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The Pseudo Orbit Tracing Property of First Return Maps

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Dedicated to Professor Kenichi Shiraiwa on his 60th birthday

§1. Introduction.

Every real flow without fixed points on a compact metric space induces a first return map on the union of sets in a certain family of local cross-sections, which was first introduced by H. Whitney [9] and after that improved by R. Bowen and P. Walters [2]. Our purpose is to investigate relationships between a real flow and its first return map with respect to the pseudo orbit tracing property.

H. B. Keynes and M. Sears [6] characterized already expansivity of a real flow by making use of a family of local cross-sections and a bijective first return map.

We denote by $(X, \boldsymbol{R})$ a real flow (abbrev. flow) without fixed points on a compact metric space X. Let d denote a metric for X and the action of $t\in\boldsymbol{R}$ on $x\in X$ is written xt. We write

$$SI=\{xt\,;\ t\in I \text{ and } x\in S\}$$

for an interval I and $S\subset X$, and

$$\varepsilon_0=\inf\{t>0\,;\ xt=x \text{ for some } x\in X\}\ .$$

Then ε_0 is a positive number since the flow $(X, \boldsymbol{R})$ has no fixed points and X is compact.

For positive numbers δ and a, a pair of doubly infinite sequences $(\{x_i\}_{i=-\infty}^{\infty}, \{t_i\}_{i=-\infty}^{\infty})$ is a *(δ, a)-chain* for $(X, \boldsymbol{R})$ if $t_i\geqq a$ and $d(x_it_i, x_{i+1})<\delta$ for all $i\in\boldsymbol{Z}$, and a pair of infinite sequences $(\{x_i\}_{i=0}^{\infty}, \{t_i\}_{i=0}^{\infty})$ is a *half (δ, a)-chain* for $(X, \boldsymbol{R})$ if $t_i\geqq a$ and $d(x_it_i, x_{i+1})<\delta$ for $i\geqq 0$. A (δ, a)-

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chain for $(X, \boldsymbol{R})$ is called the (δ, a)-*pseudo orbit*. τ_n denotes a partial sum of an infinite sequence $\{t_i\}$, i.e.

$$\tau_n=\begin{cases}\sum_0^{n-1} t_i & \text{if} \quad n\geqq 0\\ -\sum_n^{-1} t_i & \text{if} \quad n<0\,,\end{cases}$$

where $\tau_0=\sum_0^{-1} t_i=0$. For a (δ, a)-chain $(\{x_i\}, \{t_i\})$ we write

$$x_0*t=x_n(t-\tau_n) \qquad \text{if} \quad \tau_n\leqq t<\tau_{n+1}\,.$$

A (δ, a)-chain $(\{x_i\}_{i=-\infty}^{\infty}, \{t_i\}_{i=-\infty}^{\infty})$ is said to be ε-*traced* $(\varepsilon>0)$ by a point $x\in X$ if there is a strictly increasing homeomorphism $h\colon \boldsymbol{R}\to\boldsymbol{R}$ such that $h(0)=0$, $h(\boldsymbol{R})=\boldsymbol{R}$ and $d(xh(t), x_0*t)<\varepsilon$ for all $t\in\boldsymbol{R}$. That a half (δ, a)-chain $(\{x_i\}_{i=0}^{\infty}, \{t_i\}_{i=0}^{\infty})$ is ε-*traced* by a point $x\in X$ is defined similarly by restricting the time t to $t\geqq 0$.

$(X, \boldsymbol{R})$ has *POTP with respect to time a* if for any $\varepsilon>0$ there is $\delta>0$ such that every infinite (δ, a)-chain for $(X, \boldsymbol{R})$ is ε-traced by some point of X. $(X, \boldsymbol{R})$ is said to have *POTP* if $(X, \boldsymbol{R})$ has POTP with respect to time 1.

A subset $S\subset X$ is called a *local cross-section* of time $\zeta>0$ for a flow $(X, \boldsymbol{R})$ if S is closed and $S\cap x[-\zeta, \zeta]=\{x\}$ for all $x\in S$, where $\zeta<\varepsilon_0$.

If S is a local cross-section of time ζ, the action maps $S\times[-\zeta, \zeta]$ homeomorphically onto $S[-\zeta, \zeta]$. By the interior S^* of S we mean the set $S\cap\mathrm{int}(S[-\zeta, \zeta])$. Note that $S^*(-\varepsilon, \varepsilon)$ is open in X for any $\varepsilon>0$.

Throughout this paper our arguments are based on the following proposition.

PROPOSITION 1 ([6], Lemma 2.4). *There is $0<\zeta<\varepsilon_0$ satisfying that for each $\alpha>0$ we can find a finite family $\mathscr{S}=\{S_1, S_2, \cdots, S_k\}$ of pairwise disjoint local cross-sections of time ζ and diameter at most α and a family of local cross-sections $\mathscr{T}=\{T_1, T_2,\cdots, T_k\}$ with $T_i\subset S_i^*$ $(i=1,2,\cdots,k)$ such that*

$$X=T^+[0, \alpha]=T^+[-\alpha, 0]=S^+[0, \alpha]=S^+[-\alpha, 0]$$

where $T^+=\cup_{i=1}^k T_i$ and $S^+=\cup_{i=1}^k S_i$.

Hereafter let ζ and $0<\alpha<\zeta/3$ be as in the Proposition 1 and put

$$\beta=\sup\{\delta>0\ ;\ x(0, \delta)\cap S^+=\varnothing \ \text{ for } x\in S^+\}\,.$$

Obviously $0<\beta\leqq\alpha$. Let $\rho>0$ be a number such that $5\rho<\zeta$ and $2\rho<\beta$.

For $x \in T^+$ ($x \in S^+$) let $t \in \boldsymbol{R}$ be the smallest positive time such that $xt \in T^+$. Obviously $\beta \leqq t \leqq \alpha$ and a map $\varphi(x)=xt$ ($\tilde{\varphi}(x)=xt$) is well defined. It is easily checked that φ: $T^+ \to T^+$ is bijective and $\tilde{\varphi}$: $S^+ \to T^+$ is surjective.

For $S_i \in \mathscr{S}$ set $D_\rho^i = S_i[-\rho, \rho]$ and define a projective map P_ρ^i: $D_\rho^i \to S_i$ by $P_\rho^i(x)=xt$, where $|t| \leqq \rho$. Then P_ρ^i is continuous and surjective. We write

$$D_\rho^i = D_\rho \quad \text{and} \quad P_\rho^i = P_\rho$$

if there is no confusion.

The following remark is easily checked.

REMARK 2. There is an $0<a<\beta/2$ such that for $x, y \in S_i$ if $d(x, y) \leqq a$ and $xt \in T_j$ ($|t| \leqq 3\alpha$) for some T_j, then $yt \in D_\rho^j$.

Using this fact, we can set up a shadowing orbit of y relative to a φ ($\tilde{\varphi}$)-orbit of $x \in T^+$ as follows. If y is sufficiently close to x, the orbit of y will cross S_j at a time near the time when the orbit of x crosses T_j. For $x \in T_i$ and $y \in S_i$ with $d(x, y) \leqq a$, we can define a point y_1 so that $y_1 = P_\rho(yt)$, where t is the smallest positive time such that $\varphi(x)=xt$ ($\tilde{\varphi}(x)=xt$). Inductively if $d(\varphi^i(x), y_i) \leqq a$ ($d(\tilde{\varphi}^i(x), y_i) \leqq a$), then we can define a point y_{i+1} so that $y_{i+1} = P_\rho(y_i t)$, where t is the smallest positive time such that $\varphi^{i+1}(x) = \varphi^i(x)t$ ($\tilde{\varphi}^{i+1}(x) = \tilde{\varphi}^i(x)t$). Thus we obtain a time delayed orbit of y along a piece of the orbit of x. We can also construct the shadowing orbit of y as above for the orbit of x of negative powers of φ.

For simplicity we write T, S instead of T_i, S_i respectively. For $x \in T$ and $\eta > 0$ the η-stable set of x is defined by

$$W_\eta^s(x) = \{y \in S \ ; \ d(\varphi^i(x), y_i) < \eta \text{ for all } i \geqq 0\}$$

and the η-unstable set of x is defined by

$$W_\eta^u(x) = \{y \in S \ ; \ d(\varphi^i(x), y_i) < \eta \text{ for all } i \leqq 0\} .$$

The first return map φ is said to *have a canonical coordinate* if for any $\eta > 0$ there exists $\delta > 0$ such that if $d(x, y) < \delta$ ($x, y \in T^+$), then $W_\eta^s(x) \cap W_\eta^u(y) \neq \emptyset$. Given $\delta > 0$, a doubly infinite sequence $\{x_i\}_{i=-\infty}^{\infty} \subset T^+$ is called *δ-pseudo orbit of φ* if $d(\varphi(x_i), x_{i+1}) < \delta$ for all $i \in \boldsymbol{Z}$. Similarly an infinite sequence $\{x_i\}_{i=0}^{\infty} \subset S^+$ is called *δ-pseudo orbit of $\tilde{\varphi}$* if $d(\tilde{\varphi}(x_i), x_{i+1}) < \delta$ for all $i \geqq 0$. If a sequence $\{x_i\} \subset T^+$ (S^+) is a δ-pseudo orbit of φ ($\tilde{\varphi}$), we write

$$\varphi(x_i)=x_it_i \quad \text{and} \quad \tilde{\varphi}(x_i)=x_it_i$$

respectively. A δ-pseudo orbit $\{x_i\}$ of φ is said to be *ε-traced* by a point $y\in S^+$ if y satisfies the following:

(1) $d(y, x_0)<\varepsilon$,

(2) $y_i=P_\rho(y_{i-1}t_{i-1})$ and $y_{-i}=P_\rho(y_{-i+1}(-t_{-i}))$ are inductively defined for $i\geqq 1$ and they satisfy $d(y_i, x_i)<\varepsilon$ for all $i\in \boldsymbol{Z}$, where $y_0=y$.

A δ-pseudo orbit $\{x_i\}_{i=0}^{\infty}$ of $\tilde{\varphi}$ is called to be *ε-traced* by a point $y\in S^+$ if y satisfies the following:

(1) $d(y, x_1)<\varepsilon$,

(2) $y_i=P_\rho(y_{i-1}t_{i-1})$ is defined inductively and satisfies $d(y_i, x_i)<\varepsilon$ for $i\geqq 2$, where $y_1=y$.

φ ($\tilde{\varphi}$) is said to have *POTP* if for any $\varepsilon>0$ there exists $\delta>0$ such that every δ-pseudo orbit of φ ($\tilde{\varphi}$) is ε-traced by some point of S^+.

The following are our results.

THEOREM A. *If $(X, \boldsymbol{R})$ has POTP, then the bijective first return map φ obeys POTP.*

COROLLARY B. *If $(X, \boldsymbol{R})$ has POTP, then the bijective first return map φ has a canonical coordinate.*

THEOREM C. *$(X, \boldsymbol{R})$ has POTP if and only if so does $\tilde{\varphi}\colon S^+\to T^+$.*

§2. Proofs of Theorem A and Corollary B.

Let $0<\zeta<\varepsilon_0$ and $0<\alpha<\zeta/3$ be as in §1. Choose $0<a<\beta/2$ as in Remark 2 and $\rho>0$ as in §1 ($5\rho<\zeta$ and $2\rho<\beta$). Let $\mathscr{S}$ and $\mathscr{T}$ be families of local cross-sections as in Proposition 1. Before starting the proof of Theorem A, we prepare Claims 1 and 2 that suffice for our needs.

CLAIM 1. *For $\eta>0$ and $0<\mu<\zeta$ there are positive numbers ε_1, ε_2, ε_3 and ε_4 such that*

(A1) *$\varepsilon_1<\eta$ and $2\varepsilon_2<\varepsilon_1$,*

(A2) *if $d(u, v)<\varepsilon_1$ for $u, v\in S$ ($\in\mathscr{S}$), then $d(u, vt)>\varepsilon_1$ for $\mu\leqq|t|\leqq\zeta$,*

(A3) *if $d(u, v)<2\varepsilon_2$ for $u\in T$ and $v\in S$, then $d(\varphi(u), v_1)<\varepsilon_1$ ($d(\tilde{\varphi}(u), v_1)<\varepsilon_1$), $d(\varphi^{-1}(u), v_{-1})<\varepsilon_1$, where $T\in\mathscr{T}$ and $S\in\mathscr{S}$ ($T\subset S^*$),*

(A4) *if $d(u, v)\geqq\varepsilon_2$ for $u, v\in S$ ($\in\mathscr{S}$), then $d(u, vt)>\varepsilon_3$ for $|t|\leqq\mu$,*

(A5) *if $d(x, y)<\varepsilon_4$ for $x, y\in X$, then $d(xt, yt)<\varepsilon_1$ for $|t|\leqq\alpha$.*

CLAIM 2. *For any* ε, $\tau>0$ *there are positive numbers* $0<\theta<\tau$, ε_5, ε_6, ε_7 *and* ε_8 *such that for any* $x, y \in X$

(B1) $d(x, xt)<\varepsilon/4$ *for any* $|t|\leqq\theta$,

(B2) *if* $d(x, y)<\varepsilon_5$, *then* $d(xt, ys)<\varepsilon/4$ *for* $t, s\in \boldsymbol{R}$ *with* $|t|\leqq\alpha$ *and* $|t-s|<\theta$,

(B3) *if* $d(x, y)<\varepsilon_6$ *and* $xt\in T_j$ $(|t|\leqq\alpha)$ *for some* T_j, *then* $yt\in D^j$ *and* $P_\rho(yt)=y(t+s)$ *with* $|s|<\theta/4$,

(B4) *if* $d(x, y)<\varepsilon_7$, *then* $d(xt', y(t+t'))>\varepsilon_7$ *for* $\theta/2\leqq|t|\leqq\zeta$ *and* $|t'|\leqq\alpha$,

(B5) *if* $d(x, y)<\varepsilon_8$, *then* $d(xt, yt)<\varepsilon_7/2$ *for* $|t|\leqq\alpha$.

PROOF OF THEOREM A. Take η and μ such that $0<\eta<a$ and $0<\mu<\zeta-\alpha-\rho$. Fix $0<\varepsilon<\min\{\varepsilon_2, \varepsilon_3, \varepsilon_4\}$ and $0<\tau<\min\{\mu, \beta\}$.

Since $(X, \boldsymbol{R})$ has POTP and $\beta>0$, $(X, \boldsymbol{R})$ has POTP with respect to time β by Proposition 1.4 [7]. Let $0<\varepsilon'<\min\{\varepsilon/4, \varepsilon_6, \varepsilon_7/2\}$. Then there exists $0<\delta<\varepsilon'$ such that any (δ, β)-chain of $(X, \boldsymbol{R})$ is ε'-traced by some point of X. It is enough to see that δ is our asking number for $\eta>0$.

Let $\{x_i\}_{i=-\infty}^{\infty}\subset T^+$ be any δ-pseudo orbit of the first return map φ. Let $\{t_i\}_{i=-\infty}^{\infty}$ be a sequence such that $\varphi(x_i)=x_it_i$ for each $i\in\boldsymbol{Z}$. For a (δ, β)-chain $(\{x_i\}_{i=-\infty}^{\infty}, \{t_i\}_{i=-\infty}^{\infty})$ there exists $z\in X$ which ε'-traces the (δ, β)-chain. Therefore there exists a strictly increasing homeomorphism h of $\boldsymbol{R}$ such that $h(0)=0$, $h(\boldsymbol{R})=\boldsymbol{R}$ and

$$d(zh(t), x_0*t)<\varepsilon' \quad \text{for all} \quad t\in\boldsymbol{R}\,.$$

Since $x_0\in T$ $(T\in\mathscr{T})$ and $d(z, x_0)=d(zh(0), x_0)<\varepsilon'<\varepsilon_6$, we have $P_\rho(z)=zl\in S$ and $|l|<\theta/4$ by (B3). Put $z_0=zl$ and take $\xi_0\in\boldsymbol{R}$ with $h(\xi_0)=l$. Then we claim that ξ_0 does not satisfy $|h(\xi_0)-\xi_0|\geqq\theta/2$.

If $|h(\xi_0)-\xi_0|\geqq\theta/2$, then there is $t'\in\boldsymbol{R}$ such that $|h(t')-t'|=\theta/2$ and $|t'|\leqq|\xi_0|$, and

$$|t'|\leqq\theta/2+|h(\xi_0)|<\theta/2+\theta/4<\theta<\tau<\beta\leqq\alpha\,.$$

For the case $t'\geqq0$, since $d(z, x_0)<\varepsilon'<\varepsilon_7$, by (B4) we have

$$d(zh(t'), x_0*t')=d(z(t'\pm\theta/2), x_0t')>\varepsilon_7\,,$$

which contradicts $d(zh(t'), x_0*t')<\varepsilon'<\varepsilon_7$. From the fact that $d(z, x_0)<\varepsilon'$ and $d(x_0, \varphi(x_{-1}))<\delta$, it follows that

$$\begin{aligned} d(z, x_{-1}(-\tau_{-1}))&=d(z, \varphi(x_{-1}))\\ &\leqq d(z, x_0)+d(x_0, \varphi(x_{-1}))\\ &<\varepsilon'+\delta<2\varepsilon'<\varepsilon_7\,. \end{aligned}$$

Hence that $t'<0$ can not happen. This follows from the fact that $d(zh(t'), x_0*t')=d(z(t'\pm\theta/2), x_{-1}(t'-\tau_{-1}))>\varepsilon_7$.

Since $|h(\xi_0)-\xi_0|<\theta/2$, we have

$$\xi_0<h(\xi_0)+\theta/2=l+\theta/2<\theta/4+\theta/2<\theta\ ,$$
$$\xi_0>h(\xi_0)-\theta/2>-\theta\ ,$$

and so $-\theta<\xi_0<\theta$.

Now we are in the position to prove that the point $z_0=zl\in S$ $\varepsilon/2$-traces the (δ, β)-chain $(\{x_i\}_{i=-\infty}^{\infty}, \{t_i\}_{i=-\infty}^{\infty})$. Put $g(t)=h(t+\xi_0)-h(\xi_0)$ for any $t\in \boldsymbol{R}$. Then g is a strictly increasing homeomorphism of $\boldsymbol{R}$ such that $g(0)=0$, $g(\boldsymbol{R})=\boldsymbol{R}$. Thus

$$\begin{aligned} d(z_0g(t), x_0*(t+\xi_0))&=d(z(l+g(t)), x_0*(t+\xi_0))\\ &=d(zh(t+\xi_0), x_0*(t+\xi_0))\\ &<\varepsilon'<\varepsilon/4\ . \end{aligned} \tag{1}$$

For $\tau_i\leqq t<\tau_{i+1}$ $(i\in \boldsymbol{Z})$, it is enough to prove the following to obtain the conclusion.

$$d(z_0g(t), x_i(t-\tau_i))<\varepsilon/2\ .$$

Indeed, if $\tau_i\leqq t+\xi_0<\tau_{i+1}$, we have by (B1) and (1)

$$\begin{aligned} &d(z_0g(t), x_i(t-\tau_i))\\ &\quad\leqq d(z_0g(t), x_0*(t+\xi_0))+d(x_0*(t+\xi_0), x_i(t-\tau_i))\\ &\quad<\varepsilon/4+d(x_i(t+\xi_0-\tau_i), x_i(t-\tau_i))\\ &\quad<\varepsilon/4+\varepsilon/4=\varepsilon/2\ , \end{aligned}$$

and if $\tau_{i+1}\leqq t+\xi_0<\tau_{i+2}$, then

$$\begin{aligned} &d(z_0g(t), x_i(t-\tau_i))\\ &\quad\leqq d(z_0g(t), x_0*(t+\xi_0))+d(x_0*(t+\xi_0), x_i(t-\tau_i))\\ &\quad<\varepsilon/4+d(x_{i+1}(t+\xi_0-\tau_{i+1}), x_i(\tau_{i+1}-\tau_i)(t-\tau_{i+1}))\ . \end{aligned}$$

Since $d(x_{i+1}, x_i(\tau_{i+1}-\tau_i))=d(x_{i+1}, \varphi(x_i))<\delta<\varepsilon'$, we have by (B2)

$$d(x_{i+1}(t+\xi_0-\tau_{i+1}), x_i(\tau_{i+1}-\tau_i)(t-\tau_{i+1}))<\varepsilon/4\ ,$$

and hence

$$d(z_0g(t), x_i(t-\tau_i))<\varepsilon/2\ .$$

For the case that $\tau_{i-1} \leqq t+\xi_0 < \tau_i$ the analogous argument ensures that

$$d(z_0 g(t), x_i(t-\tau_i)) < \varepsilon/2 \qquad \text{for} \quad \tau_i \leqq t < \tau_{i+1} \quad (i \in \boldsymbol{Z}) .$$

To obtain Theorem A it is enough to see that z_0 η-traces the δ-pseudo orbit $\{x_i\}_{i=-\infty}^{\infty}$. To do this assume that the δ-pseudo orbit $\{x_i\}_{i=-\infty}^{\infty}$ is not η-traced by z_0 in the positive direction. Since $d(z_0, x_0) < \varepsilon/2 < \eta$, there is j such that $d(z_i, x_i) < \eta$ for $0 \leqq i < j$ and

$$d(z_j, x_j) \geqq \eta ,$$

where $z_i = P_\rho(z_{i-1} t_{i-1})$. Put $z_{i+1} = z_i u_i$ for u_i with $|t_i - u_i| \leqq \rho$. Since $2\rho < \beta$, each u_i is determined uniquely. For simplicity write

$$l_j = \sum_{n=0}^{j-1} u_n \quad \text{and} \quad \tau_j = \sum_{n=0}^{j-1} t_n .$$

Then we have either

(a) $$|g(\tau_j) - l_j| \leqq \mu ,$$

or

(b) $$|g(\tau_j) - l_j| > \mu .$$

For the both cases (a) and (b) we can derive contradictions as follows. For the case (a), from (A4) and the fact that $d(z_j, x_j) \geqq \eta > \varepsilon_2$, we have

$$\begin{aligned} \varepsilon/2 &> d(z_0 g(\tau_j), x_0 * \tau_j) \\ &= d(z_j(g(\tau_j) - l_j), x_j) > \varepsilon_3 > \varepsilon , \end{aligned}$$

which is a contradiction. For the case (b), we can find $0 < k \leqq j$ such that $|g(\tau_k) - l_k| > \mu$ and $|g(\tau_i) - l_i| \leqq \mu$ $(0 \leqq i < k)$. If there is $0 \leqq i < k$ such that $d(z_i, x_i) \geqq \varepsilon_2$, then $d(zg(\tau_i), x_i) > \varepsilon_3 > \varepsilon$ by (A4) (since $z_i = z l_i$ and $|g(\tau_i) - l_i| \lneqq \mu$). However $d(z_0 g(\tau_i), x_i) = d(z_0 g(\tau_i), x_0 * \tau_i) < \varepsilon/2$, which is impossible. Therefore we have

$$d(z_i, x_i) < \varepsilon_2 \qquad (0 \leqq i < k) . \tag{2}$$

Combing (2) and (A3), we have

$$d(z_k, \varphi(x_{k-1})) < \varepsilon_1 . \tag{3}$$

It is easily checked that (3) is inconsistent with $|g(\tau_k) - l_k| > \mu$. For, if $l_k - \mu > g(\tau_k) \geqq l_{k-1} - \mu$, then

$$\mu < l_k - g(\tau_k) \leqq l_k - l_{k-1} + \mu = u_{k-1} + \mu \leqq \alpha + \rho + \mu < \zeta .$$

By (3) and (A2) we have $d(z_0 g(\tau_k), \varphi(x_{k-1})) > \varepsilon_1$. Thus

$$\begin{aligned} \varepsilon/2 &> d(z_0 g(\tau_k), x_0 * \tau_k) \\ &= d(z_0 g(\tau_k), x_k) \\ &\geqq d(z_0 g(\tau_k), \varphi(x_{k-1})) - d(\varphi(x_{k-1}), x_k) \\ &> \varepsilon_1 - \delta > \varepsilon - \varepsilon/2 = \varepsilon/2 , \end{aligned}$$

which is impossible. If $g(\tau_k) < l_{k-1} - \mu$, then $g(\tau_k) < l_{k-1} - \mu \leqq g(\tau_{k-1})$ (since $|g(\tau_{k-1}) - l_{k-1}| \leqq \mu$) which contradicts the facts that g is strictly increasing and $\tau_{k-1} < \tau_k$. If $g(\tau_k) - l_k > \mu$, then there exists $\tau_{k-1} < t' < \tau_k$ with $g(t') = l_k + \mu$ since $g(\tau_{k-1}) \leqq l_{k-1} + \mu < l_k + \mu < g(\tau_k)$. And so

$$\mu < \mu + \tau_k - t' < \mu + (\tau_k - \tau_{k-1}) \leqq \mu + \alpha < \zeta .$$

From (3) and (A2) it follows that $d(z_k(\mu + \tau_k - t'), \varphi(x_{k-1})) > \varepsilon_1$. Since $\varphi(x_{k-1}) = x_{k-1} t_{k-1} = x_{k-1}(\tau_k - \tau_{k-1})$, we have $\varphi(x_{k-1})(t' - \tau_k) = x_{k-1}(t' - \tau_{k-1})$. Thus by (A5)

$$\begin{aligned} \varepsilon/2 &> d(z_0 g(t'), x_{k-1}(t' - \tau_{k-1})) \\ &= d(z_0(l_k + \mu), x_{k-1}(t' - \tau_{k-1})) \\ &= d(z_k(\mu + \tau_k - t')(t' - \tau_k), \varphi(x_{k-1})(t' - \tau_k)) \\ &\geqq \varepsilon_4 > \varepsilon , \end{aligned}$$

thus contradicting. Therefore the point z_0 η-traces the δ-pseudo orbit $\{x_i\}_{i=-\infty}^{\infty}$ in the positive direction.

It remains only to prove that the point z_0 η-traces the δ-pseudo orbit $\{x_i\}_{i=-\infty}^{\infty}$ in the negative direction. To do this if this is false, then there exists $j < 0$ such that $d(z_i, x_i) < \eta$ for $j < i \leqq 0$ and $d(z_j, x_j) \geqq \eta$, where $z_i = P_\rho(z_{i+1}(-t_i))$ for $j \leqq i < 0$. Let $u_i \in \boldsymbol{R}$ satisfy $z_i = z_{i+1}(-u_i)$ and $|t_i - u_i| \leqq \rho$. Put $l_j = -\sum_j^{-1} u_n$ and $\tau_j = -\sum_j^{-1} t_n$. If $|g(\tau_j) - l_j| \leqq \mu$, then

$$d(z_0 l_j, x_j) = d(z_j, x_j) \geqq \eta > \varepsilon_1 > \varepsilon_2$$

and by (A4)

$$d(z_0 g(\tau_j), x_j) > \varepsilon_3 > \varepsilon ,$$

which contradicts $d(z_0 g(\tau_j), x_0 * \tau_j) < \varepsilon/2$. Hence $|g(\tau_j) - l_j| > \mu$. Since there is $j \leqq k < 0$ such that $|g(\tau_k) - l_k| > \mu$ and $|g(\tau_i) - l_i| \leqq \mu$ $(k < i < 0)$, we have as in (2)

$$d(z_i, x_i) < \varepsilon_2 \qquad \text{for} \quad k < i \leqq 0 ,$$

from which

$$d(\varphi(x_k), z_{k+1}) \leqq d(\varphi(x_k), x_{k+1}) + d(x_{k+1}, z_{k+1}) < \delta + \varepsilon_2 < 2\varepsilon_2 .$$

From (A3) together with $(z_{k+1})_{-1}=z_k$, we have $d(z_k, x_k)<\varepsilon_1$, which is inconsistent with $|g(\tau_k)-l_k|\geqq\mu$. Therefore the point z_0 η-traces the δ-pseudo orbit $\{x_i\}_{i=-\infty}^{\infty}$ in the negative direction.

PROOF OF COROLLARY B. Let $(X, \boldsymbol{R})$ have POTP. Then φ has POTP by Theorem A. For $\eta>0$ there is $0<\delta<\eta/2$ such that any δ-pseudo orbit of φ is $\eta/2$-traced by some point of S^+.

If $d(x, y)<\delta$ for $x, y\in T^+$, then a doubly infinite sequence $\{\cdot\cdot, \varphi^{-i}(y), \cdot\cdot, \varphi^{-1}(y), x, \varphi(x), \cdot\cdot, \varphi^i(x), \cdot\cdot\}$ is a δ-pseudo orbit of φ, and hence it is $\eta/2$-traced by some point $z\in S^+$. Therefore $z\in W_\eta^s(x)$ and $d(\varphi^{-i}(y), z_{-i})<\eta/2$ for $i\geqq1$. On the other hand, since $d(x, y)<\delta$ and $d(x, z)<\eta/2$, we have

$$d(y, z)\leqq d(y, x)+d(x, z)<\delta+\eta/2<\eta/2+\eta/2=\eta ,$$

from which $z\in W_\eta^u(y)$. Therefore $W_\eta^s(x)\cap W_\eta^u(y)\neq\emptyset$, which implies that φ has a canonical coordinate.

§ 3. Proof of Theorem C.

Let $0<\zeta<\varepsilon_0$ and $0<\alpha<\zeta/3$ be as in § 1. Choose $0<a<\beta/2$ as in Remark 2 and $\rho>0$ as in § 1 ($5\rho<\zeta$ and $2\rho<\beta$).

First we prove the "only if" part. Take η and μ such that $0<\eta<a$ and $0<\mu<\zeta-\alpha-\rho$. For η and μ as before we can choose positive numbers ε_1, ε_2, ε_3 and ε_4 as in Claim 1 of § 2. For $0<\varepsilon<\min\{\varepsilon_2, \varepsilon_3, \varepsilon_4\}$ and $0<\tau<\min\{\mu, \beta\}$ as in Claim 2 of § 2 we can choose positive numbers θ, ε_5, ε_6, ε_7 and ε_8.

Since $(X, \boldsymbol{R})$ has POTP and $\beta>0$, $(X, \boldsymbol{R})$ has POTP with respect to time β from Proposition 1.4 [7]. For $0<\varepsilon'<\min\{\varepsilon/4, \varepsilon_6, \varepsilon_7/2, \varepsilon_8\}$ there exists $0<\delta<\varepsilon'$ such that any (δ, β)-chain of $(X, \boldsymbol{R})$ is ε'-traced by some point of X. It is enough to show that δ is a number with property POTP for η.

Now let $\{x_i\}_{i=0}^{\infty}$ be a δ-pseudo orbit of $\tilde{\varphi}$. Then a pair $(\{x_i\}_{i=0}^{\infty}, \{t_i\}_{i=0}^{\infty})$ is a half (δ, β)-chain of $(X, \boldsymbol{R})$, where $\tilde{\varphi}(x_i)=x_it_i$. Then there is a point $z\in X$ which ε'-traces the (δ, β)-chain. Thus there exists a strictly increasing homeomorphism h of $\boldsymbol{R}$ such that $h(0)=0$, $h(\boldsymbol{R})=\boldsymbol{R}$ and

$$d(zh(t), x_0*t)<\varepsilon' \qquad \text{for all} \quad t\in\boldsymbol{R} .$$

Since $d(z, x_0)=d(zh(0), x_0)<\varepsilon'<\varepsilon_6$, there exists $l>0$ such that $zl \in S_j$ and $|l-t_0|<\theta/4$ by (B3) ($\tilde{\varphi}(x_0)=x_0t_0 \in T_j$). Put $z_1=zl$ and take $\xi_0>0$ with $h(\xi_0)=l$. Then we have that $|h(\xi_0)-\xi_0|<\theta/2$ and so

$$-\theta<\xi_0-t_0<\theta . \tag{4}$$

Indeed, if $|h(\xi_0)-\xi_0|\geqq\theta/2$, then there is $0<t'\leqq\xi_0$ such that $|h(t')-t'|=\theta/2$. Since $h(\xi_0)=l<t_0+\theta/4<t_0+\rho/4$, we have $0<t'<t_0+\beta$. For $t_0\leqq t'<t_0+t_1$, since

$$0<t'-t_0\leqq\theta/2+h(\xi_0)-t_0<\theta/2+\theta/4<\theta<\beta ,$$

we have

$$\begin{aligned} d(zh(t'), x_0*t')&=d(z(t'\pm\theta/2), x_1(t'-t_0)) \\ &=d(zt_0(t'-t_0\pm\theta/2), x_1(t'-t_0)) , \end{aligned}$$

$$d(x_1, x_0t_0)<\delta<\varepsilon'<\varepsilon_7/2 .$$

Since $d(zt_0, x_0t_0)<\varepsilon_7/2$ by (B5), we have by (B4)

$$\varepsilon'>d(zh(t'), x_0*t')>\varepsilon_7 ,$$

thus contradicting. Since $d(z, x_0)<\varepsilon'<\varepsilon_7$, if $0<t'<t_0$, we have by (B4)

$$\varepsilon'>d(zh(t'), x_0*t')=d(z(t'\pm\theta/2), x_0t')>\varepsilon_7 ,$$

which is a contradiction. Therefore $|h(\xi_0)-\xi_0|<\theta/2$.

Put $g(t)=h(t+\xi_0)-h(\xi_0)$ for any $t\in \boldsymbol{R}$. Then g is a strictly increasing homeomorphism of $\boldsymbol{R}$ with $g(0)=0$, $g(\boldsymbol{R})=\boldsymbol{R}$, and so

$$\begin{aligned} d(z_1g(t), x_0*(t+\xi_0))&=d(z(l+g(t)), x_0*(t+\xi_0)) \\ &=d(zh(t+\xi_0), x_0*(t+\xi_0)) \\ &<\varepsilon'<\varepsilon/4 . \end{aligned} \tag{5}$$

Let $y_i=x_{i+1}$ and $s_i=t_{i+1}$ for $i\geqq0$. Put $\tilde{\tau}_i=\sum_{n=1}^{i} t_n$ $(i\geqq1)$, $\tilde{\tau}_0=0$. Using (4) and (5), we can easily check that

$$d(z_1g(t), y_i(t-\tilde{\tau}_i))<\varepsilon/2 \qquad \text{for} \quad \tilde{\tau}_i\leqq t<\tilde{\tau}_{i+1}$$

for each $i\geqq0$. Thus z_1 $\varepsilon/2$-traces the half (δ, β)-chain $(\{y_i\}_{i=0}^{\infty}, \{s\}_{i=0}^{\infty})$, which ensures that z_1 η-traces the δ-pseudo orbit $\{x_i\}_{i=0}^{\infty}$ of $\tilde{\varphi}$.

It remains to prove "if" part. Let $\mathscr{S}$ be as in Proposition 1 of §1. For a local cross section $S_i\in\mathscr{S}$ set $D_\xi=S_i[-\xi, \xi]$ $(0<\xi<\zeta)$ and define a projective map P_ξ^i: $D_\xi^i\to S_i$ by $P_\xi^i(x)=xt$, where $xt\in S_i$ and $|t|\leqq\xi$.

For $\eta>0$ we can find $0<\xi_1<\rho$ such that

(C1) $d(x, xt)<\eta/2$ for $x\in X$ and $|t|\leqq\xi_1$.

Let $a>0$ be as in Remark 2 and take $N>0$ such that $0<\eta/N<a$ and

(C2) if $d(x, y)<\eta/N$ $(x, y\in X)$ and $xt\in T_j$ $(|t|\leqq 6\alpha)$ for some T_j then $yt\in D_\xi^j$, where $\xi=\xi_1\beta/(12\alpha)$,

(C3) if $d(x, y)<\eta/N$ $(x, y\in X)$, then $d(xt, yt)<\eta/2$ for some $|t|\leqq 6\alpha$.

Since $\tilde{\varphi}$ has POTP, let $0<\delta<\eta/N$ be a number with the property of POTP of $\tilde{\varphi}$ for η/N. Choose $0<\xi_2<\xi_1/2$ and $0<\varepsilon<\delta/2$ such that

(C4) $d(x, xt)<\delta/2$ for $x\in X$ and $|t|\leqq\xi_2$,

(C5) if $d(x, y)<\varepsilon$ $(x\in T$ and $y\in X)$, then $y\in D_{\xi_2}$.

Let $0<\delta'<\min\{a, \delta\}$ be a number such that

(C6) if $d(x, y)<\delta'$ $(x, y\in X)$, then $d(xt, yt)<\varepsilon$ for $|t|\leqq\alpha$.

Let a pair $(\{x_i\}_{i=0}^\infty, \{t_i\}_{i=0}^\infty)$ be a half $(\delta', 2\alpha)$-chain of $(X, \boldsymbol{R})$. To prove that the half $(\delta', 2\alpha)$-chain is η-traced by some point of X, assume that $2\alpha\leqq t_i\leqq 4\alpha$ for any $i\geqq 0$ (cf. Proposition 1.3 [7]) and put $p_n=\max\{t; 0\leqq t\leqq t_n$ and $x_nt\in T^+\}$. Then $y_n'=x_np_n\in T$ $(T\in\mathscr{T})$ and obviously $p_n\geqq\alpha$ since $t_i\geqq 2\alpha$. Let $\zeta_n=t_n-p_n$ (note that $0\leqq\zeta_n\leqq\alpha$). Since $d(x_nt_n, x_{n+1})<\delta'$ for $n\geqq 0$, by (C6)

$$d(y_n', x_{n+1}(-\zeta_n))=d(x_nt_n(-\zeta_n), x_{n+1}(-\zeta_n))<\varepsilon ,$$

and by (C5), $x_{n+1}(-\zeta_n)\in D_{\xi_2}$. Thus we can find $q_{n+1}\in\boldsymbol{R}$ such that $x_{n+1}(-q_{n+1})=P_\rho(x_{n+1}(-\zeta_n))\in S$ and $|\zeta_n-q_{n+1}|\leqq\xi_2$. For simplicity write $y_{n+1}=x_{n+1}(-q_{n+1})$ for $n\geqq 0$. By (C4) we have

$$\begin{aligned} d(y_n', y_{n+1}) &< d(y_n', x_{n+1}(-\zeta_n))+d(x_{n+1}(-\zeta_n), x_{n+1}(-q_{n+1})) \\ &< \varepsilon+\delta/2<\delta \qquad (n\geqq 0) . \end{aligned} \tag{6}$$

We construct an infinite sequence $\{y_n\}_{n=0}^\infty\subset S^+$ such that $y_0=x_0(-q_0)\in S^+$ $(0\leqq q_0\leqq\alpha)$.

Note that $|q_n-\zeta_{n-1}|\leqq\xi_2$ $(n\geqq 1)$. From facts that $0\leqq\xi_n\leqq\alpha$ $(n\geqq 0)$ and $\xi_2<\alpha/4$ we have

$$-\alpha/4<q_n<5\alpha/4 . \tag{7}$$

Let $w_n^1, \cdots, w_n^{m_n}$ be points of $y_n[0, q_n+p_n]\cap T^+$ such that $\tilde{\varphi}(w_n^i)=w_n^{i+1}$ $(i=0, 1, \cdots, m_n-1)$, where $w_n^0=y_n\in S^+$ and $w_n^{m_n}=y_n'\in T^+$. Then by (6) we have $d(\tilde{\varphi}(w_n^i), w_n^{i+1})<\delta$ for $0\leqq i<m_n-1$ and $d(\tilde{\varphi}(w_n^{m_n-1}), w_{n+1}^0)<\delta$. Hence $\{w_n^i; 0\leqq i<m_n, n\geqq 0\}$ is a δ-pseudo orbit of $\tilde{\varphi}$.

Since $\tilde{\varphi}$ has POTP, there is a point $z\in S^+$ which (η/N)-traces the δ-pseudo orbit, and hence

$$d(w_n^i, z_{m_0+\cdots+m_{n-1}+i})<\eta/N$$

for $0\leqq i<m_n$ $(n\geqq 0)$, where $m_{-1}=0$ and $z_1=z$. Let $u_n^i\in\boldsymbol{R}$ $(0\leqq i<m_n, n\geqq 0)$

be the smallest positive time such that $\tilde{\varphi}(w_n^i)=w_n^{i+1}=w_n^i u_n^i$. Obviously $\beta\leqq u_n^i\leqq\alpha$.

Let $v_n^i\in\boldsymbol{R}$ satisfies $z_{m_0+\cdots+m_{n-1}+i}\cdot v_n^i=z_{m_0+\cdots+m_{n-1}+i+1}$ and $|u_n^i-v_n^i|\leqq\rho$. Put $v_0=\sum_{i=1}^{m_0-1}v_0^i$ and $v_n=\sum_{i=0}^{m_n-1}v_n^i$ $(n\geqq 1)$. Then we have

$$z_{m_0+\cdots+m_{n-1}+m_n}=z_{m_0+\cdots+m_{n-1}}\cdot v_n$$

for $n\geqq 0$. Since $2\alpha\leqq t_n\leqq 4\alpha$, we have $\alpha\leqq p_n\leqq t_n\leqq 4\alpha$ and from (7), $\beta\cdot m_n<q_n+p_n<6\alpha$. Hence $m_n\leqq 6\alpha/\beta$. The difference between the time v_n and the time q_n+p_n is estimated as follows:

$$|v_n-(q_n+p_n)|\leqq\sum_{i=0}^{m_n-1}|u_n^i-v_n^i|\leqq 6\alpha\xi/\beta \qquad (n\geqq 0)\ , \tag{8}$$

$$|v_0+u_0^0-(q_0+p_0)|\leqq\sum_{i=1}^{m_0-1}|u_0^i-v_0^i|\leqq 6\alpha\xi/\beta\ . \tag{9}$$

To obtain Theorem C, it is only to prove that the point $z(q_0-u_0^0)$ η-traces the half $(\delta', 2\alpha)$-chain $(\{x_i\}_{i=0}^{\infty}, \{t_i\}_{i=0}^{\infty})$. To do this we construct a piecewise linear strictly increasing homeomorphism h of $\boldsymbol{R}$ with $h(0)=0$ and $h(\boldsymbol{R})=\boldsymbol{R}$. Define a linear function h_0: $[0, t_0]\to\boldsymbol{R}$ such that

$$h_0(t)=\{(v_0+u_0^0-q_0+q_1)/t_0\}\cdot t\ .$$

Then we have by (9)

$$\begin{aligned} v_0+u_0^0-q_0+q_1&\geqq p_0-6\alpha\xi/\beta+q_1\\ &\geqq\alpha-\xi_1/2-\alpha/4>\alpha/2\ ,\end{aligned}$$

$$\begin{aligned} |h_0(t)-t|&=|(v_0+u_0^0-q_0+q_1)/t_0-1|\cdot t\\ &\leqq|v_0+u_0^0-q_0+q_1-p_0-\zeta_0|\\ &\leqq|v_0+u_0^0-(q_0+p_0)|+|q_1-\zeta_0|\\ &\leqq 6\alpha\xi/\beta+\xi_2\\ &\leqq\xi_1/2+\xi_1/2=\xi_1\ . \end{aligned}\tag{10}$$

On the other hand,

$$\begin{aligned} d(z'h_0(t), x_0t)&=d(z(q_0-u_0^0+h_0(t), y_0(q_0+t))\\ &\leqq d(z(q_0-u_0^0+h_0(t)), z(q_0-u_0^0+t))\\ &\quad+d(z(q_0-u_0^0+t), y_0(q_0+t))\ .\end{aligned}$$

By (10) and (C1) we have

$$d(z(q_0-u_0^0+h_0(t)), z(q_0-u_0^0+t))<\eta/2 \qquad (t\in[0, t_0))\ .$$

Since

$$d(y_0u_0^0,\ z)=d(w_0^1,\ z_1)<\eta/N$$

and

$$|q_0-u_0^0+t|\leqq|q_0|+|u_0^0|+|t|\leqq q_0+u_0^0+t_0$$
$$\leqq\alpha+\alpha+4\alpha=6\alpha\ ,$$

we have by (C3)

$$d(z(q_0-u_0^0+t),\ y_0(q_0+t))<\eta/2\ .$$

Therefore

$$d(z'h_0(t),\ x_0t)<\eta \qquad (t\in[0,\ t_0))\ .$$

Define a function h_n on $[\tau_n,\ \tau_{n+1}]$ $(n\geqq 1)$ by

$$h_n(t)=\{(v_n-q_n+q_{n+1})/t_n\}(t-\tau_n)+\sum_{k=0}^{n-1}(v_k-q_k+q_{k+1})+u_0^0\ ,$$

where τ_n is as in §2. Since $v_n-q_n+q_{n+1}>0$ by (8), h_n is increasing. Obviously $h_n(\tau_{n+1})=h_{n+1}(\tau_{n+1})$ for $n\geqq 0$. We claim that

$$d(z'h_n(t),\ x_n(t-\tau_n))<\eta \qquad \text{for} \quad \tau_n\leqq t<\tau_{n+1}\ .$$

Indeed, we have

$$\begin{aligned}
&d(z'h_n(t),\ x_n(t-\tau_n))\\
&\quad\leqq d\Big(z'h_n(t),\ z'\Big[\sum_{k=0}^{n-1}(v_k-q_k)+u_0^0+\Big(t-\tau_n+\sum_{k=0}^{n-1}q_{k+1}\Big)\Big]\Big)\\
&\qquad+d\Big(z'\Big[\sum_{k=0}^{n-1}(v_k-q_k)+u_0^0+\Big(t-\tau_n+\sum_{k=0}^{n-1}q_{k+1}\Big)\Big],\ x_n(t-\tau_n)\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\Big|h_n(t)-\sum_{k=0}^{n-1}(v_k-q_k)-\Big(t-\tau_n+\sum_{k=0}^{n-1}q_{k+1}\Big)\Big|\\
&\quad=|\{(v_n-q_n+q_{n+1})/t_n\}(t-\tau_n)-(t-\tau_n)|\\
&\quad\leqq|v_n-q_n+q_{n+1}-t_n|\\
&\quad\leqq|v_n-q_n-p_n+p_n+q_{n+1}-t_n|\\
&\quad\leqq|v_n-q_n-p_n|+|q_{n+1}-\xi_n|\\
&\quad\leqq 6\alpha\xi/\beta+\xi_2\leqq\xi_1\ . \qquad (11)
\end{aligned}$$

Hence by (11) and (C1) we have

$$d\left(z'h_n(t),\ z'\left[\sum_{k=0}^{n-1}(v_k-q_k)+u_0^0+\left(t-\tau_n+\sum_{k=0}^{n-1}q_{k+1}\right)\right]\right)<\eta/2\ . \tag{12}$$

On the other hand,

$$x_n(t-\tau_n)=x_n(-q_n)(t-\tau_n+q_n)=y_n(t-\tau_n+q_n)\ ,$$

$$\begin{aligned} z_{m_0+\cdots+m_{n-1}} &= z_{m_0+\cdots+m_{n-2}}v_{n-1} \\ &= z\left(\sum_{k=0}^{n-1}v_k\right) \\ &= z'\left(\sum_{k=0}^{n-1}(v_k-q_k)+u_0^0+\sum_{k=0}^{n-2}q_{k+1}\right)\ . \end{aligned}$$

Since

$$d(y_n,\ z_{m_0+\cdots+m_{n-1}})=d(w_n^0,\ z_{m_0+\cdots+m_{n-1}})<\eta/N\ ,$$

we have by (C3)

$$d\left(z'\left[\sum_{k=0}^{n-1}(v_k-q_k)+u_0^0+\left(t-\tau_n+\sum_{h=0}^{n-1}q_{k+1}\right)\right],\ x_n(t-\tau_n)\right)<\eta/2\ . \tag{13}$$

Combining (12) and (13) we have

$$d(zh_n(t),\ x_n(t-\tau_n))<\eta \qquad \text{for} \quad \tau_n\leqq t<\tau_{n+1}\ .$$

Let us put

$$h(t)=\begin{cases} h_n(t) & \text{if} \quad \tau_n\leqq t<\tau_{n+1} \\ t & \text{if} \quad t\leqq 0\ . \end{cases}$$

Then h is our requirement. We proved that for any fixed $\eta>0$ there exists $\delta'>0$ such that any half $(\delta, 2\alpha)$-chain $(\{x_i\}_{i=0}^{\infty}, \{t_i\}_{i=0}^{\infty})$ of the flow $(X, \boldsymbol{R})$ is η-traced by a point $z'\in X$.

Since $(X, \boldsymbol{R})$ has no fixed points, that $(X, \boldsymbol{R})$ has POTP is equivalent to that for any $\varepsilon>0$ there exist $\delta>0$ and $a>0$ such that any half (δ, a)-chain of $(X, \boldsymbol{R})$ is ε-traced by some point of X. Therefore if the sectional surjective map $\tilde{\varphi}$ has POTP, then the flow $(X, \boldsymbol{R})$ must have POTP. The proof is completed.

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