\}$ with respect to the norm $\|\ \|_{m,r}$).

Now, using the argument in Lawson [13, p. 34], we see that Fréchet derivative $D\sigma^{m,r}(0, 0)$ is an isomorphism on $W^{m,r}(\Omega^1_0(\mathfrak{g}_E))$. Therefore, there exist open neighbourhoods U_1, U_2 and U of 0 in $W^{m,r}(Z^1(\mathfrak{g}_E))$, $W^{m,r}(\Omega^1_{0,*}(\mathfrak{g}_E))$ and $W^{m,r}(\Omega^1_0(\mathfrak{g}_E))$, respectively such that $\sigma^{m,r}|_{U_1\times U_2}$ is a diffeomorphism from $U_1\times U_2$ onto U. In this way, we get a coordinate system (in $W^{m,r}$-sense) $(\sigma;U,U_1,U_2)$ around ∇ and call it an *admissible coordinate* around ∇. Taking $A=0$ in (2.15), we obtain the action of gauge group in $C_{E,0}$ by

$$\sigma(d^\nabla\phi, 0)=g\nabla g^{-1}\ ,\qquad \text{where}\quad g=\exp\phi\ . \tag{2.17}$$

This is an orbit through $\nabla\in C_{E,0}$ and the tangent space of this orbit at ∇ coincides with $Z^1(\mathfrak{g}_E)$.

DEFINITION 2.1. A Y-M connection $\nabla\in C_{E,0}$ is called *stable* if

$$\frac{d^2}{dt^2}\mathrm{YM}(\nabla_t)\Big|_{t=0}\geqq 0 \tag{2.18}$$

for any smooth curve ∇_t in $C_{E,0}$ with $\nabla_0=\nabla$. Moreover ∇ is *strictly stable* if in addition to (2.18)

$$\frac{d^2}{dt^2}\mathrm{YM}(\nabla_t)\Big|_{t=0}>0 \tag{2.19}$$

for any smooth curve ∇_t in $C_{E,0}$ with $\nabla_0=\nabla$ and $d\nabla_t/dt|_{t=0}\in\Omega^1_{0,*}(\mathfrak{g}_E)$.

In this stage, our definition of an asymptotic stability reads as follows:

DEFINITION 2.2. Let $m\geqq 2$ and $r>1$ with $1/r\leqq\mathrm{Min.}\{1/2,\ 1/4+m/n\}$. A Y-M connection $\nabla\in C_E$ is called *asymptotically stable* in *$W^{m,r}$-sense* if there exist open sets $0\in U_1\subset\tilde U_1$, $0\in U_2\subset\tilde U_2$ and $0\in U\subset\tilde U$ of an admissible coordinate $(\sigma^{m,r};\tilde U,\tilde U_1,\tilde U_2)$ satisfying the following properties:

For $\{d^\nabla\phi_0, A_0\}\in U_1\times U_2$, there is a unique curve $\{\phi(t), A(t)\}_{t\geq 0}\in W^{m,r}(Z^1(\mathfrak{g}_E))\times W^{m,r}(\Omega^1_{0,*}(\mathfrak{g}_E))$ such that

(i) $\{\phi(t), A(t)\} \in U_1 \times U_2$ for $t>0$;

(ii) $\nabla(t)=\nabla+\sigma^{m,r}(d^\nabla\phi(t), A(t))$ is the solution of (2.8) with the initial data $\nabla(0)=\nabla+\sigma^{m,r}(d^\nabla\phi_0, A_0)$;

(iii) The connection $\nabla(t)$ converges to ∇ up to the action of $W^{m,r}$-gauge transformation as $t\to\infty$ in $W^{m,r}(\Omega^1(\mathfrak{g}_E))$, that is,

$$\lim_{t\to\infty}(g(t)^{-1}\nabla(t)g(t)-\nabla)=0 \quad \text{in} \quad W^{m,r}(\Omega^1(\mathfrak{g}_E)) , \tag{2.20}$$

where $g(t)=\exp\phi(t)$.

Note that for $\nabla_0 \in W^{m,r}(C_E)$ with $\nabla_0-\nabla \in \tilde{U}$, we can take $\{d^\nabla\phi_0, A_0\} \in \tilde{U}_1\times\tilde{U}_2$ uniquely so that $\nabla_0=\nabla+\sigma^{m,r}(d^\nabla\phi_0, A_0)$. Hence (ii) means the solvability of the *initial value problem* of (2.8) for any ∇_0 near ∇ in $W^{m,r}$-norm.

§ 3. Gradient flow for the Y-M functional.

In this section, we shall give the explicit expression of the gradient flow of the Y-M functional (2.8) in the admissible coordinate (2.16). In what follows, the differentiation should be understood in the generalized sense (in the sense of $W^{m,r}$). We shall compute as if the quantities A, s, g, $\cdots$, etc. were sufficiently smooth, which can be easily extended to our generalized situation.

3.1. Equations of the gradient flow.

LEMMA 3.1. *Let $X=\operatorname{grad}\mathrm{YM}$ be the gradient vector field of* (2.7). *Then for any C^∞-mapping $W: C_{E,0}\to\Omega^1_0(\mathfrak{g}_E)$, we have*

$$(X(\nabla), W(\nabla))=(\delta^\nabla R^\nabla, W(\nabla)) \tag{3.1}$$

for each $\nabla\in C_{E,0}$.

PROOF. Let ∇_t be a smooth curve in $C_{E,0}$ with $\nabla_0=\nabla$ and $d\nabla_t/dt|_{t=0}=W(\nabla)$. By the definition of the gradient vector and by the straight forward calculation as Bourguignon-Lawson [3, (2.21) Theorem], we have

$$\frac{d}{dt}\mathrm{YM}(\nabla_t)|_{t=0}=(X(\nabla), W(\nabla))=(\delta^\nabla R^\nabla, W(\nabla)) .$$

Take a connection $\nabla\in C_{E,0}$ and fix it. Let us give some formulae and properties for the curvature tensors in [3, section 2].

PROPOSITION 3.2. *Let $A\in\Omega^1_0(\mathfrak{g}_E)$. We have:*

(i) *The curvature tensor $R^{\nabla+A}$ of $\nabla+A$ is given by*

$$R^{\nabla+A}{}_{ij}{}^a_b(x)=R^{\nabla}{}_{ij}{}^a_b(x)+d^{\nabla}A_{ij}{}^a_b(x)+[A,\ A]_{ij}{}^a_b(x)\ , \tag{3.2}$$

where

$$[A,\ A]_{ij}{}^a_b(x)=A_i{}^a_c(x)A_j{}^c_b(x)-A_j{}^a_c(x)A_i{}^c_b(x)\ ; \tag{3.3}$$

(ii) *The divergent $\delta^{\nabla+A}S$ for $S\in\Omega^2(\mathfrak{g}_E)$ is given by*

$$\delta^{\nabla+A}S_i{}^a_b(x)=\delta^{\nabla}S_i{}^a_b(x)+[S,\ A]_i{}^a_b(x)\ , \tag{3.4}$$

where

$$[S,\ A]_i{}^a_b(x)=S_{ij}{}^c_b(x)A^{j}{}^a_c(x)-S_{ij}{}^a_c(x)A^{j}{}^c_b(x)\ . \tag{3.5}$$

We also need the formulae for gauge actions.

PROPOSITION 3.3. *Let $g\in\mathscr{G}$. We have*

$$R^{\nabla^g}=gR^{\nabla}g^{-1}\ , \tag{3.6}$$

$$\delta^{\nabla^g}R^{\nabla^g}=g\delta^{\nabla}R^{\nabla}g^{-1}\ , \tag{3.7}$$

where $\nabla^g=g\circ\nabla\circ g^{-1}$.

Take a *Y-M connection* $\nabla\in C_{E,0}$ and fix it. Making use of these formulae, we compute (2.8). Although our calculation is done in $Z^1(\mathfrak{g}_E)\oplus\Omega^1_{0,*}(\mathfrak{g}_E)$, it still holds in $W^{m,r}(Z^1(\mathfrak{g}_E))\oplus W^{m,r}(\Omega^1_{0,*}(\mathfrak{g}_E))$. Let $\phi(t)$ and $A(t)$ be smooth curves in $\Omega^0(\mathfrak{g}_E)\cap(\ker d^{\nabla})^{\perp}$ and $\Omega^1_{0,*}(\mathfrak{g}_E)$, respectively. We consider the map

$$\sigma(t):=\sigma(d^{\nabla}\phi(t),\ A(t))=g(t)(\nabla+A(t))g(t)^{-1}-\nabla\ ,$$

where $g(t)=\exp\phi(t)$. Differentiating the above directly, we have

$$\frac{d\sigma(t)}{dt}=g(t)\left\{\frac{dA(t)}{dt}+[\nabla+A(t),\ s(t)]\right\}g(t)^{-1}\ , \tag{3.8}$$

where $s(t)=g(t)^{-1}dg(t)/dt$;

$$[\nabla+A,\ s]_i{}^a_b(x)=d^{\nabla}s_i{}^a_b(x)+[A,\ s]_i{}^a_b(x)\ ; \tag{3.9}$$

$$[A,\ s]_i{}^a_b(x)=A_i{}^a_c(x)s^c_b(x)-A_i{}^c_b(x)s^a_c(x)\ . \tag{3.10}$$

On the other hand, we obtain by Propositions 3.2-3.3

$$\begin{aligned}\delta^{\nabla+\sigma(t)}R^{\nabla+\sigma(t)}&=g(t)(\delta^{\nabla+A(t)}R^{\nabla+A(t)})g(t)^{-1}\\&=g(t)(\delta^{\nabla}R^{\nabla}+\delta^{\nabla}d^{\nabla}A(t)+[R^{\nabla},\ A(t)]+Q(A(t)))g(t)^{-1}\ ,\end{aligned} \tag{3.11}$$

where

$$Q(A)_{ib}^{a}(x)=\delta^{\nabla}[A, A]_{ib}^{a}(x)+[d^{\nabla}A, A]_{ib}^{a}(x)+[[A, A], A]_{ib}^{a}(x) . \tag{3.12}$$

See (3.3)-(3.5). Since $\delta^{\nabla}R^{\nabla}=0$ and since $\delta^{\nabla}A=0$, we have

PROPOSITION 3.4. *Let $\nabla \in C_{E,0}$ be a Y-M connection and let $\phi(t)$ and $A(t)$ be smooth curves in $\Omega^0(\mathfrak{g}_E)\cap(\ker d^{\nabla})^{\perp}$ and $\Omega^1_{0,*}(\mathfrak{g}_E)$, respectively. Then* (2.8) *can be written as*

$$\begin{cases} \dfrac{dA(t)}{dt}=-\{L^{\nabla}A(t)+Q(A(t))+[A(t), s(t)]+d^{\nabla}s(t)\} , \\ \delta^{\nabla}A(t)=0 , \end{cases} \tag{Eq$_0$}$$

where $s(t)=(\exp\phi(t))^{-1}d(\exp\phi(t))/dt$ and

$$L^{\nabla}A=(d^{\nabla}\delta^{\nabla}+\delta^{\nabla}d^{\nabla})A+[R^{\nabla}, A] . \tag{3.13}$$

(Eq$_0$) makes sense and gives the equation of the gradient flow of the functional $\mathrm{YM}^{m,r}$ defined by (2.5), whenever $m\geqq 2$ and $r\geqq n$ (see Lemma 4.4).

3.2. Reduction of (Eq$_0$) to the abstract evolution equation. We shall solve (Eq$_0$) with initial condition $A(0)=A_0$ by making use of the abstract theory of evolution equations.

In case M is of type (I) *or* (II). At first, we assume that M is of type (I). Let G^{∇} be the Green operator of $\Delta^{\nabla}:=d^{\nabla}\delta^{\nabla}+\delta^{\nabla}d^{\nabla}$ acting on $\Omega^0(\mathfrak{g}_E)$. That is, G^{∇} is the linear operator defined by $\Delta G^{\nabla}\eta^{\nabla}+H\eta^{\nabla}=\eta$ for all $\eta\in\Omega^0(\mathfrak{g}_E)$, where H is the projection onto the space $\{\pi\in\Omega^0(\mathfrak{g}_E); \Delta^{\nabla}\pi=0\}$. We denote by $G^{\nabla}(x, y)$, $(x, y)\in M\times M$, the kernel function of G^{∇} and define a linear operator P by

$$Pu=u-d^{\nabla}\eta \qquad \text{for} \quad u\in\Omega^1(\mathfrak{g}_E)$$

where $\eta(x)=(d^{\nabla}G^{\nabla}(x, \cdot), u)$ $((\cdot, \cdot)$; the inner product in $\Omega^1(\mathfrak{g}_E))$. Since Δ^{∇} is the symmetric elliptic differential operator of second order, we have

$$\|d^{\nabla}\eta\|_{m,r}\leqq C\|u\|_{m,r} \tag{3.14}$$

for all $u\in\Omega^1(\mathfrak{g}_E)$ with C independent of u. Hence P is uniquely extended to the bounded operator P_r on $L^r(\Omega^1(\mathfrak{g}_E))$. It is easy to see that $P_r^2=P_r$ and we obtain the decomposition

$$L^r(\Omega^1(\mathfrak{g}_E))=R(P_r)\oplus R(I-P_r) \quad \text{(direct sum)} \tag{3.15}$$

($R(T)$; the range of the operator T).

In case M is of type (II), we can choose such projection operator P_r onto $L^r(\Omega^1_{0,*}(\mathfrak{g}_E))$ as the one constructed by Fujiwara-Morimoto [7]. Now, set

$$X_r := R(P_r) .$$

Then it follows from Ebin [4] (in case (I)) and Fujiwara-Morimoto [7] (in case (II)) that

$$X_r = L^r(\Omega^1_{0,*}(\mathfrak{g}_E)) , \quad X_r^* \text{ (the dual space of } X_r) = X_{r'} ,$$
$$R(I-P_r) \subset \{d^\nabla s;\ s \in W^{1,r}(\Omega^0(\mathfrak{g}_E))\}$$

where $r' = r/(r-1)$.

We next define the operator L_r on X_r by $L_r = P_r L^\nabla$ with definition domain $D(L_r) = W^{2,r}(\Omega^1(\mathfrak{g}_E)) \cap X_r$ in case M is of type (I) or $D(L_r) = \{A \in W^{2,r}(\Omega^1(\mathfrak{g}_E));\ A|_{\partial M} = 0\} \cap X_r$ in case M is of type (II). Then we have by Ebin [4] and Fujiwara-Morimoto [7] L_r^* (the adjoint operator) $= L_{r'}$.

Now, let us reduce (Eq$_0$) to the abstract equations on X_r. Applying P_r to both sides of the first equation (Eq$_0$), we have

$$\frac{dA}{dt} + L_r A + P_r(Q(A) + [A, s]) = 0 \qquad \text{in} \quad X_r .$$

Note that $P_r A = A$ since $\delta^\nabla A = 0$ and that $P_r d^\nabla s = 0$. Moreover, applying δ^∇ to both sides of the same equation, we get

$$\Delta^\nabla s = \delta^\nabla(Q(A) + [A, s] + [R^\nabla, A]) + \delta^{\nabla^2} d^\nabla A .$$

We choose such s as

$$s = G^\nabla \delta^\nabla(Q(A) + [A, s] + [R^\nabla, A]) + G^\nabla \delta^{\nabla^2} d^\nabla A .$$

After all, we get the following system of equations for $\{A, s\}$:

$$\begin{cases} \dfrac{dA}{dt} + L_r A + P_r(Q(A) + [A, s]) = 0 & \text{in} \quad X_r,\ t > 0 , \\ s = G^\nabla \delta^\nabla(Q(A) + [A, s]) + G^\nabla J^\nabla A , & t > 0 , \\ A(0) = A_0 , \end{cases} \qquad \text{(Eq}_1\text{)}$$

where $J^\nabla A := \delta^\nabla([R^\nabla, A]) + \delta^{\nabla^2} d^\nabla A$. Note that $J^\nabla A \equiv 0$ when M is of type (II) and ∇ is the flat connection.

REMARK 3.5. Since the Green operator G^∇ maps $\Omega^0(\mathfrak{g}_E)$ onto the orthogonal complement of the subspace $\{\pi \in \Omega^0(\mathfrak{g}_E);\ \Delta^\nabla \pi = 0\}$, we see s defined by (Eq$_1$) belongs to $L^r((\ker d^\nabla)^\perp)$. See also Proposition 4.3.

In case M is of type (III). Next we consider the case of Theorem C. We define the projection operator P by

$$Pu_{ib}^{a}(x) := u_{ib}^{a}(x) - \sum_{j=1}^{n} \int_{R^n} \frac{\partial^2 \Gamma(x-y)}{\partial y^i \partial y^j} u_{jb}^{a}(y)dy$$

for $u = (u_{ib}^{a}(x)) \in L^r(\Omega^1(\mathfrak{g}_E))$, where $\Gamma(x) = (1/2\pi)\log|x|$ $(n=2)$, $=\{(n-2)\mathrm{vol}(S^{n-1})\}^{-1}|x|^{2-n}$ $(n \geqq 3)$. By Calderón-Zygmund theorem for the singular integral operators, we see P is a projection operator on $L^r(\Omega^1(\mathfrak{g}_E))$ and $P(L^r(\Omega^1(\mathfrak{g}_E))) = L^r(\Omega^1_{0,*}(\mathfrak{g}_E))$. Since $R^\nabla \equiv 0$, we have $L^\nabla = -\Delta = \sum_{j=1}^{n} (\partial/\partial x^j)^2$ and hence P commutes with L^∇. In the similar manner as in the cases of (I) and (II), we have

$$\begin{cases} \dfrac{dA}{dt} - \Delta A + P(Q(A) + [A, s]) = 0 \quad \text{in} \quad X_r := L^r(\Omega^1_{0,*}(\mathfrak{g}_E)), \quad t>0, \\ s = d\Gamma * (Q(A) + [A, s]), \qquad t>0, \\ A(0) = A_0, \end{cases} \tag{Eq$_2$}$$

where $*$ denotes the convolution operator (not the Hodge star operator).

Conversely, if $A \in \Omega^1_{0,*}(\mathfrak{g}_E)$ and $s \in \Omega^0(\mathfrak{g}_E)$ satisfies (Eq$_1$) or (Eq$_2$), it is easy to see that $\{A, s\}$ is the smooth solution of (Eq$_0$). Therefore, in what follows, we shall investigate the solvability of (Eq$_1$) and (Eq$_2$) in X_r.

§4. Proof of Theorems A and B.

In this section, we restrict ourselves to the cases (I) and (II). Then we may solve (Eq$_1$). The operator L_r introduced in the preceding section plays an important role in (Eq$_1$).

LEMMA 4.1. *Suppose that* $r \geqq 2$. *Let* $\nabla \in C_{E,0}$ *be a strictly stable Y-M connection if M is of type* (I) *and be the flat connection if M is of type* (II). *Then we have the following*:

(i) *The resolvent* $\rho(-L_r)$ *of* $-L_r$ *contains the right half-plane* $\{\lambda \in C; \mathrm{Re}\,\lambda \geqq 0\}$. *In particular,* $0 \in \rho(-L_r)$;

(ii) *There is a positive constant* M_r *such that*

$$\|(L_r + \lambda)^{-1}\|_{B(X_r)} \leqq M_r(1+|\lambda|)^{-1} \tag{4.1}$$

for all $\mathrm{Re}\,\lambda \geqq 0$, *where* $\|\ \|_{B(X_r)}$ *denotes the norm of bounded linear operators on* X_r.

For the proof, see the Appendix.

An immediate consequence of (4.1) reads:

LEMMA 4.2. *Under the assumption of Lemma* 4.1, *we have*

(i) $-L_r$ *generates a uniformly bounded holomorphic semi-group* $\{e^{-tL_r}\}_{t\geqq 0}$ *of class* C_0 *in* X_r;

(ii) *Let* L_r^{∇} *be the operator defined by* (3.13) *with the domain* $D(L_r^{\nabla})\equiv W^{2,r}(\Omega^1(\mathfrak{g}_E))$ *in case* M *is of type* (I) *or with the domain* $D(L_r^{\nabla})\equiv\{A\in W^{2,r}(\Omega^1(\mathfrak{g}_E));\ A|_{\partial M}=0\}$ *in case* M *is of type* (II). *Then for the definition domains of the fractional powers of* L_r *and* L_r^{∇}, *we have the continuous injection*

$$D(L_r^{\alpha})\subset D((L_r^{\nabla})^{\beta})\qquad for\quad 0<\beta<\alpha\,. \tag{4.2}$$

Indeed, since $0\in\rho(-L_r)$, we see by (3.14) that there is a constant $C>0$ with

$$\|L_r^{\nabla}A\|_{0,r}\leqq C\|L_rA\|_{0,r}\qquad \text{for all}\quad A\in D(L_r)\,.$$

Hence we get (4.2) by Krein [12, Chapter 1, Lemma 7.3]. Moreover, since $D((L_r^{\nabla})^{\beta})$ is continuously imbedded into the space of Bessel potential $W^{2\beta,r}(\Omega^1(\mathfrak{g}_E))$ (see Fujiwara [6]), it follows from (4.2) that there is a constant $C=C(\alpha,\beta)$ for $0<\beta<\alpha$ such that

$$\|A\|_{2\beta,r}\leqq C\|L_r^{\alpha}A\|_{0,r} \tag{4.3}$$

for all $A\in D(L_r^{\alpha})$. Therefore, in order to prove Theorems A and B, it suffices to show

PROPOSITION 4.3. *Let* $k=1,2,\cdots$, $r>n$ *and* $\gamma>0$ *and let* $A_0\in D(L_r^{k/2+\gamma})$. *There exists a positive constant* λ_0 *such that if* $\|L_r^{k/2+\gamma}A_0\|_r\leqq\lambda_0$, *there is a unique solution* $\{A,s\}$ *of* (Eq_1) *with*

$$A\in C([0,\infty);D(L_r^{k/2+\gamma}))\cap C((0,\infty);D(L_r^{k/2+\gamma}))\cap C^1((0,\infty);D(L_r^{k/2}))\,,$$
$$s\in C([0,\infty);W^{k,r}(\Omega^0(\mathfrak{g}_E)))\cap C((0,\infty);W^{k+1,r}(\Omega^0(\mathfrak{g}_E)))$$

satisfying

$$\|L_r^{k/2+\alpha}A(t)\|_{0,r}=o(t^{\gamma-\alpha})\qquad as\ t\downarrow 0\quad for\ \gamma\leqq\alpha<1-\gamma/2\,, \tag{4.4}$$

$$\|s(t)\|_{k+1,r}=o(t^{-(1+n/r)/2})\qquad as\ t\downarrow 0\,. \tag{4.5}$$

Moreover, such a solution $\{A,s\}$ *satisfies the asymptotic behavior*

$$\|L_r^{k/2+\alpha}A(t)\|_{0,r}=O(t^{\gamma-\alpha})\qquad as\ t\to\infty\quad for\ \gamma\leqq\alpha<1-\gamma/2\,, \tag{4.6}$$

$$\|s(t)\|_{k+1,r}=O(t^{-(1+n/r)/2})\qquad as\ t\to\infty\,. \tag{4.7}$$

Here and in what follows, we shall consider $D(L_r^{\beta})$, $\beta\geqq 0$, *as the Banach* *space with the norm* $\|L_r^\beta A\|_{0,r}$ *(not the graph norm).*

REMARK. As is stated in Remark 3.5, we see $s \in C([0, \infty); W^{k,r}((\ker d^\nabla)^\perp)) \cap C((0, \infty); W^{k+1,r}((\ker d^\nabla)^\perp))$. Further note that $W^{k+1,r}(\Omega^1_{0,*}(\mathfrak{g}_E)) \subset D(L_r^{k/2+\gamma})$ for $0 \leqq \gamma \leqq 1/2$ and $D(L_r^{k/2+\alpha}) \subset W^{k+1,r}(\Omega^1(\mathfrak{g}_E))$ for $\alpha \geqq 1/2$. Therefore, taking $m=k+1$ in Proposition 4.3 and then using Proposition 2.2, we obtain $\{\phi(t), A(t)\}_{t\geq 0}$ satisfying the conditions (i)-(iii) of Definition 2.2 for any $\{d^\nabla\phi_0, A_0\} \in U_1 \times U_2$. Hence Theorems A and B follow.

In what follows, we shall denote $P=P_r$, $L=L_r$ and $\| \ \|_r = \| \ \|_{0,r}$ for simplicity. We shall denote by C various constants which may change from line to line. In particular, $C=C(*, *, \cdots)$ will denote a constant depending only on the quantities appearing in the parentheses. To prove this Proposition, we need:

LEMMA 4.4. *Let* $k=1, 2, \cdots$. *There exist a constant* $C=C(p, \varepsilon_1)$ *for any* $p>r$ *and any* $\varepsilon_1>0$ *and constants* $C=C(\varepsilon_i)$ *for any* $\varepsilon_i>0$ $(i=2, 3)$ *such that*

$$\begin{aligned}\|Q(A)-Q(\bar{A})\|_{k,r} &\leqq C(p, \varepsilon_1)(\|L^{k/2+n(1/r-1/p)/2+\varepsilon_1}(A-\bar{A})\|_r\|L^{k/2+n/2p+1/2+\varepsilon_1}\bar{A}\|_r \\ &\qquad + \|L^{k/2+n(1/r-1/p)/2+\varepsilon_1}A\|_r\|L^{k/2+n/2p+1/2+\varepsilon_1}(A-\bar{A})\|_r) \\ &\quad + C(\varepsilon_2)(\|L^{k/2+n/3r+\varepsilon_2}A\|_r^2 + \|L^{k/2+n/3r+\varepsilon_2}\bar{A}\|_r^2)\|L^{k/2+n/3r+\varepsilon_2}(A-\bar{A})\|_r\,, \end{aligned} \tag{4.8}$$

$$\|[A, s]-[\bar{A}, \bar{s}]\|_{k,r} \leqq C(\varepsilon_3)(\|L^{k/2+\varepsilon_3}A\|_r\|s-\bar{s}\|_{k+1,r} + \|\bar{s}\|_{k+1,r}\|L^{k/2+\varepsilon_3}(A-\bar{A})\|_r \tag{4.9}$$

for all $A, \bar{A} \in D(L^{k/2+n/2p+1/2+\varepsilon})$ $(\varepsilon = \max_i \varepsilon_i)$ *and all* $s, \bar{s} \in W^{k+1,r}(\Omega^0(\mathfrak{g}_E))$.

PROOF. It suffices to prove when $\bar{A}=\bar{s}=0$, because we can reduce the desired result to such a case by the triangle inequalities. By the Hölder inequality, we have

$$\|Q(A)\|_{k,r} \leqq C(\|A\|_{k,p}\|A\|_{k+1,q} + \|A\|_{k,3r}^3)\,, \tag{4.10}$$

where $1/p+1/q=1/r$, $r<p$, $q<\infty$. Taking $\alpha=1/2+n/2p$, $\beta=n(1/r-1/p)/2$ and $\theta=n/3r$, we have by (4.3) the following continuous imbeddings (see Bergh-Lofström [2, Theorem 6.5.1]):

$$D(L^{k/2+\alpha+\varepsilon_1}) \subset W^{k+2\alpha,r}(\Omega^1(\mathfrak{g}_E)) \subset W^{k+1,q}(\Omega^1(\mathfrak{g}_E))\ ;$$
$$D(L^{k/2+\beta+\varepsilon_1}) \subset W^{k+2\beta,r}(\Omega^1(\mathfrak{g}_E)) \subset W^{k,p}(\Omega^1(\mathfrak{g}_E))\ ;$$
$$D(L^{k/2+\theta+\varepsilon_2}) \subset W^{k+2\theta,r}(\Omega^1(\mathfrak{g}_E)) \subset W^{k,3r}(\Omega^1(\mathfrak{g}_E))$$

for $\varepsilon_i>0$ $(i=1, 2)$. Hence we get by (4.10)

$$\|Q(A)\|_{k,r}\leqq C_{\varepsilon_1}\|L^{k/2+\beta+\varepsilon_1}A\|_r\|L^{k/2+\alpha+\varepsilon_1}A\|_r+C_{\varepsilon_2}\|L^{k/2+\theta+\varepsilon_2}A\|_r^3 .$$

Similarly, since $D(L^{k/2+\varepsilon_3})\subset W^{k,r}(\Omega^1(\mathfrak{g}_E))$ and $W^{k+1,r}(\Omega^0(\mathfrak{g}_E))\subset W^{k,\infty}(\Omega^0(\mathfrak{g}_E))$ (by $r>n$), we have

$$\|[A, s]\|_{k,r}\leqq\|A\|_{k,r}\|s\|_{k,\infty}\leqq C_{\varepsilon_3}\|L^{k/2+\varepsilon_3}A\|_r\|s\|_{k+1,r} .$$

In particular, since

$$\begin{aligned}&\|[d^\nabla A, A]\|_{k,r}+\|\delta^\nabla[A, A]\|_{k,r}\\&\quad\leqq\|A\|_{k+1,r}\|A\|_{k,\infty}\leqq\|A\|_{k+1,r}^2\leqq C_{\varepsilon_1}\|L^{k/2+1/2+\varepsilon_1}A\|_r^2 \quad \text{(by (4.3))}\end{aligned}$$

for $\varepsilon_1>0$ with $C_{\varepsilon_1}>0$ independent of A, we have

$$\begin{aligned}&\|Q(A)-Q(\bar{A})\|_{k,r}\\&\leqq C_{\varepsilon_1}\{(\|L^{k/2+1/2+\varepsilon_1}A\|_r+\|L^{k/2+1/2+\varepsilon_1}\bar{A}\|_r)\|L^{k/2+1/2+\varepsilon_1}(A-\bar{A})\|_r\}\\&\quad+C_{\varepsilon_2}\{(\|L^{k/2+n/3r+\varepsilon_2}A\|_r^2+\|L^{k/2+n/3r+\varepsilon_2}\bar{A}\|_r^2)\|L^{k/2+n/3r+\varepsilon_2}(A-\bar{A})\|_r\} . \qquad (4.11)\end{aligned}$$

LEMMA 4.5. *Let $k=1, 2, \cdots$. For $A\in W^{k,r}(\Omega^1(\mathfrak{g}_E))$, we have $G^\nabla\delta^\nabla A$, $G^\nabla J^\nabla A\in W^{k+1,r}(\Omega^0(\mathfrak{g}_E))$ and*

$$\|G^\nabla\delta^\nabla A\|_{k+1,r}\leqq C\|A\|_{k,r} , \qquad \|G^\nabla J^\nabla A\|_{k+1,r}\leqq C\|A\|_{k,r} \qquad (4.12)$$

with C independent of A.

PROOF. Since $d^\nabla d^\nabla A=[R^\nabla, A]$ for $A\in\Omega^1(\mathfrak{g}_E)$ (by the Ricci formula), we see that J^∇ is a bounded operator from $W^{k,r}(\Omega^1(\mathfrak{g}_E))$ into $W^{k-1,r}(\Omega^0(\mathfrak{g}_E))$. Hence (4.12) follows from the general theory of the elliptic differential operators of the second order. See, for example, Aubin [1].

PROOF OF PROPOSITION 4.3. At first we consider the following integral equation:

$$\begin{aligned}A(t)&=e^{-tL}A_0-\int_0^t e^{-(t-\tau)L}P(Q(A(\tau))+[A(\tau), s(\tau)])d\tau\\ s(t)&=G^\nabla\delta^\nabla(Q(A(t))+[A(t), s(t)])+G^\nabla J^\nabla A(t) .\end{aligned} \qquad (\text{I.E.}_1)$$

(i) *Existence.* We want to construct the solution of (I.E.$_1$) by successive approximation, according to the scheme

$$\begin{aligned}&A_0(t)=e^{-tL}A_0 , \qquad s_0(t)\equiv 0 ,\\&A_{j+1}(t)=A_0(t)-\int_0^t e^{-(t-\tau)L}P(Q(A_j(\tau))+[A_j(\tau), s_j(\tau)])d\tau , \qquad (4.13)\\&s_{j+1}(t)=G^\nabla\delta^\nabla(Q(A_j(t))+[A_j(t), s_j(t)])+G^\nabla J^\nabla A_{j+1}(t) ,\end{aligned}$$

where $j=0, 1, \cdots$. Then for $\gamma \leqq \alpha < 1-\gamma/2$, there exists $\{K_{\alpha,j}, M_j, N_j\}_{j=0}^{\infty}$ such that

$$\|L^{k/2+\alpha}A_j(t)\|_r \leqq K_{\alpha,j}t^{\gamma-\alpha} \qquad \text{for} \quad t>0\,, \tag{4.14}$$

$$\|s_j(t)\|_{k+1,r} \leqq M_j t^{-(1+n/r)/2} \qquad \text{for} \quad t>0\,, \tag{4.15}$$

$$\|s_j(t)\|_{k,r} \leqq N_j \qquad \text{for} \quad t \geqq 0\,. \tag{4.16}$$

Indeed, by (4.1) we have

$$\|L^{\alpha}e^{-tL}\|_{B(X_r)} \leqq C_{\alpha}t^{-\alpha}\,, \qquad \alpha>0$$

for all $t>0$ with $C_{\alpha}>0$ independent of t. Hence (4.14)-(4.16) are true for $j=0$ if we choose

$$K_{\alpha,0}=\sup_{t>0} t^{\alpha-\gamma}\|L^{\alpha-\gamma}e^{-tL}L^{k/2+\gamma}A_0\|_r\,, \qquad M_0=N_0=0\,.$$

Suppose that (4.14)-(4.16) are true for j. Without loss of generality, we may assume that $0<\gamma<(1-n/r)/3$. Taking $\varepsilon_1=\gamma/4$, $\varepsilon_2=\gamma/2-(1-n/r)/3$ and $\varepsilon_3=(1-n/r-\gamma)/2$ in (4.11) and (4.9) respectively, we have $\varepsilon_i>0$ $(i=1, 2, 3)$ and

$$\begin{aligned}
&\|L^{k/2+\alpha}A_{j+1}(t)\|_r \\
&\leqq K_{\alpha,0}t^{\gamma-\alpha}+\int_0^t \|L^{\alpha+\gamma/2}e^{-(t-\tau)L}\|_{B(X_r)} \\
&\qquad \times \|L^{k/2-\gamma/2}P(Q(A_j(\tau))+[A_j(\tau), s_j(\tau)])\|_r d\tau \\
&\leqq K_{\alpha,0}t^{\gamma-\alpha}+C\int_0^t (t-\tau)^{-\alpha-\gamma/2}(\|L^{k/2+1/2+\gamma/4}A_j(\tau)\|_r^2 \\
&\qquad +\|L^{k/2+1/3+\gamma/2}A_j(\tau)\|_r^3+\|s_j(\tau)\|_{k+1,r}\|L^{k/2+(1-n/r-\gamma)/2}A_j(\tau)\|_r)d\tau \\
&=K_{\alpha,0}t^{\gamma-\alpha}+CB(1-\alpha-\gamma/2, 3\gamma/2) \\
&\qquad \times(K_{1/2+\gamma/4,j}^2+K_{1/3+\gamma/2,j}^3+M_jK_{(1-n/r-\gamma)/2,j})t^{\gamma-\alpha}\,,
\end{aligned} \tag{4.17}$$

where $B(\cdot, \cdot)$ is the beta function. By Lemmas 4.4 and 4.5 ($\varepsilon_1=\varepsilon_3=\gamma$, $\varepsilon_2=(1-n/r)/6$) and (4.3), we have

$$\begin{aligned}
&\|s_{j+1}(t)\|_{k+1,r} \\
&\quad \leqq C(\|L^{k/2+(n/r-n/p)/2+\gamma}A_j(t)\|_r\|L^{k/2+(1+n/p)/2+\gamma}A_j(t)\|_r \\
&\qquad +\|L^{k/2+(1+n/r)/6+\gamma}A_j(t)\|_r^3+\|L^{k/2+\gamma}A_j(t)\|_r\|s_j(t)\|_{k+1,r} \\
&\qquad +\|L^{k/2+(1+n/r)/2+\gamma}A_{j+1}(t)\|_r)\,.
\end{aligned}$$

It follows from the assumption on j and (4.17) that

$$\begin{aligned}&\|s_{j+1}(t)\|_{k+1,r}\\&\quad\leqq C(K_{(1+n/r)/2+\gamma,0}+K^2_{1/2+\gamma/4,j}+K^3_{1/3+\gamma/2,j}+M_jK_{(1-n/r-\gamma)/2,j}\\&\qquad+K_{(n/r-n/p)/2+\gamma,j}K_{(1+n/p)/2+\gamma,j}+K^3_{(1+n/r)/6+\gamma,j}+M_jK_{\gamma,j})\\&\qquad\times t^{-(1+n/r)/2}\,,\end{aligned}\tag{4.18}$$

$$\begin{aligned}&\|s_{j+1}(t)\|_{k,r}\\&\quad\leqq C(K_{\gamma,0}+K^2_{1/2+\gamma/4,j}+K^3_{1/3+\gamma/2,j}+M_jK_{(1-n/r-\gamma)/2,j}\\&\qquad+K^2_{\gamma,j}+K^3_{\gamma,j}+N_jK_{\gamma,j})\qquad\text{for}\quad t\geqq 0\,.\end{aligned}\tag{4.19}$$

By (4.17)-(4.19), we see that (4.14)-(4.16) are satisfied with j replaced by $j+1$, with

$$\begin{aligned}K_{\alpha,j+1}=K_{\alpha,0}+CB(1-\alpha-\gamma/2,\,3\gamma/2)(&K^2_{1/2+\gamma/4,j}+K^3_{1/3+\gamma/2,j}\\&+M_jK_{(1-n/r+\gamma)/2,j})\,,\end{aligned}\tag{4.20}$$

$$\begin{aligned}M_{j+1}=C(&K_{(1+n/r)/2+\gamma,0}+K^2_{1/2+\gamma/4,j}+K^3_{1/3+\gamma/2,j}+M_jK_{(1-n/r+\gamma)/2,j}\\&+K_{n/2r-n/2p+\gamma,j}K_{1/2+n/2p+\gamma,j}+K^3_{1/6+n/6r+\gamma,j}+M_jK_{\gamma,j})\,,\end{aligned}\tag{4.21}$$

$$\begin{aligned}N_{j+1}=C(&K_{\gamma,0}+K^2_{1/2+\gamma/4,j}+K^3_{1/3+\gamma/2,j}+M_jK_{(1-n/r+\gamma)/2,j}\\&+K^2_{\gamma,j}+K^3_{\gamma,j}+K_{\gamma,j}N_j)\,.\end{aligned}\tag{4.22}$$

Let $S=\{\gamma,\ (1-n/r-\gamma)/2,\ n/2-n/2p+\gamma,\ 1/6+n/6r+\gamma,\ 1/3+\gamma/2,\ 1/2+\gamma/4,\ 1/2+n/2p+\gamma\}$. We define $\{K_j\}_{j=0}^\infty$ and $\{F_j\}_{j=0}^\infty$ by $K_j\equiv\max_{\alpha\in S}K_{\alpha,j}$ and $F_j\equiv\max\{K_j,M_j,N_j\}$. Then it follows from (4.20)-(4.22) that

$$F_{j+1}\leqq C(F_0+F_j^2+F_j^3)\qquad\text{for}\quad j=0,1,\cdots.$$

As is well known, for such sequence $\{F_j\}_{j=0}^\infty$, there exists a positive, monotone decreasing function $F(\lambda)$ of $\lambda>0$ such that $F_j\leqq F(\lambda)$ for all $j=0,1,\cdots$ if $F_0\leqq\lambda$. Moreover, we have $\lim_{\lambda\to 0}F(\lambda)=0$.

Now, we choose $\lambda_1>0$ so that $F(\lambda_1)<1$ and assume that $F_0\leqq\lambda_1$. Under this assumption, we obtain from (4.14)-(4.16)

$$\begin{aligned}&\|L^{k/2+\alpha}A_j(t)\|_r\leqq F(\lambda_1)t^{\gamma-\alpha}\quad(\alpha\in S)\,,\qquad t>0\,,\\&\|s_j(t)\|_{k+1,r}\leqq F(\lambda_1)t^{-(1+n/r)/2}\,,\qquad t>0\,,\\&\|s_j(t)\|_{k,r}\leqq F(\lambda_1)\,,\qquad t\geqq 0\,,\end{aligned}$$

for all $j=0,1,\cdots$. Set $B_j(t)=A_{j+1}(t)-A_j(t)$ and $u_j(t)=s_{j+1}(t)-s_j(t)$. In the similar manner of (4.17)-(4.19), we have by Lemmas 4.4 and 4.5

$$\begin{aligned}&\|L^{k/2+\alpha}B_j(t)\|_r\\&\quad\leqq CF\int_0^t(t-\tau)^{-\alpha-\gamma/2}(\tau^{3\gamma/4-1/2}\|L^{k/2+1/2+\gamma/4}B_{j-1}\|_r\end{aligned}$$

$$+\tau^{\gamma-2/3}\|L^{k/2+1/3+\gamma/2}B_{j-1}\|_r+\tau^{3\gamma/2-1/2+n/2r}\|u_{j-1}\|_{k+1,r}$$
$$+\tau^{-1/2-n/2r}\|L^{k/2+(1-n/r-\gamma)/2}B_{j-1}\|_r)d\tau \tag{4.23}$$

for $\gamma\leqq\alpha<1-\gamma/2$,

$$\begin{aligned}\|u_j(t)\|_{k+1,r} &\leqq CF(t^{n/2p-n/2r}\|L^{k/2+1/2+n/2p+\gamma}B_{j-1}\|_r \\ &\quad +t^{-n/2p-1/2}\|L^{k/2+n/2r-n/2p+\gamma}B_{j-1}\|_r \\ &\quad +t^{-1/3-n/3}\|L^{k/2+1/6+n/6r+\gamma}B_{j-1}\|_r \\ &\quad +\|u_{j-1}\|_{k+1,r}+t^{-1/2-n/2r}\|L^{k/2+\gamma}B_{j-1}\|_r) \\ &\quad +CF\int_0^t(t-\tau)^{(1-n/r-3\gamma)/2-1}(\tau^{3\gamma/4-1/2}\|L^{k/2+1/2+\gamma/4}B_{j-1}\|_r \\ &\quad +\tau^{\gamma-2/3}\|L^{k/2+1/3+\gamma/2}B_{j-1}\|_r+\tau^{3\gamma/2-1/2+n/2r}\|u_{j-1}\|_{k+1,r} \\ &\quad +\tau^{-1/2-n/2r}\|L^{k/2+(1-n/r-\gamma)/2}B_{j-1}\|_r)d\tau \quad \text{(by (4.23))},\end{aligned}$$

$$\begin{aligned}\|u_j(t)\|_{k,r} &\leqq CF(\|u_{j-1}\|_{k,r}+\|L^{k/2+\gamma}B_{j-1}\|_r) \\ &\quad +CF\int_0^t(t-\tau)^{-3\gamma/2}(\tau^{3\gamma/4-1/2}\|L^{k/2+1/2+\gamma/4}B_{j-1}\|_r \\ &\quad +\tau^{\gamma-2/3}\|L^{k/2+1/3+\gamma/2}B_{j-1}\|_r+\tau^{3\gamma/2-1/2+n/2r}\|u_{j-1}\|_{k+1,r} \\ &\quad +\tau^{-1/2-n/2r}\|L^{k/2+(1-n/r-\gamma)/2}B_{j-1}\|_r)d\tau ,\end{aligned}$$

where $F=F(\lambda_1)$. Note that $F^2<F$, since $0<F<1$. By a direct calculation, we have for $j=0$

$$\|L^{k/2+\alpha}B_0(t)\|_r\leqq CB(1-\alpha-\gamma/2,\ 3\gamma/2)Ft^{\gamma-\alpha} \qquad (\gamma\leqq\alpha<1-\gamma/2,\ t>0),$$
$$\|u_0(t)\|_{k+1,r}\leqq CFt^{-1/2-n/2r} \qquad (t>0),$$
$$\|u_0(t)\|_{k,r}\geqq CF \qquad (t\geqq 0).$$

Therefore, by induction, we obtain

$$\|L^{k/2+\alpha}B_j(t)\|_r\leqq(\bar{C}F)^{j+1}t^{\gamma-\alpha} \qquad (\alpha\in S,\ t>0), \tag{4.24}$$
$$\|u_j(t)\|_{k+1,r}\leqq(\bar{C}F)^{j+1}t^{-1/2-n/2r} \qquad (t>0), \tag{4.25}$$
$$\|u_j(t)\|_{k,r}\geqq(\bar{C}F)^{j+1} \qquad (t\geqq 0), \tag{4.26}$$

where $\bar{C}\equiv 8C\max_{\alpha\in S}B(1-\alpha-\gamma/2,\ 3\gamma/2)$ (C; the constant in (4.23)). Moreover, substituting (4.24) into (4.23) again, we get

$$\|L^{k/2+\alpha}B_j(t)\|_r\leqq CB(1-\alpha-\gamma/2,\ 3\gamma/2)(\bar{C}F)^j t^{\gamma-\alpha} \qquad (t>0) \tag{4.27}$$

for $\gamma\leqq\alpha<1-\gamma/2$.

Now, we take $\lambda_*>0$ so that

$$F(\lambda_*)<1/\bar{C} \tag{4.28}$$

and assume that

$$F_0 \leqq \lambda_* . \tag{4.29}$$

Since $L^{k/2+\alpha}A_j(t)=\sum_{i=1}^{j-1} L^{k/2+\alpha}B_i(t)$ and $s_j(t)=\sum_{i=1}^{j-1} u_i(t)$, it follows from (4.25)-(4.27) that there exist $\{A, s\}$:

$$A \in C([0, \infty); D(L^{k/2+\gamma})) \cap C((0, \infty); D(L^{k/2+\alpha})) \qquad (\gamma<\alpha<1-\gamma/2) ,$$
$$s \in C([0, \infty); W^{k,r}(\Omega^0(\mathfrak{g}_E))) \cap C((0, \infty); W^{k+1,r}(\Omega^0(\mathfrak{g}_E)))$$

such that

$$L^{k/2+\alpha}A_j(t) \rightarrow L^{k/2+\alpha}A(t) \textit{ in } L^r(\Omega^1(\mathfrak{g}_E))$$
$$\textit{uniformly } t \in [0, \infty) \textit{ for } 0 \leqq \alpha \leqq \gamma ,$$
$$\textit{uniformly } t \in [\varepsilon, \infty) \textit{ for } \gamma<\alpha<1-\gamma/2 ,$$

$$s_j(t) \rightarrow s(t) \textit{ in } W^{k,r}(\Omega^0(\mathfrak{g}_E)) \textit{ uniformly } t \in [0, \infty)$$
$$\textit{in } W^{k+1,r}(\Omega^0(\mathfrak{g}_E)) \textit{ uniformly } t \in [\varepsilon, \infty)$$

for any $\varepsilon<0$. The limits $\{A, s\}$ satisfy

$$\|L^{k/2+\alpha}A(t)\|_r \leqq F_\alpha(\lambda_*)t^{\gamma-\alpha} \qquad (t>0) , \tag{4.30}$$

$$\|s(t)\|_{k+1,r} \leqq F(\lambda_*)t^{-1/2-n/2r} \qquad (t>0) , \tag{4.31}$$

$$\|s(t)\|_{k,r} \leqq F(\lambda_*) \qquad (t \geqq 0) . \tag{4.32}$$

Note that $F_\alpha(\lambda_*)$ is dominated by $F(\lambda_*)$ for $\alpha \in S$. Hence by Lemma 4.4, we have that $\|Q(A_j(t))\|_{k,r}$ and $\|[A_j(t), s_j(t)]\|_{k,r}$ are dominated by $t^{3\gamma/2-1}$ for all j and that

$$Q(A_j(t)) \rightarrow Q(A(t)) , \qquad [A_j(t), s_j(t)] \rightarrow [A(t), s(t)]$$

in $W^{k,r}(\Omega^1(\mathfrak{g}_E))$. Taking $j \rightarrow \infty$ in (4.13), we see by the Lebesgue dominated convergence theorem that $\{A, s\}$ is a solution of (I.E.$_1$).

Now we shall consider the condition (4.29). Since

$$\|L^{k/2+\alpha}e^{-tL}A_0\|_r \leqq \|L^{\alpha-\gamma}e^{-tL}\|_{B(X_r)}\|L^{k/2+\gamma}A_0\|_r \leqq C_\alpha t^{\gamma-\alpha}\|L^{k/2+\gamma}A_0\|_r ,$$

(4.29) is satisfied if $\|L^{k/2+\gamma}A_0\|_r$ is sufficiently small. Hence we have just proved the existence of λ_0 and the solution $\{A, s\}$ of (I.E.$_1$) with decay properties (4.6) and (4.7).

To see the behaviour of $\{A(t), s(t)\}$ at $t=0$, we need to return to the approximation solution $\{A_j(t), s_j(t)\}$. Since

$$\sup_{0<t} t^{\alpha-\gamma}\|L^{k/2+\alpha}A_0(t)\|_r \leqq \sup_{0<t} t^{\alpha-\gamma}\|L^{\alpha-\gamma}e^{-tL}L^{k/2+\gamma}A_0\|_r$$

and $t^{\alpha-\gamma}L^{\alpha-\gamma}e^{-tL}\to 0$ strongly as $t\downarrow 0$, there is $T_\varepsilon^*>0$ for any $\varepsilon>0$ such that

$$\sup_{0<t<T_\varepsilon^*} t^{\alpha-\gamma}\|L^{k/2+\alpha}A_0(t)\|_r<\varepsilon\ .$$

In the similar manner of (4.17) and (4.18), we see that

$$\sup_{0<t<T_\varepsilon^*} t^{\alpha-\gamma}\|L^{k/2+\alpha}A_j(t)\|_r<C_\alpha\varepsilon\ ,$$
$$\sup_{0<t<T_\varepsilon^*} t^{1/2+n/2r}\|s_j(t)\|_{k+1,r}<C\varepsilon$$

and that $\{t^{\alpha-\gamma}L^{k/2+\alpha}A_j(t)\}_{j=0}^\infty$ and $\{t^{1/2+n/2r}s_j(t)\}_{j=0}^\infty$ are *uniformly convergent sequences* in $L^r(\Omega^1(\mathfrak{g}_E))$ and in $W^{k,r}(\Omega^0(\mathfrak{g}_E))$ for $t\in[0, T_\varepsilon^*]$, respectively. Hence the limit $\{A(t), s(t)\}$ has the properties (4.4) and (4.5) near $t=0$.

(ii) *Uniqueness.* Let $\{\bar{A}, \bar{s}\}$ be another solution of (I.E.$_1$) with properties (4.4) and (4.5). Then we can take a constant $0<\tilde{F}(t_0)<1$ such that

$$\|L^{k/2+\alpha}A(t)\|_r,\ \|L^{k/2+\alpha}\bar{A}(t)\|_r\leqq\tilde{F}(t_0)t^{\gamma-\alpha}\qquad(\alpha\in S-\{\gamma\})\ ,$$
$$\|s(t)\|_{k+1,r},\ \|\bar{s}(t)\|_{k+1,r}\leqq\tilde{F}(t_0)t^{-1/2-n/2r}$$

for all $t\in(0, t_0]$ and that $\tilde{F}(t_0)\to 0$ as $t_0\to 0$. Taking $B=A-\bar{A}$ and $u=s-\bar{s}$, we obtain by induction in the similar manner as above

$$\|L^{k/2+\alpha}B(t)\|_r\leqq 2\tilde{F}(t_0)(2\bar{C}\tilde{F}(t_0))^j t^{\gamma-\alpha}\qquad(\alpha\in S-\{\gamma\})\ , \tag{4.33}$$

$$\|u(t)\|_{k+1,r}\leqq 2\tilde{F}(t_0)(2\bar{C}\tilde{F}(t_0))^j t^{-1/2-n/2r} \tag{4.34}$$

for $t\in(0, t_0]$ and $j=0, 1, \cdots$, where $\bar{C}$ is the same constant in (4.24)-(4.26). In the above, we should note that $\|L^{k/2+\gamma}A_0\|_r$ is sufficiently small and that $\sup_{0\leqq t\leqq t_0}\|L^{k/2+\gamma}B(t)\|_r\leqq\tilde{F}(t_0)$ for small t_0, since $B\in C([0, t_0]; D(L^{k/2+\gamma}))$ with $B(0)=0$.

Now, we choose $t_0>0$ so that $2\bar{C}\tilde{F}(t_0)<1$. Letting $j\to\infty$ in (4.33) and (4.34), we have

$$B(t)\equiv u(t)\equiv 0\qquad\text{on}\quad t\in[0, t_0]\ .$$

Note that $u\in C([0, \infty); W^{k,r}(\Omega^0(\mathfrak{g}_E)))$. Repeating this argument on $[t_0, \infty)$, we find a sequence $t_0<t_1<\cdots$ such that $B(t)\equiv u(t)\equiv 0$ on $[0, t_j]$ for any $j=0, 1, \cdots$. Since $L^{k/2+\alpha}B_0(t)$ and $u(t)$ are continuous functions on $[t_0, \infty)$

with values in $L^r(\Omega^1(\mathfrak{g}_E))$ and $W^{k+1,r}(\Omega^0(\mathfrak{g}_E))$, respectively, there exists $\eta>0$ such that $t_{j+1}-t_j\geqq\eta$ for all j (see Fujita-Kato [5, p. 286, Proposition I]).

(iii) *Differentiability of* $A(t)$. It remains to show that $A\in C((0,\infty); D(L^{k/2+1}))\cap C^1((0,\infty); D(L^{k/2}))$ and that $\{A, s\}$ satisfies the first equation of (Eq_1). By Lemma 4.2 (i) and the general theory of the holomorphic semi-group (see Tanabe [16, Theorem 3.3.4]), it suffices to show that $L^{k/2}P(Q(A)+[A, s])$ is a Hölder continuous function of $t\in(0,\infty)$ with values in X_r. By Lemma 4.4, we may prove the following:

LEMMA 4.6. *Let* $\{A, s\}$ *be the solution of* (I.E.$_1$) *constructed as above. Then for any* $\varepsilon>0$, *we have*

(1) $L^{k/2+\alpha}A$ *is a uniformly Hölder continuous functions with values in* X_r *on* $[\varepsilon,\infty)$ *for* $0\leqq\alpha<1-\gamma/2$;

(2) s *is a uniformly Hölder continuous function with values in* $W^{k+1,r}(\Omega^0(\mathfrak{g}_E))$ *on* $[\varepsilon,\infty)$.

PROOF. (1) By Lemma 4.1, we have for $\eta>0$ and $0<\theta<1$ $(e^{-\eta L}-1)L^{-\theta}\in \boldsymbol{B}(X_r)$ and

$$\|(e^{-\eta L}-1)L^{-\theta}\|_{B(X_r)}\leqq C_\theta\eta^\theta\ . \tag{4.35}$$

It suffices to prove the assertion for

$$\tilde{A}(t):=\int_0^t e^{-(t-\tau)L}P(Q(A(\tau))+[A(\tau), s(\tau)])d\tau\ .$$

By a direct calculation, we have for $\eta>0$

$$\begin{aligned}&L^{k/2+\alpha}\tilde{A}(t+\eta)-L^{k/2+\alpha}\tilde{A}(t)\\&=\int_t^{t+\eta}L^{k/2+\alpha}e^{-(t+\eta-\tau)L}P(Q(A(\tau))+[A(\tau), s(\tau)])d\tau\\&\quad+\int_0^t L^{k/2+\alpha}(e^{-\eta L}-1)e^{-(t-\tau)L}P(Q(A(\tau))+[A(\tau), s(\tau)])d\tau\\&\equiv I_1^\alpha(t)+I_2^\alpha(t)\ .\end{aligned}$$

In the similar manner of (4.17), we get by (4.30) and (4.31)

$$\begin{aligned}\|I_1^\alpha(t)\|_r&\leqq CF(\lambda_*)\int_t^{t+\eta}(t+\eta-\tau)^{-\alpha-\gamma/2}\tau^{3\gamma/2-1}d\tau\\&=CF(\lambda_*)B(1-\alpha-\gamma/2, 3\gamma/2)\eta^{\gamma-\alpha}\\\text{or}\quad&=CF(\lambda_*)\varepsilon^{3\gamma/2-1}(1-\alpha-\gamma/2)^{-1}\eta^{1-\alpha-\gamma/2}\ ,\end{aligned}$$

according to the case $0\leqq\alpha<\gamma$ or $\gamma\leqq\alpha<1-\gamma/2$. Taking $0<\theta<1-\alpha-\gamma/2$, we have by (4.35)

$$\begin{aligned}\|I_2^\alpha(t)\|_r &\leqq \int_0^t \|(e^{-\eta L}-1)L^{-\theta}\|_{B(X_r)}\|L^{\alpha+\theta+\gamma/2}e^{-(t-\tau)L}\|_{B(X_r)} \\ &\qquad\times \|L^{k/2-\gamma/2}P(Q(A(\tau))+[A(\tau), s(\tau)])\|_r d\tau \\ &\leqq C\eta^\theta F(\lambda_*)B(1-\alpha-\theta-\gamma/2, 3\gamma/2)t^{-\alpha-\theta+\gamma} \\ &\leqq CF(\lambda_*)B(1-\alpha-\theta-\gamma/2, 3\gamma/2)\varepsilon^{-\alpha-\theta+\gamma}\eta^\theta\end{aligned}$$

for all $t\in[\varepsilon, \infty)$. Hence we obtain the assertion of (1).

(2) Similarly, it follows from (4.30) and (4.31) that for $\eta>0$

$$\begin{aligned}&\|s(t+\eta)-s(t)\|_{k+1,r} \\ &\quad\leqq C\sup_{0\leqq\tau}\|L^{k/2+\gamma}A(\tau)\|_r\|s(t+\eta)-s(t)\|_{k+1,r} \\ &\qquad+CF(\lambda_*)\{\varepsilon^{-1/2-n/2r}\|L^{k/2+\gamma}(A(t+\eta)-A(t))\|_r \\ &\qquad\qquad+\varepsilon^{-n/2p-1/2}\|L^{k/2+(n/r-n/p)/2+\gamma}(A(t+\eta)-A(t))\|_r \\ &\qquad\qquad+\varepsilon^{n/2p-n/2r}\|L^{k/2+(1+n/p)/2+\gamma}(A(t+\eta)-A(t))\|_r \\ &\qquad\qquad+\varepsilon^{-1/2-n/2r}\|L^{k/2+(1+n/r)/6+\gamma}(A(t+\eta)-A(t))\|_r\} \\ &\qquad+C\|L^{k/2+\gamma}(A(t+\eta)-A(t))\|_r\,.\end{aligned}$$

Since $C\sup_{0\leqq\tau}\|L^{k/2+\gamma}A(\tau)\|_r<CF(\lambda_*)<1$ by (4.30), it follows from the above inequality and the assertion (1) that s has the desired property.

§5. Proof of Theorem C.

In this section, we consider the case when $M=\boldsymbol{R}^n$ and ∇ is the flat connection. As is mentioned in section 3, we can obtain the asymptotic stability by solving (Eq_2). Our result now reads:

PROPOSITION 5.1. *Let $m=2, 3, \cdots$ and let $A_0\in W^{m,n}(\Omega^1_{0,*}(\mathfrak{g}_E))$. There exists a positive constant λ_0 such that if $\|A_0\|_{m,n}\leqq\lambda_0$, there is a unique solution $\{A, s\}$ of (Eq_2) with*

$$\begin{aligned}A\in C([0,\infty); W^{m,n}(\Omega^1_{0,*}(\mathfrak{g}_E)))\cap C^1((0,\infty); W^{m,n}(\Omega^1_{0,*}(\mathfrak{g}_E))) \\ \cap C((0,\infty); W^{m+2,n}(\Omega^1_{0,*}(\mathfrak{g}_E)))\,, \\ s\in C([0,\infty); W^{m-1,n}(\Omega^0(\mathfrak{g}_E)))\cap C((0,\infty); W^{m,n}(\Omega^0(\mathfrak{g}_E)))\end{aligned}$$

satisfying

$$t^{(1-n/r)/2}A\in BC([0,\infty); W^{m,r}(\Omega^1_{0,*}(\mathfrak{g}_E))) \quad \textit{for} \quad n<r\,, \tag{5.1}$$

$$t^{1/2}A\in BC([0,\infty); W^{m+1,r}(\Omega^1_{0,*}(\mathfrak{g}_E)))\,, \tag{5.2}$$

$$t^{1/2}s\in BC((0,\infty); W^{m,n}(\Omega^0(\mathfrak{g}_E))) \tag{5.3}$$

all with values zero at $t=0$ *(*$BC([0,\infty); X)$*; the set of continuous, uniformly bounded functions on* $[0,\infty)$ *with values in* X*). Moreover,* A *has the* $W^{m,n}$*-decay property*

$$\|A(t)\|_{m,n}\to 0 \qquad as \quad t\to\infty . \tag{5.4}$$

To show this proposition, we shall make use of the implicit function theorem combined with the (L^r, L^q)-estimates for the solutions of the heat equation. Since $L^\nabla=-\Delta=-\sum_{j=1}^n(\partial/\partial x^j)^2$ commutes with the projection operator P, the evolution operator e^{-tL} can be represented explicitly as

$$e^{-tL}A_{ia}^{\,b}(x)=\frac{1}{(4\pi t)^{n/2}}\int_{R^n}e^{-|x-y|^2/4t}A_{ia}^{\,b}(y)dy$$

for $A=(A_{ia}^{\,b}(x))\in X_r$ $(1<r<\infty)$. Then we have

LEMMA 5.2 ((L^n, L^q)-estimates). *For* $q\leqq n$*, there is a constant* $C=C(q, n)$ *such that*

$$\|e^{-tL}A\|_{m,n}\leqq Ct^{-(n/q-1)/2}\|A\|_{m,q} ,$$
$$\|e^{-tL}A\|_{m+1,n}\leqq Ct^{-n/2q}\|A\|_{m,q}$$

for all $A\in W^{m,n}(\Omega^1_{0,*}(\mathfrak{g}_E))$.

PROOF OF PROPOSITION 5.1. In the similar manner of the proof of Proposition 4.3, we construct at first the following integral equation:

$$\begin{aligned} A(t)&=e^{-tL}A_0-\int_0^t e^{-(t-\tau)L}P(Q_1(A(\tau))+Q_2(A(\tau))+[A(\tau), s(\tau)])d\tau ,\\ s(t)&=d\Gamma*(Q_1(A(t))+Q_2(A(t))+[A(t), s(t)]) , \end{aligned} \tag{I.E.$_2$}$$

where $Q(A)=Q_1(A)+Q_2(A)$ with $Q_1(A)\sim A\cdot\partial A$ and $Q_2(A)\sim A^3$. Now, we define the function spaces Y and Z and the norms $\|\ \|_Y$ and $\|\ \|_Z$ as follows:

$$Y\equiv\{A\in BC([0,\infty); W^{m,n}(\Omega^1_{0,*}(\mathfrak{g}_E))); t^{1/2}A\in BC([0,\infty); W^{m+1,n}(\Omega^1_{0,*}(\mathfrak{g}_E)))\} ,$$
$$Z\equiv\{s\in BC([0,\infty); W^{m-1,n}(\Omega^0(\mathfrak{g}_E))); t^{1/2}s\in BC([0,\infty); W^{m+1,n}(\Omega^0(\mathfrak{g}_E)))\} ,$$

$$\|A\|_Y\equiv\sup_{t\geqq 0}\|A(t)\|_{m,n}+\sup_{t\geqq 0}t^{1/2}\|A(t)\|_{m+1,n} ,$$
$$\|s\|_Z\equiv\sup_{t\geqq 0}\|s(t)\|_{m-1,n}+\sup_{t\geqq 0}t^{1/2}\|s(t)\|_{m,n} .$$

Then Y and Z are Banach spaces with the norms $\|\ \|_Y$ and $\|\ \|_Z$, respectively. We set $W\equiv Y\times Z$. We can consider W as Banach space by inducing the product topology.

Let us define a map f on $W^{m,n}(\Omega^1_{0,*}(\mathfrak{g}_E))\times W$ by $f(A_0, A, s)\equiv\{B, u\}$:

$$B(t)=A(t)-e^{-tL}A_0+\int_0^t e^{-(t-\tau)L}P(Q_1(A(\tau))+Q_2(A(\tau))+[A(\tau), s(\tau)])d\tau\ , \qquad (5.5)$$

$$u(t)=s(t)-d\Gamma*(Q_1(A(t))+Q_2(A(t))+[A(t), s(t)])\ , \qquad (5.6)$$

for $A_0\in W^{m,n}(\Omega^1_{0,*}(\mathfrak{g}_E))$ and $\{A, s\}\in W$. Then we have

LEMMA 5.3. *f is a continuous map from $W^{m,n}(\Omega^1_{0,*}(\mathfrak{g}_E))\times W$ into W. For each $A_0\in W^{m,n}(\Omega^1_{0,*}(\mathfrak{g}_E))$, $f(A_0, \cdot)$ is a map of class C^1 from W into itself.*

PROOF. Let B and u be the functions defined by (5.5) and (5.6), respectively. We shall prove at first $\{B, u\}\in W$.

(i) $B\in Y$. Since $e^{-tL}A_0\in Y$ by Lemma 5.2, it suffices to show that $\tilde{B}\in Y$, where $\tilde{B}(t)=\int_0^t e^{-(t-\tau)L}P(Q_1(A)+Q_2(A)+[A, s])d\tau$. By the interpolation inequality (see Tanabe [16, Lemma 1.2.2])

$$\|A\|_{m,r}\leqq C\|A\|_{m,n}^{n/r}\|A\|_{m+1,n}^{1-n/r} \qquad \text{for} \quad n\leqq r\ , \qquad (5.7)$$

we see $t^{(1-n/r)/2}A\in BC([0, \infty); W^{m,r}(\Omega^1_{0,*}(\mathfrak{g}_E)))$ for $A\in Y$. In particular, we get $\sup_{t\geqq 0} t^{1/4}\|A(t)\|_{m,2n}\leqq C\|A\|_Y$. Hence by Lemma 5.2 and the Hölder inequality, we have

$$\begin{aligned}\|\tilde{B}(t)\|_{m,n}&\leqq C\int_0^t (t-\tau)^{-1/2}(\|A(\tau)\|_{m,n}\|A(\tau)\|_{m+1,n}+\|A(\tau)\|_{m,n}\|s(\tau)\|_{m,n})d\tau\\ &\quad+C\int_0^t (t-\tau)^{-1/4}\|A(\tau)\|_{m,2n}^3 d\tau\\ &\leqq 2C\beta(\|A\|_Y^2+\|A\|_Y\|s\|_Z+\|A\|_Y^3) \qquad (5.8)\end{aligned}$$

for all $t\geqq 0$, where $\beta\equiv\max\{B(1/2, 1/2), B(3/4, 1/4)\}$. Similarly, we have

$$\|\tilde{B}(t)\|_{m+1,n}\leqq C(\|A\|_Y^2+\|A\|_Y^3+\|A\|_Y\|s\|_Z)B(1/4, 1/4)t^{-1/2}\ . \qquad (5.9)$$

(ii) $s\in Z$. It follows from the Hardy-Littlewood-Sobolev inequality (see Reed-Simon [15, p. 31]) that $d\Gamma*$ is a bounded operator from $W^{m,n/2}(\Omega^0(\mathfrak{g}_E))$ into $W^{m,n}(\Omega^0(\mathfrak{g}_E))$. We have therefore by the Hölder inequality and (5.7)

$$\|s(t)\|_{m,n}\leqq C(\|A\|_Y^2+\|A\|_Y^3+\|A\|_Y\|s\|_Z)t^{-1/2}\ , \qquad (5.10)$$

$$\|s(t)\|_{m-1,n}\leqq C(\|A\|_Y^2+\|A\|_Y^3+\|A\|_Y\|s\|_Z)\ . \qquad (5.11)$$

Since the continuity and the differentiability of f follow easily from (5.8)-(5.11), we may omit details. This completes the proof.

Since $f(0, 0, 0)=\{0, 0\}$ and the Fréchet derivative $f_{\{A,s\}}(0, 0, 0)=$identity on W (see (5.8)-(5.11)), it follows from the implicit function theorem that there is a *unique continuous* mapping w in a neighbourhood $U_{\lambda}^{m,n}\equiv \{A_0 \in W^{m,n}(\Omega^1_{0,*}(\mathfrak{g}_E));\ \|A_0\|_{m,n}<\lambda\}$ of 0; $w: U_{\lambda}^{m,n}\to W$ such that

$$w(0)=\{0, 0\} \quad \text{and} \quad f(A_0, w(A_0))=\{0, 0\} \tag{5.12}$$

Representing $w(A_0)=\{A(A_0), s(A_0)\}$, we see by (5.12) that $\{A(A_0), s(A_0)\}$ is a unique solution of (I.E.$_2$). As in the preceding section, we can show $\{A(A_0), s(A_0)\}$ is actually the solution of (Eq$_2$) with initial value A_0 by using the decay properties (5.1)-(5.3).

Now it remains to show the $W^{m,n}$-decay property (5.4). Since the map $A_0\to A(A_0)$ is a continuous one from $W^{m,n}(\Omega^1_{0,*}(\mathfrak{g}_E))$ into Y and since the space $\{B\in \Omega^1_{0,*}(\mathfrak{g}_E);\ B$ has a compact support in $M\}$ is dense in $W^{m,n}(\Omega^1_{0,*}(\mathfrak{g}_E))$, there is $\bar{A}_0 \in \Omega^1_{0,*}(\mathfrak{g}_E)$ with compact support for any $\varepsilon>0$ such that

$$\sup_{t\geqq 0} \|A(A_0)(t)-A(\bar{A}_0)(t)\|_{m,n}<\varepsilon . \tag{5.13}$$

On the other hand, for such a solution $A(\bar{A}_0)$, we can show $\|A(\bar{A}_0)(t)\|_{m,n}\to 0$ as $t\to\infty$. See, e.g., Kato [10, Theorem 4]. Hence by (5.13)

$$\begin{aligned}&\lim_{t\to\infty}\sup \|A(A_0)(t)\|_{m,n}\\ &\quad \leqq \sup_{t>0} \|A(A_0)(t)-A(\bar{A}_0)(t)\|_{m,n}+\lim_{t\to\infty} \|A(\bar{A}_0)(t)\|_{m,n}<\varepsilon .\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we obtain the desired result.

Appendix.

PROOF OF LEMMA 4.1. When M is of type (II) and ∇ is the flat connection, L_r is essentially equal to the Stokes operator for incompressible fluids. Therefore, the assertion of this Lemma follows from Giga [8]. We may prove another case.

Since ∇ is a strictly positive Y-M connection, L_2 is a positive definite self-adjoint operator in X_2 and the assertion of this Lemma is valid for $r=2$. Now we shall prove for $r>2$.

For a moment, let us assume that there is a constant $\mu_r>0$ *such that* $\{\mathrm{Re}\,\lambda\geqq\mu_r\}\subset\rho(-L_r)$ *and* (4.1) *is satisfied for* $\mathrm{Re}\,\lambda\geqq\mu_r$. By [16, Remark 3.3.2], we may show for $\mathrm{Re}\,\lambda\geqq 0$ with $|\lambda|\leqq R_r$ $(R_r>0)$. Suppose the contrary. Then for each $j=1, 2, \cdots$, there exist $A_j\in D(L_r)$ with $\|A_j\|_r=1$ and $\mathrm{Re}\,\lambda_j\geqq 0$ with $|\lambda_j|\leqq R$ such that $(1+|\lambda_j|)\geqq j\|(L_r+\lambda_j)A_j\|_r$. Since $|\lambda_j|\leqq R_r$ for all j, we have

$$(L_r+\lambda_j)A_j \to 0 \quad \text{in } X_r \quad \text{as } j\to\infty . \tag{A.1}$$

Moreover, since

$$\|L_rA_j\|_r \leqq \|(L_r+\lambda_j)A_j\|_r + |\lambda_j|\,\|A_j\|_r \leqq \|(L_r+\lambda_j)A_j\|_r + R_r \quad (\text{by } \|A_j\|_r \equiv 1) ,$$

we see $\{L_rA_j\}_{j=1}^{\infty}$ is a bounded sequence in X_r.

On the other hand, it follows from (3.14) and a priori estimate for L^{∇} that

$$\|A_j\|_{2,r} \leqq C(\|L_rA_j\|_r + \|A_j\|_r)$$

with C independent of j. Therefore $\{A_j\}_{j=1}^{\infty}$ is a bounded sequence in $W^{2,r}(\Omega^1(\mathfrak{g}_E))$. By the Rellich theorem, there exist a subsequence of $\{A_j\}_{j=1}^{\infty}$, which we denote by $\{A_j\}_{j=1}^{\infty}$ itself for simplicity, and $A\in W^{2,r}(\Omega^1(\mathfrak{g}_E))$ such that $A_j\to A$ strongly in $W^{1,r}(\Omega^1(\mathfrak{g}_E))$. Obviously $\|A\|_r=1$. Nevertheless, since (4.1) is true for $r=2$, we see

$$(1+|\lambda_j|)\|A_j\|_2 \leqq C\|(L_2+\lambda_j)A_j\|_2 \leqq C\|(L_r+\lambda_j)A_j\|_r$$

with C independent of j. By (A.1), $A_j\to 0$ in X_2 and hence $A=0$. This contradicts $\|A\|_r=1$.

Now we prove the existence of $\mu_r>0$ as above. For $A\in D(L_r)$ and $\operatorname{Re}\lambda\geqq 0$, set $B\equiv(L_r+\lambda)A$. Then we have

$$L_r^{\nabla}A+\lambda A = B+dV , \qquad \text{where} \quad V(x)=(d^{\nabla}G(x,\cdot),\, L_r^{\nabla}A) . \tag{A.2}$$

Since $\delta^{\nabla}A=0$, we get by the Ricci formula

$$\begin{aligned} V(x) &= (d^{\nabla}G^{\nabla}(x,\cdot),\, \delta^{\nabla}d^{\nabla}A+[R^{\nabla}, A]) \\ &= ([R^{\nabla}, G^{\nabla}(x,\cdot)],\, d^{\nabla}A)+(d^{\nabla}G^{\nabla}(x,\cdot),\, [R^{\nabla}, A]) \\ &= (\delta^{\nabla}[R^{\nabla}, G^{\nabla}(x,\cdot)],\, A)+(d^{\nabla}G^{\nabla}(x,\cdot),\, [R^{\nabla}, A]) . \end{aligned}$$

As in (3.14), we have

$$\|d^{\nabla}V\|_r \leqq C\|R^{\nabla}\|_{1,\infty}\|A\|_r . \tag{A.3}$$

Since ∇ is strictly positive, it follows from (A.2) that

$$(1+|\lambda|)\|A\|_r+\|\nabla^2A\|_r \leqq C\|B+d^{\nabla}V\|_r ,$$

with C independent of λ or A. Hence by (A.3)

$$(1+|\lambda|-C\|R^{\nabla}\|_{1,\infty})+\|\nabla^2A\|_r \leqq C\|B\|_r .$$

Taking $\mu_r\equiv 2C\|R^{\nabla}\|_{1,\infty}$, we obtain

$$\|A\|_r \leqq 2C(1+|\lambda|)^{-1}\|B\|_r \quad \text{for} \quad \mathrm{Re}\,\lambda \geqq \mu_r .$$

References

[1] T. AUBIN, *Nonlinear Analysis on Manifolds. Monge-Ampere Equations*, Springer (1982).

[2] J. BERGH and J. LOFSTORM, *Interpolation Spaces. An Introduction*, Springer (1976).

[3] J. BOURGUIGNON and H. B. LAWSON, JR., Stability and isolation phenomena for Yang-Mills fields, Comm. Math. Phys., **79** (1981), 189–230.

[4] D. G. EBIN, *Espace des mêtriques riemanniennes et mouvement des fluides via les variétés d'applications*, Cent. Math. Ecole Poly. et Univ. Paris VII (1972).

[5] H. FUJITA and T. KATO, On the Navier-Stokes initial value problem I, Arch. Rational Mech. Anal., **16** (1964), 269–315.

[6] D. FUJIWARA, On the asymptotic behaviour of the Green operators for elliptic boundary problems and the pure imaginary powers of some second order operators, J. Math. Soc. Japan, **21** (1969), 481–521.

[7] D. FUJIWARA and H. MORIMOTO, An L_r-theorem of the Helmholtz decomposition of vector fields, J. Fac. Sci. Univ. Tokyo Sect. I, **24** (1977), 685–700.

[8] Y. GIGA, Analyticity of semigroup generated by the Stokes operator in L_r-spaces, Math. Z., **178** (1981), 297–329.

[9] Y. GIGA and T. MIYAKAWA, Solution in L_r of the Navier-Stokes initial value problem, Arch. Rational Mech. Anal., **89** (1985), 267–281.

[10] T. KATO, Strong L^p-solution of the Navier-stokes equations in $\boldsymbol{R}^m$, with applications to the weak solutions, Math. Z., **187** (1984), 471–480.

[11] K. KONO and T. NAGASAWA, Weakly asymptotical stability of Yang-Mills' gradient flow, Tokyo J. Math., **11** (1988), 339–357.

[12] S. G. KREIN, *Linear Differential Equations in Banach Space*, Amer. Math. Soc. (1971).

[13] H. B. LAWSON, JR., *The Theory of Gauge Fields in Four Dimensions*, C. B. M. S. Regional Conf. Series, **58** (1985), Amer. Math. Soc.

[14] H. OMORI, *Infinite Dimensional Lie Transformation Groups*, Lecture Notes in Math., **427** (1974), Springer.

[15] M. REED and B. SIMON, *Method of Modern Mathematical Physics, vol II: Fourier-Analysis, Self-Adjointness*, Academic Press (1975).

[16] H. TANABE, *Equation of Evolution*, Pitman (1979).

Present Address:
HIDEO KOZONO
DEPARTMENT OF APPLIED PHYSICS, FACULTY OF ENGINEERING, NAGOYA UNIVERSITY
NAGOYA 464, JAPAN

YOSHIAKI MAEDA
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY, KEIO UNIVERSITY
YOKOHAMA 223, JAPAN