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On an Isomorphism between Specht Module and Left Cell of $\mathfrak{S}_n$

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Dedicated to Professor Nagayoshi Iwahori on his 60th birthday

Introduction.

The irreducible representations of the symmetric group $\mathfrak{S}_n$ are classically known and they are parametrized by the partitions of n. One of their realizations is so called Specht module which is defined over $\boldsymbol{Z}$ and has a natural base (Young's natural base). On the other hand, Kazhdan and Lusztig [3] constructed another realization, the W-graph representation which is obtained from a left cell of $\mathfrak{S}_n$ and it also has a natural $\boldsymbol{Z}$-base (the vertices of the W-graph). We give in this paper an explicit isomorphism between the above two modules and show that the base change matrix is uni-triangular for some ordering of the base. Using this isomorphism, if a partition λ satisfies certain condition, we can construct the W-graph of the left cell of $\mathfrak{S}_n$ (without some edges which connect vertices having the same I-set) corresponding to λ not using the Kazhdan-Lusztig polynomials $P_{y,w}$ but using the relations in Specht module (the Garnir relations) inductively.

§1. The λ-diagram.

1.1. Notations. Let $P(n)$ be the set of partitions $\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_r)$, where $\lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_r>0$ and $\sum_{i=1}^{r}\lambda_i=n$. For a partition λ, the Young frame of λ is the arrangement of n squares; the first row λ_1, the second row $\lambda_2, \cdots$, the last row λ_r parts, and line up to the left. Young tableau of shape λ has the frame λ and each square is numbered from 1 to n. A Young tableau of shape λ is called standard if its numbering is increasing from left to right in each row and from top to bottom in

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each column. We write Tab_λ the set of all Young tableaux of shape λ and STab_λ the set of all standard Young tableaux of shape λ. The transposition of the partition λ is denoted by ${}^t\lambda$. For a Young tableau D of shape λ, ${}^tD$ denotes the transposition of D whose shape is ${}^t\lambda$. We denote by $D_{\lambda,\mathrm{Top}}$ the standard Young tableau of shape λ, whose numbering is 1 to λ_1 for the first row, λ_1+1 to $\lambda_1+\lambda_2$ for the second row and so on. We set $D_{\lambda,\mathrm{Bot}}={}^t(D_{{}^t\lambda,\mathrm{Top}})$. For a Young tableau D, we put

$$I(D)=\{i \mid 1\leqq i\leqq n-1,\ i+1 \text{ is in a lower position than } i \text{ in } D\}\,,$$
$$I_0(D)=\{i\in I(D) \mid i+1 \text{ is in the left side of } i \text{ in } D\}\,,$$
$$I_1(D)=\{i\in I(D) \mid i+1 \text{ is directly below } i \text{ in } D\}\,.$$

Then we have

LEMMA 1. *For a standard Young tableau D of shape λ, $\lambda\in P(n)$,*
(1) $I(D)=I_0(D)\amalg I_1(D)$ *(disjoint union).*
(2) $I(D)\amalg I({}^tD)=\{1, 2, \cdots, n-1\}$.
(3) $I_0(D)=\varnothing$ *if and only if* $D=D_{\lambda,\mathrm{Bot}}$.
(4) $I_0({}^tD)=\varnothing$ *if and only if* $D=D_{\lambda,\mathrm{Top}}$.

PROOF. (1) The equality follows immediately from the definitions.

(2) We see easily that $i+1$ is either in A or in B (see Fig. 1). Hence $i+1\in A$ iff $i\in I({}^tD)$ and $i+1\in B$ iff $i\in I(D)$.

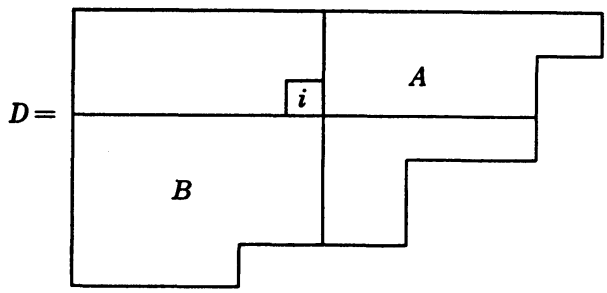


FIGURE 1

(3) The if part clearly follows. On the other hand, if $I_0(D)=\varnothing$, $i+1$ must be directly below i or in the right side of i, hence D must have the form $D_{\lambda,\mathrm{Bot}}$.

(4) Note that ${}^tD_{\lambda,\mathrm{Top}}=D_{{}^t\lambda,\mathrm{Bot}}$. Then (4) follows from (3).

1.2. The Robinson-Schensted correspondence. The following fact is well known.

PROPOSITION 1 (cf. [4]). *There is a bijection from $\mathfrak{S}_n$ to $\amalg_{\lambda\in P(n)}(\mathrm{STab}_\lambda\times\mathrm{STab}_\lambda)$ which is called Robinson-Schensted's correspondence. By this, if x corresponds to $(P(x), Q(x))$, then $P(x^{-1})=Q(x)$.*

We denote by s_i the transposition $(i, i+1)\in\mathfrak{S}_n$. The set $S=\{s_i\}_{1\leq i\leq n-1}$ generates $\mathfrak{S}_n$ as a Coxeter group. Let $l(x)$ be the length of $x\in\mathfrak{S}_n$ with respect to this generator set S.

LEMMA 2. *For x in $\mathfrak{S}_n$, we have*
(1) $l(s_ix)=l(x)-1$ *if and only if* $i\in I(P(x))$,
(2) $l(xs_i)=l(x)-1$ *if and only if* $i\in I(Q(x))$.

PROOF. (1) Suppose $x\in\mathfrak{S}_n$ has a form $x=\begin{pmatrix}1 & 2\cdots n\\ a_1 & a_2\cdots a_n\end{pmatrix}$. Then $l(s_ix)=l(x)-1$ iff i appears after $i+1$ in $a_1, a_2, \cdots, a_n$, which is equivalent to the condition $i\in I(P(x))$ by the construction of $P(x)$.
(2) As $l(x^{-1})=l(x)$, (2) follows from (1).

1.3. For $D\in\mathrm{STab}_\lambda$, we define $x(D)\in\mathfrak{S}_n$ by

$$x(D)=\begin{pmatrix}1 & 2 & 3 & \cdots & n\\ a_1 & a_2 & a_3 & \cdots & a_n\end{pmatrix} \qquad \text{if} \quad D=\begin{array}{|ll}a_s & \\ \vdots & \\ a_2 & \vdots \\ a_1 & a_{s+1}\end{array}$$

LEMMA 3. *For $D\in\mathrm{STab}_\lambda$, we have*
(1) *By the Robinson-Schensted correspondence, $x(D)$ corresponds to $(D, D_{\lambda,\mathrm{Bot}})$.*
(2) $l(s_ix(D))=l(x(D))-1$ *if and only if* $i\in I(D)$.

PROOF. (1) It is clear from the definition of $x(D)$ and the construction of the Robinson-Schensted correspondence.
(2) Clear from (1) and Lemma 2.

LEMMA 4.
(1) $l(x(D_{\lambda,\mathrm{Top}}))=\dfrac{n^2-\sum\lambda_i^2}{2}-\dfrac{\sum(i-1)\lambda_i(\lambda_i-1)}{2}$.
(2) $l(x(D_{\lambda,\mathrm{Bot}}))=\sum(i-1)\lambda_i$.

1.4. For $\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_r)\in P(n)$, we put

$$N=l(x(D_{\lambda,\mathrm{Top}}))-l(x(D_{\lambda,\mathrm{Bot}}))=\frac{n(n+1)}{2}-\frac{\sum i\lambda_i(\lambda_i+1)}{2}.$$

For $0\leqq k\leqq N$, we define a subset $\mathscr{D}_k^\lambda$ of STab_λ inductively from N to 0. We put

$$\mathscr{D}_N^\lambda=\{D_{\lambda,\mathrm{Top}}\}\ ,$$

and for $0\leqq k<N$, we put

$$\mathscr{D}_k^\lambda=\{D\in \mathrm{STab}_\lambda \mid D=s_jD' \text{ for some } D'\in\mathscr{D}_{k+1}^\lambda,\ j\in I_0(D')\}\ ,$$

where s_jD' is the standard tableau interchanging j and $j+1$ in D.

PROPOSITION 2.
(1) $\coprod_{0\leq k\leq N}\mathscr{D}_k^\lambda=\mathrm{STab}_\lambda$ (*disjoint union*).
(2) $\mathscr{D}_0^\lambda=\{D_{\lambda,\mathrm{Bot}}\}$.

PROOF. For $D\in\mathscr{D}_k^\lambda$, we can show inductively that

$$l(x(D))=l(x(D_{\lambda,\mathrm{Top}}))-(N-k)\ .$$

Hence if $k\neq l$, $\mathscr{D}_k^\lambda\cap\mathscr{D}_l^\lambda=\varnothing$ i.e. disjoint. For $D\in\mathrm{STab}_\lambda$, compare it with $D_{\lambda,\mathrm{Top}}$ for the top row from left to right then for the second row in the same manner and so on. If the first difference is a in D, we must have $a-1\in I_0({}^tD)$ and $D'=s_{a-1}D\in\mathrm{STab}_\lambda$. For this D', instead of D, do the same operation. Continuing this, we finally get $D_{\lambda,\mathrm{Top}}$. If it takes m times, then $D\in\mathscr{D}_{N-m}^\lambda$. If we do the same operation on ${}^tD$ (compare with ${}^tD_{\lambda,\mathrm{Bot}}$), we get ${}^tD_{\lambda,\mathrm{Bot}}$ after k times interchange. Transposing all of them, if $D\in\mathscr{D}_l^\lambda$ then $D_{\lambda,\mathrm{Bot}}\in\mathscr{D}_{l-k}^\lambda$. Therefore we get (1) and (2).

For $D\in\mathrm{STab}_\lambda$, we define $\mathrm{ht}(D)=k$ if $D\in\mathscr{D}_k^\lambda$.

1.5. Partial orders on STab_λ. We define three partial orders on STab_λ.

(1) For $D, D'\in\mathrm{STab}_\lambda$, if $\mathrm{ht}(D')=\mathrm{ht}(D)+1$ and $D'=(i, j)D$, for some transposition (i, j), we define $D<D'$. Let $\leqq$ be the transitive closure of this relation, thus $\leqq$ is a partial order on STab_λ.

(2) We define a partial order $\trianglelefteq$ on STab_λ as follows. For D, $D'\in\mathrm{STab}_\lambda$,

$$D\trianglelefteq D' \text{ if and only if } m_{ir}(D)\leqq m_{ir}(D') \text{ for all } i \text{ and all } r\ ,$$

where $m_{ir}(D)=\#\{j \mid j\leqq i \text{ and } j \text{ appears in the first } r \text{ rows of } D\}$ (cf. James [2; 3.10]). This order can be interpreted as follows. For $D\in\mathrm{STab}_\lambda$ and $1\leqq i\leqq n$, we define $D_{\leq i}$ the sub-diagram of D whose entries are less than or equal to i. Then this is also a standard Young tableau of size i. We also define for a Young tableau D, $\lambda(D)=(\lambda_1(D), \lambda_2(D), \cdots, \lambda_r(D))$ where $\lambda_i(D)$ is the length of i-th row of D, hence D has a Young frame of shape $\lambda(D)$. For two partitions λ, $\mu\in P(n)$, we define $\lambda\trianglelefteq\mu$ iff

$\lambda_1 \leqq \mu_1$, $\lambda_1+\lambda_2 \leqq \mu_1+\mu_2, \cdots, \lambda_1+\lambda_2+\cdots+\lambda_s \leqq \mu_1+\mu_2+\cdots+\mu_s$ and $r \geqq s$ $(\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_r),\ \mu=(\mu_1, \mu_2, \cdots, \mu_s))$. Then $D \trianglelefteq D'$ iff $\lambda(D_{\leqq i}) \leqq \lambda(D'_{\leqq i})$ for all $1 \leqq i \leqq n$.

(3) We define the third partial order $\lesseqqgtr$ on STab_λ. $D \lesseqqgtr D'$ if and only if $x(D) \leqq x(D')$, where $\leqq$ is the Bruhat order on $\mathfrak{S}_n$. Then we have

PROPOSITION 3. *For $D, D' \in \mathrm{STab}_\lambda$,*

$$D \leqq D' \textit{ if and only if } D \trianglelefteq D', \textit{ and these imply } D \lesseqqgtr D' .$$

PROOF. Clearly $D \leqq D'$ implies $D \lesseqqgtr D'$. The partial order $\leqq$ is generated by the relation $D_1 < D_2$, $D_2=(p, q)D_1$. If p is in the k-th row and q is in the l-th row in D_1, and if $p>q$, then $k<l$. It is easy to see that

$$m_{ir}(D_1)=m_{ir}(D_2) \quad \text{for} \quad i<q \text{ or } i \geqq p \text{ or } r<k \text{ or } r \geqq l ,$$

and

$$m_{ir}(D_1)+1=m_{ir}(D_2) \quad \text{for} \quad q \leqq i<p \text{ and } k \leqq r<l ,$$

therefore $D_1 \triangleleft D_2$. Hence $D \leqq D'$ implies $D \trianglelefteq D'$. On the other hand, using the lemma below inductively, we can prove that $D \trianglelefteq D'$ implies $D \leqq D'$.

LEMMA 5. *For $D, D' \in \mathrm{STab}_\lambda$ such that $D \triangleleft D'$, there exists $D'' \in \mathrm{STab}_\lambda$ such that $\mathrm{ht}(D'')=\mathrm{ht}(D)+1$ and $D<D'' \trianglelefteq D'$.*

PROOF. Compare D and D' for the first row from left to right then for the second row in the same manner and so on. If the first difference is a in D and b in D' in the p-th row, then $a>b$ by $D \triangleleft D'$. A corner of a Young tableau is a square that there is no square attached to the right of and below it. We put

$$C_D=\{j \mid b \leqq j<a,\ j \text{ is in a corner of } D_{\leqq a-1} ,$$
$$\text{and } j \text{ is a lower square than } a \text{ in } D\} .$$

As b is in the lower position than a in D, $C_D \neq \varnothing$. Let $c \in C_D$ be in the heighest position among C_D. Then $b \leqq c<a$, and if we put $D''=(a, c)D$ then $D'' \in \mathrm{STab}_\lambda$ and $\mathrm{ht}(D'')=\mathrm{ht}(D)+1$ by the definition of c, therefore $D<D''$. We must show $D'' \trianglelefteq D'$. If c is in the q-th row of D,

$$m_{ir}(D'')=m_{ir}(D) \leqq m_{ir}(D') \quad \text{if} \quad i<c \text{ or } i \geqq a \text{ or } r<p \text{ or } r \geqq q ,$$

and

$$m_{ir}(D'')=m_{ir}(D)+1 \quad \text{if} \quad c\leqq i<a \text{ and } p\leqq r<q\,.$$

Therefore it is enough to show that

$$(1.5.1) \qquad m_{ir}(D)<m_{ir}(D') \quad \text{if} \quad c\leqq i<a \text{ and } p\leqq r<q\,.$$

We fix i, $c\leqq i<a$, and move r from p to $q-1$. If the first negation of (1.5.1) occurs for $r=r_0$, then $m_{ir_0}(D)=m_{ir_0}(D')$. We define s to be the smallest j such that $\lambda_j(D_{\leqq a-1})=\lambda_q(D_{\leqq a-1})$. For $p\leqq k<s$,

$$m_{ik}(D)=m_{b-1,k}(D)\leqq m_{b-1,k}(D')<m_{b,k}(D')\leqq m_{ik}(D')\,.$$

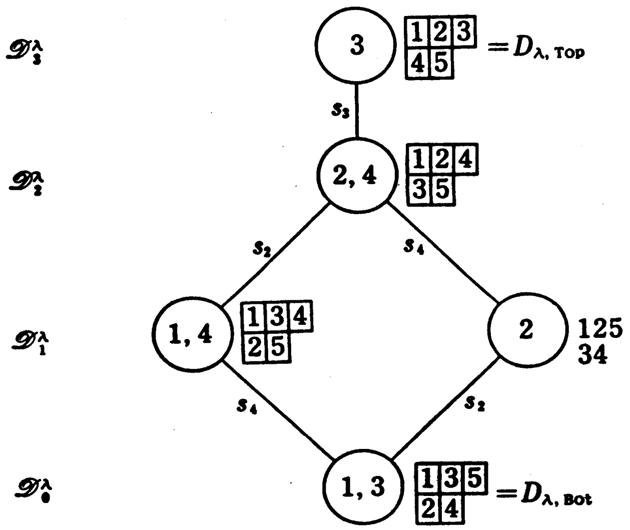
Therefore r_0 must satisfy $s\leqq r_0<q$. But then,

$$\lambda_{r_0+1}(D_{\leqq i})=\lambda_{r_0}(D_{\leqq i})>\lambda_{r_0}(D'_{\leqq i})\geqq\lambda_{r_0+1}(D'_{\leqq i})\,,$$

hence $m_{i,r_0+1}(D)>m_{i,r_0+1}(D')$ which contradicts the fact $D\lhd D'$. Therefore we have (1.5.1) and finally we get $D''\trianglelefteq D'$.

1.6. We define the λ-diagram to be the Hasse diagram of $(\mathrm{STab}_\lambda, \leqq)$ whose vertices have two data D and $I(D)$. If $D\in\mathscr{D}_k^\lambda$, $D'\in\mathscr{D}_{k+1}^\lambda$ and $s_jD'=D$ then we label the edge $\{D, D'\}$ as s_j. An edge may have no label (cf. Example 5.1).

EXAMPLE. $\lambda=(3, 2)$.



§ 2. The left cells of $\mathfrak{S}_n$.

By Kazhdan and Lusztig [3], the left cells of $\mathfrak{S}_n$ are determined as follows. For $D\in\mathrm{STab}_\lambda$, we define

$$X_D=\{x\in\mathfrak{S}_n \mid Q(x)=D \text{ for the Robinson-Schensted correspondence}\} .$$

We denote by Γ_D the W-graph corresponding to X_D with edge and multiplicity defined by $\mu(x, y)$ (cf. [3]). Then $\{\Gamma_D \mid D\in \mathrm{STab}_\lambda,\ \lambda\in P(n)\}$ is the set of all left cells of $\mathfrak{S}_n$ and if $D, D'\in\mathrm{STab}_\lambda$, then Γ_D and $\Gamma_{D'}$ are isomorphic as W-graphs ([3; Theorem 1.4]). We particularly consider $\Gamma^\lambda=\Gamma_{D_{\lambda,\mathrm{Bot}}}$ on $X^\lambda=X_{D_{\lambda,\mathrm{Bot}}}$.

PROPOSITION 4. *The λ-diagram can be identified with the induced subgraph which is obtained from the Hasse diagram of the Bruhat order of $\mathfrak{S}_n$ by restricting vertices to X^λ.*

PROOF. For $D\in\mathrm{STab}_\lambda$, $x(D)$ corresponds to $(D, D_{\lambda,\mathrm{Bot}})$. For $D, D'\in\mathrm{STab}_\lambda$, $l(x(D'))=l(x(D))+1$ if and only if $\mathrm{ht}(D')=\mathrm{ht}(D)+1$, and in this case $x(D')>x(D)$ if and only if $D'>D$ (cf. Hiller [1; Definition 6.0]).

LEMMA 6. *If $x\leqq y$ and $l(y)-l(x)\geqq 2$, then*

$$\mu(x, y)\neq 0 \textit{ implies } \mathscr{L}(y)\subset\mathscr{L}(x) ,$$

where $\mathscr{L}(x)=\{s\in S \mid sx<x\}$.

PROOF. By [3; (2.3.g)], for $s\in S$,

$$P_{x,y}=P_{sx,y} \quad \text{if} \quad x<y,\ sy<y,\ sx>x ,$$

where $P_{x,y}$ is the Kazhdan-Lusztig polynomial.

$$\begin{aligned}\text{The degree of } P_{sx,y} &\leqq \frac{1}{2}(l(y)-l(sx)-1)\\ &=\frac{1}{2}(l(y)-l(x)-2)\\ &<\frac{1}{2}(l(y)-l(x)-1) .\end{aligned}$$

Therefore if $\mu(x, y)\neq 0$ and $sy<y$ then we must have $sx<x$. Hence $\mathscr{L}(y)\subset\mathscr{L}(x)$.

COROLLARY 1. *Γ^λ is obtained from λ-diagram by adding some edges $\{D, D'\}$ for $D\in\mathscr{D}_k^\lambda$, $D'\in\mathscr{D}_l^\lambda$, $l-k=\mathrm{odd}\geqq 3$, $I(D')\subset I(D)$.*

PROOF. For $D\in\mathrm{STab}_\lambda$, $I(D)=\{i \mid s_i\in\mathscr{L}(x(D))\}$, thus it is clear from the lemma above.

For $x\in X^\lambda$ we denote by g_D the base element of Γ^λ corresponding to x, where $D=P(x)$. Thus $\{g_D\}_{D\in \mathrm{STab}_\lambda}$ is the natural $\boldsymbol{Z}$-base for Γ^λ.

§ 3. Specht modules.

In this section we will quote some results of James [2].

3.1. We define an equivalence relation $\sim$ on Tab_λ by $D\sim D'$ if and only if i-th rows of D and D' are equal as sets for all i. We define

$$\mathrm{Tabloid}_\lambda=\mathrm{Tab}_\lambda/\sim \text{ (the set of equivalence classes under } \sim) .$$

We denote by $[D]$ the equivalence class containing D. And we put

$$M_\lambda = \text{the free } \boldsymbol{Z}\text{-module on } \mathrm{Tabloid}_\lambda .$$

Then $\mathfrak{S}_n$ acts naturally on M_λ. We also define for $D\in \mathrm{Tab}_\lambda$

$$V_D=\{x\in\mathfrak{S}_n \mid x \text{ stabilizes each column of } D\} ,$$
$$k_D=\sum_{x\in V_D}\mathrm{sgn}(x)x\in \boldsymbol{Z}[\mathfrak{S}_n] ,$$
$$e_D=k_D[D]\in M_\lambda ,$$
$$S^\lambda=\langle e_D\rangle_{\boldsymbol{Z},D\in \mathrm{Tab}_\lambda}\subset M_\lambda .$$

S^λ is called the Specht module corresponding to the partition λ.

LEMMA 7. *The $\boldsymbol{Z}[\mathfrak{S}_n]$ module S^λ is a cyclic module generated by any e_D, $D\in \mathrm{Tab}_\lambda$.*

It is also known that

PROPOSITION 5 ([2; Theorem 8.4]). *$\{e_D\}_{D\in\mathrm{STab}_\lambda}$ is a $\boldsymbol{Z}$-base of S^λ.*

The action of $\mathfrak{S}_n$ on S^λ with respect to this base $\{e_D\}_{D\in\mathrm{STab}_\lambda}$ is determined by the action of $s_i=(i, i+1)$ on e_D for $D\in \mathrm{STab}_\lambda$.

PROPOSITION 6 (James [2; 25.1]). *The action of s_i on e_D is calculated as follows ($D\in\mathrm{STab}_\lambda$):*

(1) *If $i\in I_1(D)$, then $s_ie_D=-e_D$.*

(2) *If $i\in I_0(D)\cup I_0({}^tD)$, then $s_ie_D=e_{s_iD}$.*

(3) *If $i\in I_1({}^tD)$, then $s_ie_D=e_D+\sum_{D'<D}a_{D'}D'$.*

The case (3) is calculated by the Garnir relations.

3.2. The Garnir relations. For $D\in\mathrm{Tab}_\lambda$, let X be a subset of the i-th column of D and Y be a subset of the $(i+1)$-th column of D. We define, for $Z\subset\{1, 2, \cdots, n\}$, $\mathfrak{S}_Z$ to be the permutation group on the set Z. We fix a coset representative of $\mathfrak{S}_{X\cup Y}/\mathfrak{S}_X\times\mathfrak{S}_Y$ and denote it by

$\{\sigma_i\}_{1\leq i\leq r}$. Then we define

$$G_{X,Y}=\sum_{1\leq i\leq r}\mathrm{sgn}(\sigma_i)\sigma_i\in \boldsymbol{Z}[\mathfrak{S}_n]\ .$$

PROPOSITION 7 (James [2; Theorem 7.2]). *If* $|X\cup Y|>$ *the number of squares in the i-th column of D, then*

$$G_{X,Y}e_D=0$$

which is called the Garnir relation.

§ 4. The isomorphism.

4.1. Now we can state the main result of this paper. We consider two $\boldsymbol{Z}[\mathfrak{S}_n]$ modules with natural bases, $S^\lambda=\langle e_D\rangle_{\boldsymbol{Z},D\in \mathrm{STab}_\lambda}$ and $\Gamma^\lambda=\langle g_D\rangle_{\boldsymbol{Z},D\in \mathrm{STab}_\lambda}$.

THEOREM. *There exists a unique isomorphism* φ *from* S^λ *to* Γ^λ *such that*

$$\varphi(e_D)=g_D+\sum_{D'<D}a_{D',D}g_{D'}\qquad (a_{D',D}\in \boldsymbol{Z})\ .$$

PROOF. Let W_I be the subgroup of $\mathfrak{S}_n$ generated by a subset $I\subset S$. For $\mu\in P(n)$, we set $I(\mu)=I(D_{t_{\mu,\mathrm{Bot}}})$ and regard it as a subset of S by identifying i with s_i. Then $W_{I(\mu)}$ is the horizontal subgroup of $D_{\mu,\mathrm{Top}}$. We also set $S^\lambda_{\boldsymbol{Q}}=S^\lambda\otimes\boldsymbol{Q}$ and $\Gamma^\lambda_{\boldsymbol{Q}}=\Gamma^\lambda\otimes\boldsymbol{Q}$. By Stanley [5; 4.1]

$$\langle \mathrm{Ind}^{\mathfrak{S}_n}_{W_{I(\mu)}}1,\ S^\lambda_{\boldsymbol{Q}}\rangle=\#\{D\in \mathrm{STab}_\lambda\,|\,I(D)\subset\widehat{I(\mu)}\}$$

where $\widehat{I(\mu)}=\{1, 2,\cdots, n-1\}-I(\mu)$. This multiplicity is the Kostka number $K_{\lambda,\mu}$. On the other hand

$$\begin{aligned}\langle \mathrm{Ind}^{\mathfrak{S}_n}_{W_{I(\mu)}}1,\ \Gamma^\lambda_{\boldsymbol{Q}}\rangle&=\langle 1,\ \Gamma^\lambda_{\boldsymbol{Q}}|_{W_{I(\mu)}}\rangle\\&=\#\{D\in \mathrm{STab}_\lambda\,|\,I(D)\cap I(\mu)=\varnothing\}\\&=\#\{D\in \mathrm{STab}_\lambda\,|\,I(D)\subset\widehat{I(\mu)}\}\ .\end{aligned}$$

As the matrix $(K_{\lambda,\mu})_{\lambda,\mu\in P(n)}$ is invertible, we get $S^\lambda_{\boldsymbol{Q}}\simeq\Gamma^\lambda_{\boldsymbol{Q}}$ as $\boldsymbol{Q}[\mathfrak{S}_n]$ modules. The -1 eigenspace of s_i in $\Gamma^\lambda_{\boldsymbol{Q}}$ is $\sum_{i\in I(D)}\boldsymbol{Q}g_D$, and it is easy to see that if $I(D)\supset I(D_{\lambda,\mathrm{Bot}})$ for $D\in\mathrm{STab}_\lambda$, then $D=D_{\lambda,\mathrm{Bot}}$. Therefore the common -1 eigenspace of $I(D_{\lambda,\mathrm{Bot}})$ in $\Gamma^\lambda_{\boldsymbol{Q}}$ is $\boldsymbol{Q}g_{D_{\lambda,\mathrm{Bot}}}$. But since $e_{D_{\lambda,\mathrm{Bot}}}$ is a common -1 eigenvector of $I(D_{\lambda,\mathrm{Bot}})$ in $S^\lambda_{\boldsymbol{Q}}$, there is an isomorphism $\tilde{\varphi}$: $S^\lambda_{\boldsymbol{Q}}\xrightarrow{\sim}\Gamma^\lambda_{\boldsymbol{Q}}$ such that $\tilde{\varphi}(e_{D_{\lambda,\mathrm{Bot}}})=g_{D_{\lambda,\mathrm{Bot}}}$.

By Lemma 7, we can define $\varphi=\tilde{\varphi}|_{S^\lambda}$: $S^\lambda\to\Gamma^\lambda$, which is a $\boldsymbol{Z}[\mathfrak{S}_n]$ homomorphism. We will show that this homomorphism φ has the required property by induction on $\mathrm{ht}(D)$. If $\mathrm{ht}(D)=0$, then $D=D_{\lambda,\mathrm{Bot}}$ and by the construction $\varphi(e_D)$ has the required form. If $\mathrm{ht}(D)>0$, take $j\in I_0(D)$ and define $D'=s_jD$. Then by induction

$$\varphi(e_{D'})=g_{D'}+\sum_{D''\prec D'}a_{D'',D'}g_{D''}\ .$$

As φ is a $\boldsymbol{Z}[\mathfrak{S}_n]$ homomorphism,

$$\varphi(e_D)=\varphi(s_je_{D'})=s_j\varphi(e_{D'})\ .$$

On the other hand

$$s_jg_{D'}=g_D+g_{D'}+\sum_{D^*\prec D'}b_{D^*}g_{D^*}$$

and

$$s_jg_{D''}=\begin{cases}-g_{D''} & \text{if}\quad j\in I(D'')\\ g_{D''}+\text{lower} & \text{if}\quad j\in I_1({}^tD'')\\ g_{D^\circ}+g_{D''}+\text{lower} & \text{if}\quad j\in I_0({}^tD'')\end{cases}$$

where $D^\circ=s_jD''$ and $D''<D^\circ\prec D$. Therefore $\varphi(e_D)$ also has the required form. φ is an isomorphism by this form, and uniqueness is clear.

4.2. Construction of left cells. For $D\in\mathrm{STab}_\lambda$ we define

$$B_D=\{D'\in\mathrm{STab}_\lambda\mid\mathrm{ht}(D)-\mathrm{ht}(D')=\mathrm{odd}\geqq 3\ \text{and}\ I(D')\supset I(D)\}$$

and

$$C(D)=\Big(\bigcup_{D'\in B_D}I(D')\Big)-I(D)\subset\{1,2,\cdots,n-1\}\ .$$

For $\lambda\in P(n)$, we define a condition (C_λ) as follows:

(C_λ) For all $D\in\mathrm{STab}_\lambda-\{D_{\lambda,\mathrm{Bot}}\}$, there exists $j\in I_0(D)$ such that $j\notin C(s_jD)$.

REMARK. For $n\leqq 7$, all partitions $\lambda\in P(n)$ satisfy the condition (C_λ). But there are partitions which do not satisfy the condition for $n\geqq 8$, e.g. $\lambda=(2,2,2,1,1)$.

COROLLARY 2. *If a partition λ satisfies the condition (C_λ), we can construct sub-W-graph of the left cell Γ^λ from the λ-diagram using the Garnir relations.*

PROOF. We construct Γ^λ inductively on ht(D), $D \in \mathrm{STab}_\lambda$. For ht($D$)$<3$, λ-diagram is equal to Γ^λ. For $k \geqq 3$, assume that Γ^λ is already constructed for ht(D)$<k$, so the action of s_i on g_D is already defined for ht(D)$<k$ and all i. Take $D \in \mathscr{D}_k^\lambda$. If $C(D)=\varnothing$, then no edges are added and take another $D \in \mathscr{D}_k^\lambda$. For $i \in C(D)$, we have two cases.

(1) If $i \in I_0({}^tD)$, put $D'=s_iD$. By the condition (C_λ) for D', there exists $j \in I_0(D')$ such that $j \notin C(s_jD')$. Then $\varphi(e_{D'})=s_j\varphi(e_{s_jD'})$ is already determined in terms of g_{D°, $D^\circ \leqq D'$. On the other hand $\varphi(e_D)=g_D+$ lower is also determined. As $\varphi(e_{D'})=s_i\varphi(e_D)$, we get $s_ig_D=g_{D'}+g_D+\sum_{D''<D}\alpha_{D''}g_{D''}$. Then add all the edges $\{D, D''\}$ for $\alpha_{D''}\neq 0$ and define $\mu(D'', D)=\alpha_{D''}$.

(2) If $i \in I_1({}^tD)$, then $s_ie_D=e_D+\sum_{D^*\lhd D}\beta_{D^*}e_{D^*}$ (calculated by the Garnir relations). $\varphi(e_D)$ is already determined and as $s_i\varphi(e_D)=\varphi(s_ie_D)$, we get $s_ig_D=g_D+\sum_{D''<D}\gamma_{D''}g_{D''}$. Then add all the edges $\{D, D''\}$ for $\gamma_{D''}\neq 0$ and define $\mu(D'', D)=\gamma_{D''}$.

Do this operation for all $i \in C(D)$. For all $D \in \mathscr{D}_k^\lambda$ do the same operation and we get Γ^λ for ht(D)$\leqq k$. Thus inductively we get Γ^λ. (If there are vertices with the same I-set, some edges may be added to become the left cell.)

§5. Examples.

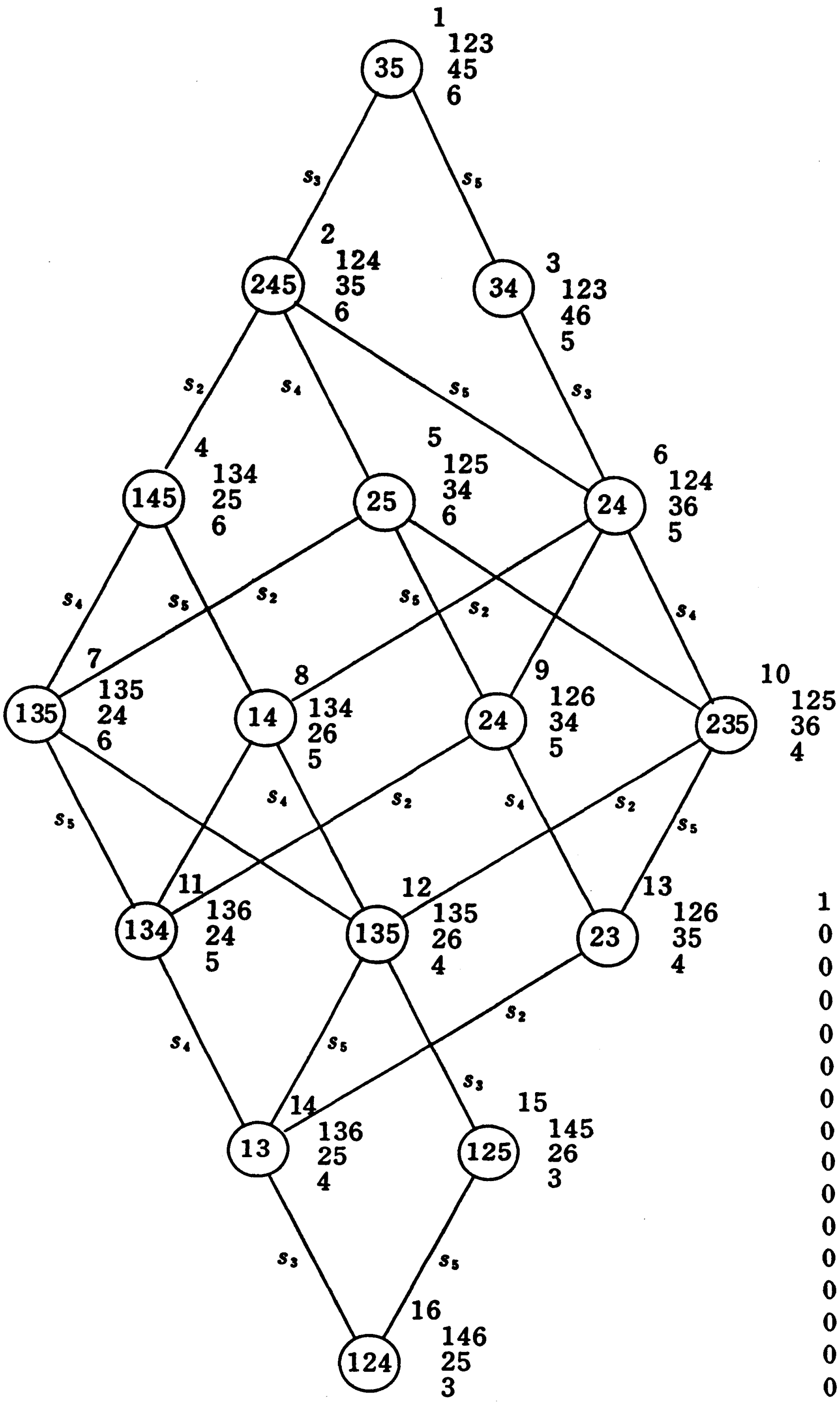
In this section we calculate W-graphs (the left cells) and transition matrices for $\lambda=(3, 2, 1)$ and for all partitions of $n=7$ using Corollary 2.

5.1. $\lambda=(3, 2, 1)$. The λ-diagram is indicated below. We label the Young tableaux $D_1, D_2, \cdots, D_{16}$ from top to bottom. We also write $e_i=e_{D_i}$ and $g_j=g_{D_j}$. As the action of s_i on g_D is already known for ht(D)$\leqq 2$, we can calculate $\varphi(e_D)$ for ht(D)$\leqq 3$ inductively:

$$\varphi(e_{16})=g_{16} \qquad \text{by definition of } \varphi\ ,$$
$$\varphi(e_{15})=s_5\varphi(e_{16})=s_5g_{16}=g_{15}+g_{16}\ ,$$
$$\varphi(e_{14})=s_3\varphi(e_{16})=s_3g_{16}=g_{14}+g_{16}\ ,$$
$$\varphi(e_{13})=s_2\varphi(e_{14})=s_2(g_{14}+g_{16})=g_{13}+g_{14}\ ,$$
$$\vdots$$
$$\varphi(e_7)=s_5\varphi(e_{11})=g_7+g_{11}+g_{12}+g_{14}\ .$$

We define the action of s_i on g_D for ht(D)$=3$. $C(D_9)=\{1\}$ and this is the case (2) of Corollary 2.

$$s_1e_9=e_9-e_{11}-e_{16} \qquad \text{by Garnir relation}\ ,$$



λ-diagram for $\lambda=(3, 2, 1)$

1	11	111	1111	110	01	1
0	10	111	1111	111	11	1
0	01	001	0110	100	00	1
0	00	100	1100	110	10	1
0	00	010	1011	111	11	1
0	00	001	0111	111	10	1
0	00	000	1000	110	10	0
0	00	000	0100	110	11	1
0	00	000	0010	101	10	1
0	00	000	0001	011	10	0
0	00	000	0000	100	10	0
0	00	000	0000	010	11	1
0	00	000	0000	001	10	0
0	00	000	0000	000	10	1
0	00	000	0000	000	01	1
0	00	000	0000	000	00	1

transition matrix for $\lambda=(3, 2, 1)$

and

$$s_1\varphi(e_9)=\varphi(e_9-e_{11}-e_{16})=g_9+g_{13} .$$

On the other hand

$$s_1\varphi(e_9)=s_1g_9+s_1(g_{11}+e_{13}+g_{14}+g_{16})=s_1g_9-g_{11}+g_{13}-g_{16} .$$

Therefore $s_1g_9=g_9+g_{11}+g_{16}$ and we get $\mu(D_{16}, D_9)=1$. $C(D_8)=\{2\}$ and this is the case (1) of Corollary 2. $s_2e_8=e_6=s_4e_{10}$ and the action of s_4 on g_{10} is known:

$$\begin{aligned}s_4\varphi(e_{10})&=s_4(g_{10}+g_{12}+g_{13}+g_{14})\\&=g_6+g_8+g_9+g_{10}+g_{11}+g_{12}+g_{13}+g_{14}+g_{16} .\end{aligned}$$

On the other hand

$$\begin{aligned}s_2\varphi(e_8)&=s_2g_8+s_2(g_{11}+g_{12}+g_{14}+g_{15}+g_{16})\\&=s_2g_8+g_9+g_{10}+g_{11}+g_{12}+g_{13}+g_{14} ,\end{aligned}$$

therefore $s_2g_8=g_6+g_8+g_{16}$ and we get $\mu(D_{16}, D_8)=1$. As we defined all the actions of s_i on g_D for $\mathrm{ht}(D)\leqq 3$, we can calculate $\varphi(e_D)$ for $\mathrm{ht}(D)=4$. In this way we get transition matrix and W-graph simultaneously. The added edges are

D_8——D_{16} case (1),
D_9——D_{16}, D_5——D_{15}, D_3——D_{11}, D_1——D_7, D_1——D_{10} case (2),

with multiplicities 1. This W-graph is also the left cell.

5.2. $n=7$. We indicate only half of the cases. For transposed partition ${}^t\lambda$, the Young tableaux are ${}^tD$ with the same numbering. The edges are the same and the I-sets are $I({}^tD)=\{1, 2, \cdots, 6\}-I(D)$. The transition matrix is indicated in lower triangle, e.g. for ${}^t\lambda=(2, 2, 1, 1, 1)$, $\varphi(e_8)=g_8+g_6+g_5+g_3$. Dotted edges are added for the W-graph to become the left cell.

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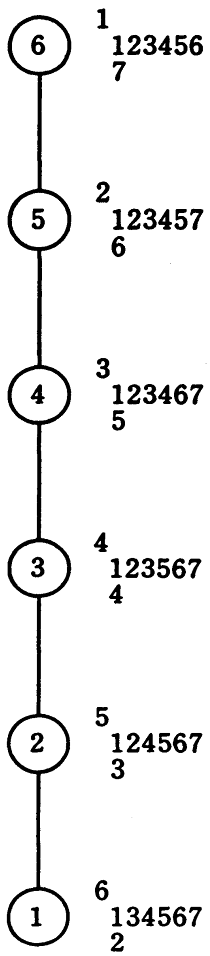
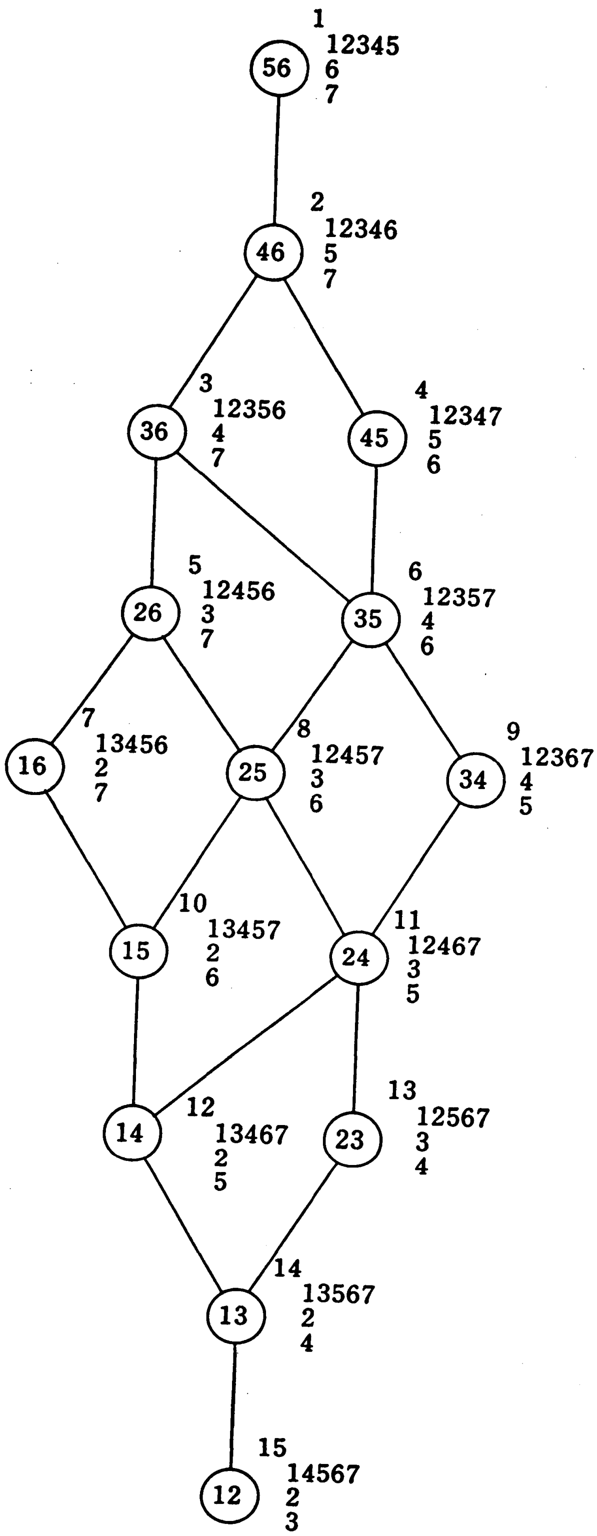
$\lambda=(7)$
1
${}^t\lambda=(1, 1, 1, 1, 1, 1, 1)$



$\lambda=(6,1)$

1	1	1	1	1	1
1	1	1	1	1	1
0	1	1	1	1	1
0	0	1	1	1	1
0	0	0	1	1	1
0	0	0	0	1	1

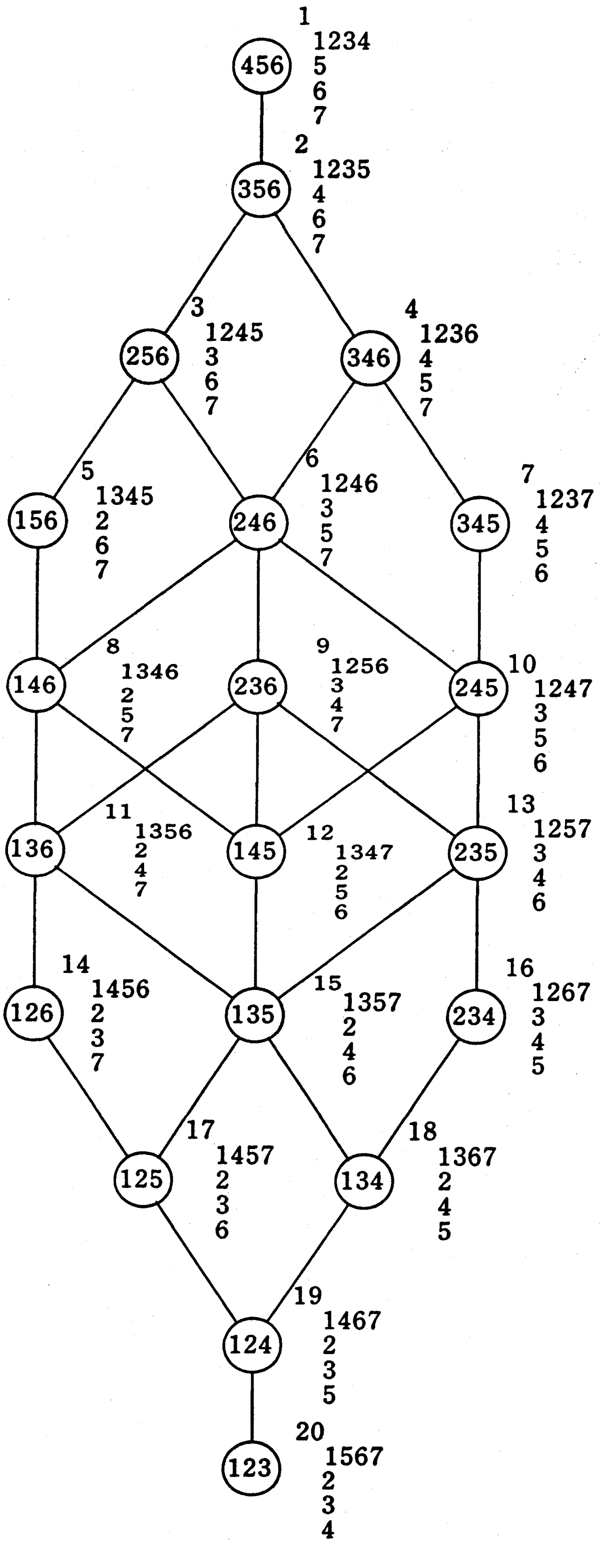
${}^t\lambda=(2,1,1,1,1,1)$



$\lambda=(5,1,1)$

1	1	10	10	100	00	00	0	0
1	1	11	11	110	10	00	0	0
0	1	10	11	111	11	10	0	0
1	1	01	01	010	10	00	0	0
0	0	10	10	110	11	11	1	0
0	1	11	01	011	11	10	0	0
0	0	00	10	100	10	10	1	1
0	0	10	11	010	11	11	1	0
0	0	01	01	001	01	10	0	0
0	0	00	10	110	10	10	1	1
0	0	00	01	011	01	11	1	0
0	0	00	00	010	11	10	1	1
0	0	00	00	001	01	01	1	0
0	0	00	00	000	01	11	1	1
0	0	00	00	000	00	01	1	1

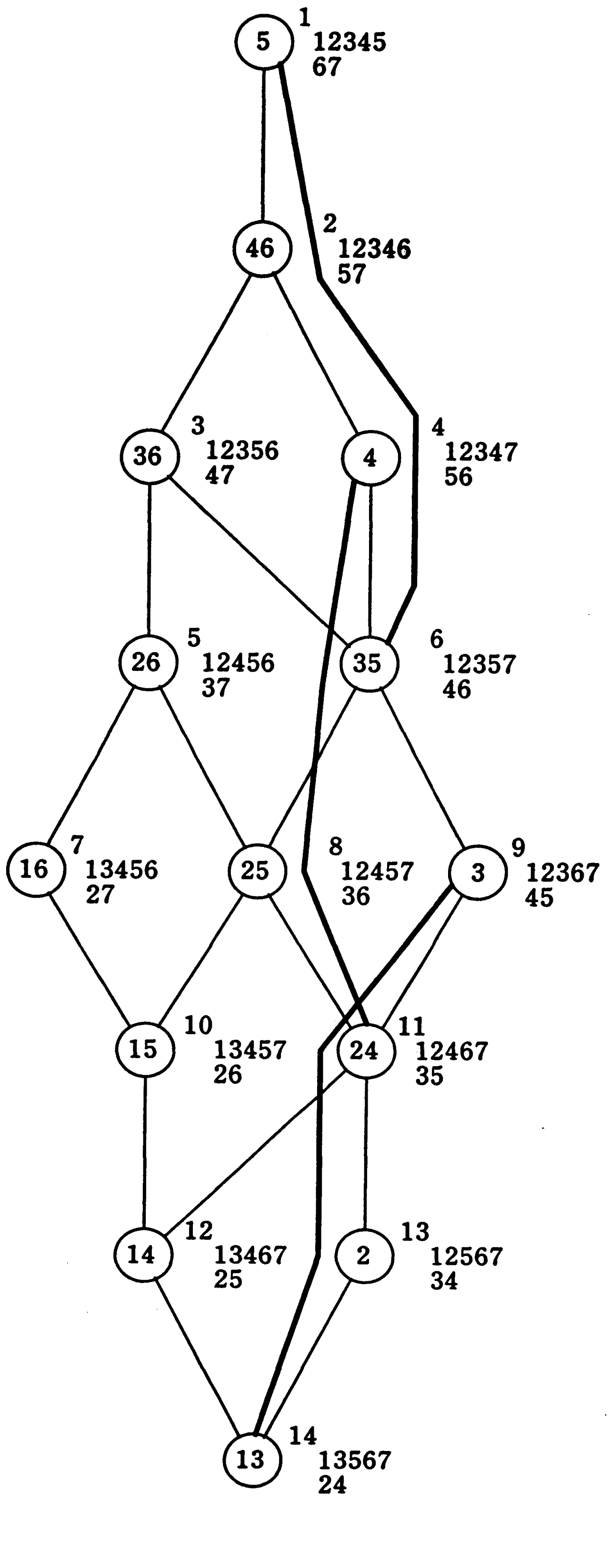
${}^t\lambda=(3,1,1,1,1)$



$\lambda=(4,1,1,1)$

1	1	10	100	000	000	000	00	0	0
1	1	11	110	100	000	000	00	0	0
0	1	10	110	110	100	000	00	0	0
1	1	01	011	101	010	000	00	0	0
0	0	10	100	100	100	100	00	0	0
0	1	11	010	111	111	010	00	0	0
1	1	01	001	001	010	000	00	0	0
0	0	10	110	100	110	110	10	0	0
0	0	01	010	010	101	011	01	0	0
0	1	11	011	001	011	010	00	0	0
0	0	00	010	110	100	110	11	1	0
0	0	10	110	101	010	010	10	0	0
0	0	01	011	011	001	011	01	0	0
0	0	00	000	010	100	100	10	1	1
0	0	00	010	111	111	010	11	1	0
0	0	00	001	001	001	001	01	0	0
0	0	00	000	010	101	110	10	1	1
0	0	00	000	001	011	011	01	1	0
0	0	00	000	000	001	011	11	1	1
0	0	00	000	000	000	001	01	1	1

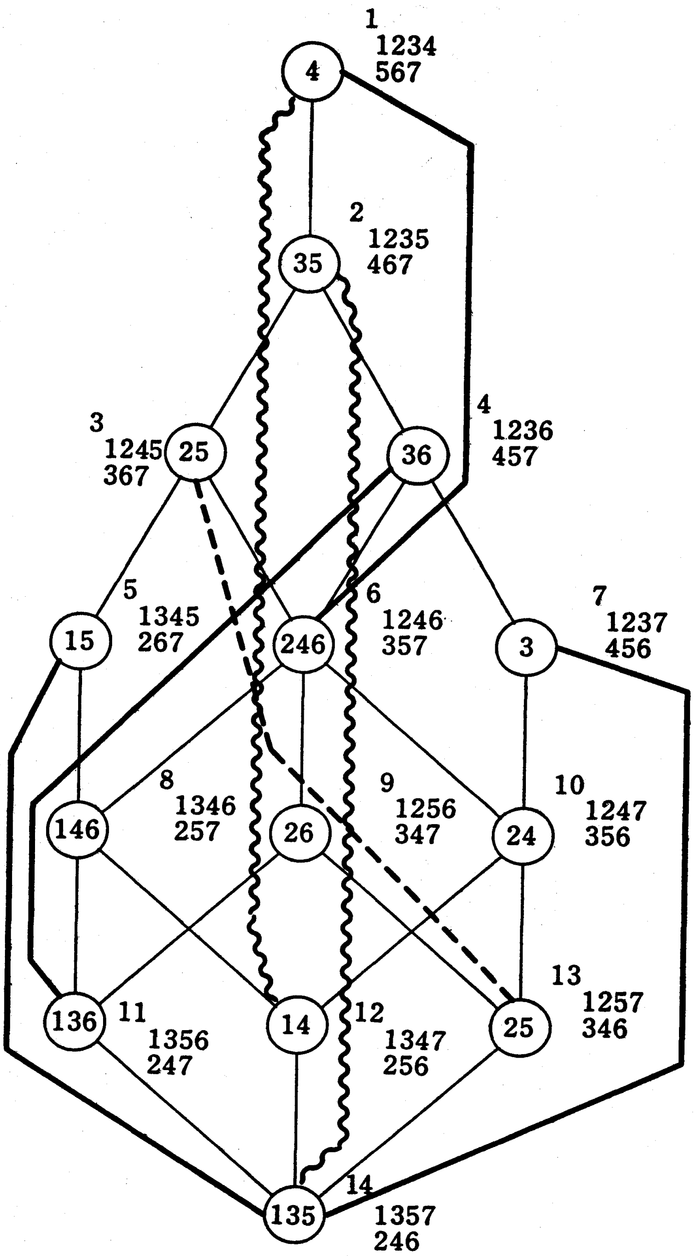
${}^t\lambda=(4,1,1,1)$



$\lambda=(5,2)$

1	1	11	12	121	12	11	1
1	1	11	11	111	12	11	1
0	1	10	11	111	11	11	1
0	1	01	01	011	12	11	1
0	0	10	10	110	11	11	1
1	1	11	01	011	11	11	1
0	0	00	10	100	10	10	1
0	0	10	11	010	11	11	1
1	0	00	01	001	01	11	1
0	0	00	10	110	10	10	1
0	0	01	01	011	01	11	1
0	0	00	00	010	11	10	1
0	0	01	00	000	01	01	1
0	0	00	00	001	01	11	1

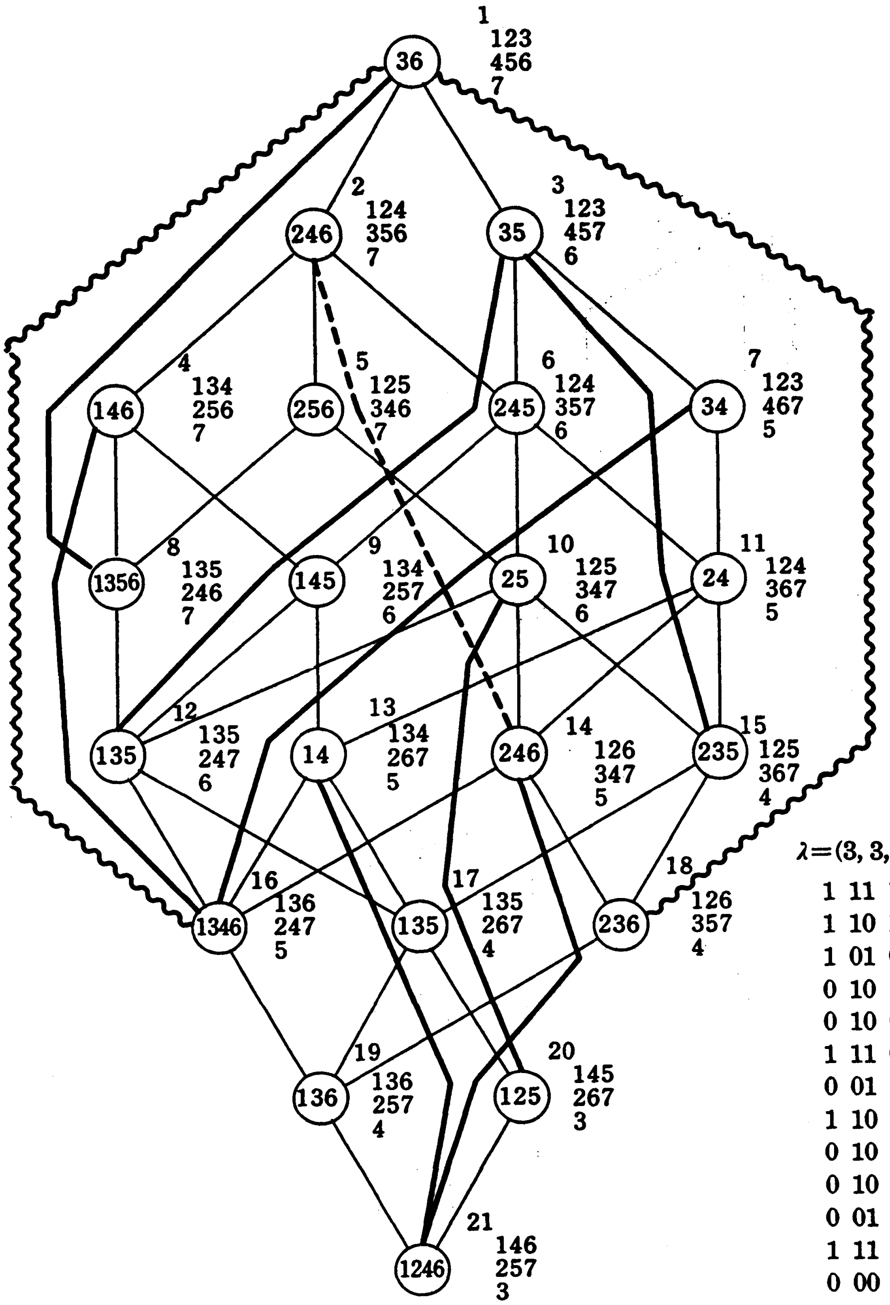
${}^t\lambda=(2,2,1,1,1)$



$\lambda=(4,3)$

1	1	11	121	111	111	2
1	1	11	111	111	111	2
0	1	10	110	111	111	1
0	1	01	011	111	111	1
0	0	10	100	100	110	1
1	1	11	010	111	111	1
0	0	01	001	001	011	1
0	0	10	110	100	110	1
1	0	00	010	010	101	1
0	0	01	011	001	011	1
0	0	01	010	110	100	1
1	0	00	010	101	010	1
0	0	10	010	011	001	1
1	1	11	111	111	111	1

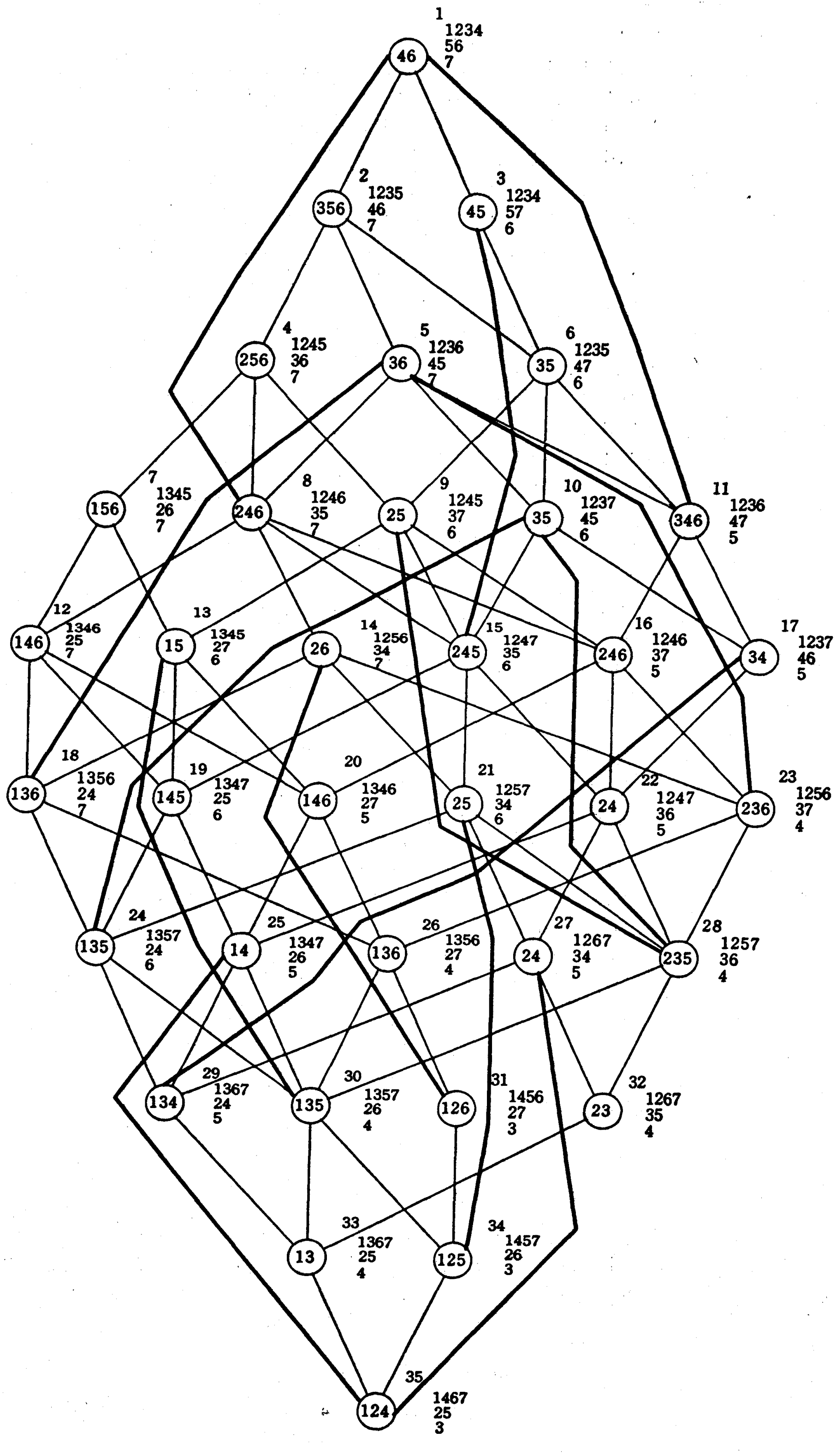
${}^t\lambda=(2,2,2,1)$



$\lambda=(3, 3, 1)$

1	11	1111	1111	1111	211	11	1
1	10	1110	1111	1111	110	01	1
1	01	0011	0111	1111	110	01	1
0	10	1000	1100	1100	110	00	0
0	10	0100	1010	1001	010	01	0
1	11	0010	0111	1111	111	11	1
0	01	0001	0001	0110	100	00	1
1	10	1100	1000	1000	010	00	0
0	10	1010	0100	1100	110	10	1
0	10	0110	0010	1011	111	11	1
0	01	0011	0001	0111	111	10	1
1	11	1110	1110	1000	110	10	0
0	00	0010	0101	0100	110	11	1
0	10	0110	0011	0010	101	10	1
0	01	0011	0011	0001	011	10	0
1	11	1111	1111	1110	100	10	0
0	01	0010	0111	1101	010	11	1
1	11	0010	0011	0011	001	10	0
1	11	1011	0111	1111	111	10	1
0	01	0000	0010	1001	010	01	1
1	01	0001	0011	1111	111	11	1

${}^t\lambda=(3, 2, 2)$


1
1234
56
7
46
2
1235
46
7
356
3
1234
57
6
45
4
1245
36
7
256
5
1236
45
7
36
6
1235
47
6
35
7
1345
26
7
156
8
1246
35
7
246
9
1245
37
6
25
10
1237
45
6
35
11
1236
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5
346
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7
146
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27
6
15
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1256
34
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15
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6
245
16
1246
37
5
246
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34
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136
19
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6
145
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27
5
146
21
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235
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1367
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134
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4
135
31
1456
27
3
126
32
1267
35
4
23
33
1367
25
4
13
34
1457
26
3
125
35
1467
25
3
124

$\lambda=(4, 2, 1)$

1	11	111	12111	111210	111101	10101	0110	01	0
1	10	111	11111	111111	111111	11111	1110	01	1
1	01	001	00110	010200	010100	10001	0100	01	0
0	10	100	11100	111110	111111	11112	1211	11	1
0	10	010	01011	101111	111111	11111	1110	01	1
1	11	001	00111	010111	011110	11011	1100	01	1
0	00	100	10000	110000	111000	11100	1200	11	1
1	10	110	01000	101110	111111	11111	1111	11	1
0	10	101	00100	010110	011111	11112	1201	11	1
0	10	011	00010	000101	010110	11011	1100	01	1
1	11	011	00001	000011	001010	01010	1000	00	1
0	00	100	11000	100000	111000	11100	1100	10	1
0	00	100	10100	010000	011000	11100	1210	11	1
1	00	000	01000	001000	100101	10111	1111	11	1
1	11	111	01110	000100	010110	11011	1101	11	1
1	10	111	01101	000010	001011	01111	1101	10	1
1	10	011	00011	000001	000010	01010	1000	00	1
0	00	010	01000	101000	100000	10100	1100	10	0
0	00	100	11100	110100	010000	11000	1100	10	1
0	00	100	11100	110010	001000	01100	1110	11	1
1	01	000	01000	001100	000100	10011	1101	11	1
1	11	111	01111	000111	000010	01011	1101	10	1
1	00	010	01001	001010	000001	00101	0101	10	0
0	00	010	01010	101100	110100	10000	1100	10	0
0	00	100	11100	110110	011010	01000	1100	11	1
0	00	010	01000	101010	101001	00100	0110	11	1
0	01	000	00000	000100	000110	00010	1001	10	1
1	01	011	01111	001111	000111	00001	0101	10	0
0	00	000	00010	000101	010110	11010	1000	10	0
0	00	010	01110	111110	111111	11101	0100	11	1
0	00	010	00000	001000	100001	00100	0010	01	1
0	01	001	00110	000100	000110	00011	0001	10	0
0	00	000	00110	010101	010110	11011	1101	10	1
0	00	010	00010	001000	100101	10101	0110	01	1
0	00	000	00010	000001	000110	11011	1101	11	1

${}^t\lambda=(3, 2, 1, 1)$

5.3. **Remark.** The algorithm of Corollary 2 can be ameliorated and we get W-graphs for all partitions of $n\leqq 9$. We have a microcomputer program to calculate these W-graphs. In these cases, the transition matrices are all nonnegative.

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