TOKYO J. MATH.
VOL. 13, No. 1, 1990

Geodesics in Minimal Immersions of S^3 into S^{24}

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(Communicated by K. Ogiue)

In the present paper we consider geodesics which are obtained as images of great circles of $S^3(1)$ induced by an isometric minimal immersion $f: S^3(1) \to S^{24}(r)$, $r^2=1/8$, namely, geodesics of $f(S^3(1))$. $S^{24}(r)$ being regarded as a hypersphere of $\boldsymbol{R}^{25}$, we can consider such geodesics as curves in $\boldsymbol{R}^{25}$ with curvatures k_1, k_2, k_3. It is found that these are constants which depend on the choice of the geodesic except the case where f is a standard minimal immersion [8]. Equations satisfied by k_1, k_2, k_3 and the necessary and sufficient condition for an isometric minimal immersion to have a geodesic which is a circle are obtained.

Though we concentrate our topic upon the case $S^3(1) \to S^{24}(1)$, in the beginning part of the paper some properties of minimal immersions of spheres into spheres in general are recollected with some additional results.

§1. Introduction.

Isometric minimal immersions of spheres into spheres were studied by M. do Carmo and N. Wallach [1]. They established a theorem which is fundamental to the study of such immersions. In [1] we can see that such immersions can be regarded as $f: S^m(1) \to S^{n-1}(r)$ where n and r depend on m and a natural number s which is the order of the spherical harmonics on $S^m(1)$ inducing f, thus

$$n = n(m, s) = \frac{(2s+m-1)(s+m-2)!}{s!(m-1)!},$$

$$r^2 = (r(m, s))^2 = \frac{m}{s(s+m-1)}.$$

In the present paper the set of such isometric minimal immersions is denoted by IMI(m, s). From an immersion $f \in$ IMI(m, s) we get a set of immersions by the action of the group of isometries of $S^{n-1}(r)$. This set is called the equivalence class of f and is denoted by eq(f). The vector space $W(m, s)$ of Do Carmo and Wallach is the space spanned by such equivalence classes. To any such eq(f) there corresponds just one point of $W(m, s)$ but this point lies in a compact convex body $L(m, s)$ in

Received September 18, 1989

$W(m, s)$. If we take a point in the interior of $L(m, s)$, we get an equivalence class eq(f) where f is a full immersion into $S^{n-1}(r)$, but, if we take a point of $\partial L(m, s)$, we get an equivalence class eq(f) where f sends $S^m(1)$ into a sphere of dimension less than $n-1$. We consider only cases $m \geqq 3$, $s \geqq 4$ since every $f \in \mathrm{IMI}(m, s)$ is a standard minimal immersion if $m<3$ or $s<4$.

We can regard $W(m, s)$ as a linear space of some tensors [2], [4]. Any point C of $W(m, s)$ is a harmonic bi-symmetric tensor of bi-degree (s, s), namely C is a tensor of degree $2s$ satisfying the following conditions (i), (ii), (iii). In addition C satisfies the condition (iv).

(i) $C(v_1, \cdots, v_s; v_{s+1}, \cdots, v_{2s})$ is symmetric both in $v_1, \cdots, v_s$ and in $v_{s+1}, \cdots, v_{2s}$,

(ii) $C(v, \cdots, v; w, \cdots, w)=C(w, \cdots, w; v, \cdots, v)$,

(iii) $\sum_{i=1}^{m+1} C(e_i, e_i, v, \cdots, v; w, \cdots, w)=0$,

(iv) $C(w, w, v, \cdots, v; v, \cdots, v)=0$.

Here $v_1, \cdots, v_{2s}, v, w$ are arbitrary vectors of $\boldsymbol{R}^{m+1}$ in which $S^m(1)$ is embedded as the unit sphere and $\{e_1, \cdots, e_{m+1}\}$ is an orthonormal basis of $\boldsymbol{R}^{m+1}$.

REMARK. We use the following indices and adopt the summation convention if possible.

$$A, B, C, \cdots=1, \cdots, n\,; \quad h, i, j, \cdots=1, \cdots, m+1\,.$$

If $f \in \mathrm{IMI}(m, s)$, the point C of $W(m, s)$ corresponding to eq(f) is given by

$$C=\sum_A F^A \otimes F^A-\sum_A H^A \otimes H^A \tag{1.1}$$

where any one of F^A and H^A $(A=1, \cdots, n)$ is a symmetric harmonic tensor of degree s in $\boldsymbol{R}^{m+1}$. The role of such tensors is as follows. Let $\{\tilde{e}_1, \cdots, \tilde{e}_n\}$ be an orthonormal basis of $\boldsymbol{R}^n$ where $S^{n-1}(r)$ is embedded as a hypersphere of radius r, and let u be the position vector of the point of $S^m(1)$ in $\boldsymbol{R}^{m+1}$. Then $F^A(u, \cdots, u)\tilde{e}_A=i \circ f(u)$ where i is the embedding of $S^{n-1}(r)$ into $\boldsymbol{R}^n$. On the other hand, if $h \in \mathrm{IMI}(m, s)$ is a standard minimal immersion, then $H^A(u, \cdots, u)\tilde{e}_A=i \circ h(u)$. Thus $C \in W(m, s)$ can be written in the form (1.1) if and only if $C \in L(m, s)$ [1], [2]. Such a tensor C is called the associate of f (or eq(f)) or is said to be associated with f (or eq(f)).

§§ 2, 3 are written as preliminaries. We define $C_{q,r}$ and $u_{q,r}$ and state the property of the unit bi-symmetric tensor U in § 2 where some results are given which are not stated in [2]. In § 3 we define vector fields

$V_p\colon \boldsymbol{R}\to\boldsymbol{R}^n$ and functions $V_{q,r}\colon \boldsymbol{R}\to\boldsymbol{R}$ on a geodesic which is obtained as $f(u)$ where $u\colon \boldsymbol{R}\to\boldsymbol{R}^{m+1}$ denotes a great circle of $S^m(1)$. The relation of $V_{q,r}$ to $C_{q,r}$ and $u_{q,r}$ obtained there is the pivot in our calculation. § 4 is devoted to cases $f\in \mathrm{IMI}(3, 4)$. There we have only constant curvatures k_1, k_2, k_3 which depend on the geodesic considered. If $k_2\neq 0$, then $k_3\neq 0$ and

$$(k_1)^2+(k_2)^2+(k_3)^2=20\ , \qquad (k_1)^2(k_3)^2=64\ .$$

If $k_2=0$, then the geodesic is a circle on a 2-plane of the ambient $\boldsymbol{R}^{25}$. However, such geodesics exist only in some isometric minimal immersions and the point C associated with such an immersion belongs to $\partial L(3, 4)$. In § 5 we study the value of $C_{4,0}(v, w)$ which $C\in L(3, 4)$ can take when $\{v, w\}$ is a set of orthonormal vectors. It is found that the range is $[-1/15, 1/10]$.

§ 2. Property of the unit bi-symmetric tensor U.

The set of harmonic bi-symmetric tensor of bi-degree (s, s), namely, the set of tensors satisfying the conditions (i), (ii), (iii) of § 1, is denoted by $B(m, s)$. Let $B\in B(m, s)$. If we define a function $b\colon \boldsymbol{R}^{m+1}\times\boldsymbol{R}^{m+1}\to\boldsymbol{R}$ by

$$b(v, w)=B(v, \cdots, v; w, \cdots, w)\ ,$$

then b determines just one $B\in B(m, s)$, namely, if B_1 and B_2 belong to $B(m, s)$ and satisfy

$$B_1(v, \cdots, v; w, \cdots, w)=B_2(v, \cdots, v; w, \cdots, w)\ ,$$

then $B_1=B_2$.

The tensor $U\in B(m, s)$ that satisfies

$$U(v, \cdots, v; w, \cdots, w) =\sum_{p=0}^{\sigma} u_p\langle v, w\rangle^{s-2p}\langle v, v\rangle^p\langle w, w\rangle^p\ , \qquad u_0=1 \tag{2.1}$$

identically for vectors v, w of $\boldsymbol{R}^{m+1}$ is called the unit tensor or the unit element of $B(m, s)$. Here $\sigma=[s/2]$ is the largest natural number satisfying $2\sigma\leqq s$.

U may be defined by

$$\begin{aligned} U(v_1, \cdots, v_s; w_1, \cdots, w_s) &=\mathscr{S}_v\mathscr{S}_w\sum_{p=0}^{\sigma} u_p\langle v_1, v_2\rangle\cdots\langle v_{2p-1}, v_{2p}\rangle \\ &\qquad \langle w_1, w_2\rangle\cdots\langle w_{2p-1}, w_{2p}\rangle \\ &\qquad \langle v_{2p+1}, w_{2p+1}\rangle\cdots\langle v_s, w_s\rangle \end{aligned} \tag{2.2}$$

where $\mathscr{S}_v$ (resp. $\mathscr{S}_w$) means symmetrization with respect to $v_1, \cdots, v_s$ (resp. $w_1, \cdots, w_s$) [2].

As U is harmonic, we have, for $p=1, \cdots, \sigma$,

$$(s-2p+2)(s-2p+1)u_{p-1}+2p(2s+m-2p-1)u_p=0 . \tag{2.3}$$

This and $u_0=1$ determine $u_1, \cdots, u_\sigma$. As we have defined a and c' by

$$a=\sum_{p=0}^{\sigma} u_p , \qquad ac'=r^2 \tag{2.4}$$

in [2], the relation between U and the tensors H^A of a standard minimal immersion is

$$c'U=\sum_A H^A \otimes H^A . \tag{2.5}$$

If $m=3$ and $s=4$, we have

$$a=\frac{5}{16} , \qquad c'=\frac{2}{5} . \tag{2.6}$$

Let v and w be vectors of $\boldsymbol{R}^{m+1}$. As in [5] we define $B_{p,q}(v, w)$ by

$$B_{p,q}(v, w)=B(v, \cdots, v, w, \cdots, w; v, \cdots, v, w, \cdots, w) \tag{2.7}$$

where in the right hand side w appears p times before the semicolon and q times after the semicolon. As an application (2.7) defines $U_{p,q}(v, w)$ and $C_{p,q}(v, w)$. $C_{p,q}(v, w)$ vanishes if $p+q \leqq 3$ or $p+q \geqq 2s-3$ [2], [4].

LEMMA 2.1. *Let B be any element of $B(m, s)$ and let $B_{p,q}(v, w)$ be defined by* (2.7). *Then replacing v by $v+xw$ and w by $w+yv$ where x, $y \in \boldsymbol{R}$ we get*

$$\begin{aligned} B_{p,q}(v+xw,\ w+yv) &= B_{p,q}(v, w) \\ &\quad +((s-p)B_{p+1,q}(v, w)+(s-q)B_{p,q+1}(v, w))x \\ &\quad +(pB_{p-1,q}(v, w)+qB_{p,q-1}(v, w))y \\ &\quad +[*] \end{aligned} \tag{2.8}$$

where [*] *is a polynomial in x and y containing only terms of degree higher than one.*

From (2.1) or (2.2) we can see that, if the set $\{v, w\}$ is orthonormal, then $U_{p,q}(v, w)$ does not depend on the choice of the orthonormal set, so that we can write $U_{p,q}(v, w)=u_{p,q}$. Clearly $u_{p,q}$ vanishes if $p+q$ is odd and we have

$$u_{p,q}=u_{q,p}=u_{s-p,s-q}=u_{s-q,s-p} . \tag{2.9}$$

Taking an orthonormal set $\{a, b\}$ of vectors in $\boldsymbol{R}^{m+1}$, we can express a great circle of $S^m(1)$ in the form $u: \boldsymbol{R}\to\boldsymbol{R}^{m+1}$ where

$$u(t)=a \cos t+b \sin t . \tag{2.10}$$

Thus we have

$$u'(t)=-a \sin t+b \cos t , \quad u''(t)=-u(t) , \quad \|u(t)\|=\|u'(t)\|=1 .$$

If we take any element B of $B(m, s)$, we have $B_{p,q}(u(t), u'(t))$ by (2.7) and, as an application of (2.8), we get the following lemma.

LEMMA 2.2. *Let $B\in B(m, s)$ and $u(t)$ be given by* (2.10). *Then we have*

$$\begin{aligned}\frac{dB_{p,q}(u(t), u'(t))}{dt} &\\ =(s-p)B_{p+1,q}&(u, u')+(s-q)B_{p,q+1}(u, u')\\ &-pB_{p-1,q}(u, u')-qB_{p,q-1}(u, u') ,\end{aligned} \tag{2.11}$$

where $u(t)$ and $u'(t)$ are abbreviated to u and u'.

This result can be applied to U and $C\in W(m, s)$. $\{u(t), u'(t)\}$ being orthonormal, we have $U_{p,q}(u(t), u'(t))=u_{p,q}$. Hence we get, in view of (2.11),

$$(s-p)u_{p+1,q}+(s-q)u_{p,q+1}=pu_{p-1,q}+qu_{p,q-1} . \tag{2.12}$$

When we want to find the value of $u_{p,q}$, we can use (2.12) in addition to (2.2).

If $m=3$ and $s=4$, we get

$$\begin{aligned} &u_{0,0}=\frac{5}{16} , \quad u_{2,0}=-\frac{5}{48} , \quad u_{1,1}=\frac{5}{32} ,\\ &u_{4,0}=\frac{1}{16} , \quad u_{1,3}=-\frac{3}{32} , \quad u_{2,2}=\frac{19}{144} ,\\ &u_{4,2}=u_{2,0} , \quad u_{3,3}=u_{1,1} , \quad u_{4,4}=u_{0,0} . \end{aligned}$$

§3. **Vector fields V_p and functions $V_{q,r}$ on a geodesic.**

When an orthonormal set $\{a, b\}$ of vectors of R^{m+1} is given, we get a great circle of $S^m(1)$ such that

$$u(t)=a \cos t+b \sin t\ . \tag{3.1}$$

The image $f(u(t))$ by $f \in \mathrm{IMI}(m, s)$ describes a geodesic of $f(S^m(1))$. Conversely any geodesic of $f(S^m(1))$ can be expressed by $f(u(t))$ where $u(t)$ is given by (3.1) with $\{a, b\}$ depending on the choice of the geodesic. Thus a geodesic of $f(S^m(1))$ parametrized by its arc length can be expressed by

$$X^A(t)=F^A(u(t), \cdots, u(t))$$

in R^n where F^A are harmonic tensors explained in §1.

Let us define functions $F_p^A(t)$ by

$$F_p^A(t)=F^A(u(t), \cdots, u(t), u'(t), \cdots, u'(t)) \tag{3.2}$$

where in the right hand side $u(t)$ appears $s-p$ times and $u'(t)$ appears p times. Then we get

$$\frac{dF_p^A(t)}{dt}=(s-p)F_{p+1}^A(t)-pF_{p-1}^A(t) \tag{3.3}$$

by virtue of $u''=-u$. Besides, we have

$$F_p^A(t+\pi/2)=(-1)^p F_{s-p}^A(t)\ . \tag{3.4}$$

Let us now define V_p by

$$V_p(t)=F_p^A(t)\tilde{e}_A\ . \tag{3.5}$$

Then from (3.3) and (3.4) we get

$$\frac{dV_p}{dt}=(s-p)V_{p+1}-pV_{p-1}\ , \tag{3.6}$$

$$V_p(t+\pi/2)=(-1)^p V_{s-p}(t)\ . \tag{3.7}$$

Differentiating $X(t)=V_0(t)$ repeatedly with respect to t, we get, by virtue of (3.6),

$$\begin{aligned} &X(t)=V_0(t)\ , \\ &\frac{dX(t)}{dt}=sV_1(t)\ , \end{aligned} \tag{3.8}$$

$$\frac{d^2X(t)}{dt^2} = -s\,V_0(t) + s(s-1)\,V_2(t)\ ,$$

$$\frac{d^3X(t)}{dt^3} = (-3s^2+2s)\,V_1(t) + s(s-1)(s-2)\,V_3(t)\ ,$$

$$\cdots\cdots\cdots\cdots$$

which may be written

(3.9)
$$\frac{d^{2p}X(t)}{dt^{2p}} = \sum_{q=0}^{p} a_{p,q}\,V_{2q}(t)\ ,$$
$$\frac{d^{2p+1}X(t)}{dt^{2p+1}} = \sum_{q=0}^{p} b_{p,q}\,V_{2q+1}(t)\ .$$

$V_{q,r}(t)$ is defined by

(3.10)
$$V_{q,r}(t) = \langle V_q(t), V_r(t)\rangle = \sum_A F_p^A(t) F_r^A(t)\ .$$

We have

(3.11)
$$V_{q,r}(t+\pi/2) = (-1)^{q+r}\,V_{s-q,s-r}(t)\ .$$

From (1.1), (2.5) and $U_{q,r}(u(t), u'(t)) = u_{q,r}$ we get

(3.12)
$$V_{q,r}(t) = C_{q,r}(u(t), u'(t)) + c' u_{q,r}\ .$$

If $q+r \leqq 3$ or $q+r \geqq 2s-3$, then we have

$$V_{q,r}(t) = c' u_{q,r}$$

because of $C_{q,r}(v, w) = 0$ [4].

§4. Geodesics in isometric minimal immersions $S^3(1) \to S^{24}(r)$.

The Frenet formula of a geodesic in this case, considered as a curve in R^{25}, is written as follows,

$$\frac{dX}{dt} = i_1\ ,$$

$$\frac{di_1}{dt} = k_1 i_2\ ,$$

$$\frac{di_2}{dt} = -k_1 i_1 + k_2 i_3\ ,$$

$$\frac{di_3}{dt} = -k_2 i_2 + k_3 i_4\ ,$$

$$\frac{di_4}{dt} = -k_3 i_3$$

where $\{i_1, i_2, i_3, i_4\}$ is an orthonormal set of vectors in R^{25} depending on t. This formula stops as above since we have

$$\frac{dX}{dt} = 4V_1 ,$$

$$\frac{d^2X}{dt^2} = -4V_0 + 12V_2 ,$$

$$\frac{d^3X}{dt^3} = -40V_1 + 24V_3 ,$$

$$\frac{d^4X}{dt^4} = 40V_0 - 192V_2 + 24V_4 ,$$

$$\frac{d^5X}{dt^5} = 544V_1 - 480V_3 .$$

Eliminating V_1 and V_3 we get

$$\frac{d^5X}{dt^5} + 20\frac{d^3X}{dt^3} + 64\frac{dX}{dt} = 0 . \tag{4.1}$$

First we have

$$i_1 = 4V_1 ,$$
$$k_1 i_2 = -4V_0 + 12V_2 ,$$

hence

$$(k_1)^2 = 16V_{0,0} - 96V_{2,0} + 144V_{2,2} .$$

Now, in view of (3.12) we have

$$V_{0,0} = c'u_{0,0} , \quad V_{2,0} = c'u_{2,0} , \quad V_{2,2} = c'u_{2,2} + C_{2,2}(u, u')$$

where $C_{p,q}(u(t), u'(t))$ is abbreviated to $C_{p,q}(u, u')$. As $s=4$, $C_{q,r}$ vanishes if $q+r \neq 4$ and this results in $dC_{q,r}(u, u')/dt = 0$ since we have (2.11). Thus k_1 is a constant.

As we get $C_{4,0} = -4C_{3,1} = 6C_{2,2}$ from $C_{3,0} = C_{2,1} = 0$, we can put

$$V_{2,2} = c'u_{2,2} + \frac{1}{6}C_{4,0}(u, u') .$$

As we have $c' = 2/5$, we get

$$V_{0,0}=\frac{1}{8}\,, \qquad V_{2,0}=-\frac{1}{24}\,,$$
$$V_{2,2}=\frac{19}{360}+\frac{1}{6}C_{4,0}(u,\ u')\,,$$

hence

$$(k_1)^2=\frac{68}{5}+24C_{4,0}(u,\ u')\,. \tag{4.2}$$

As any geodesic is a curve on $S^{24}(r)$, $r^2=1/8$, k_1 must satisfy $(k_1)^2\geqq 8$.

As k_1 is a constant, we get from $d(k_1 i_2)/dt=-40\,V_1+24\,V_3$

$$-(k_1)^2 i_1+k_1k_2i_3=-40\,V_1+24\,V_3\,, \tag{4.3}$$

hence

$$(k_1)^2((k_1)^2+(k_2)^2)=1600\,V_{1,1}-1920\,V_{3,1}+576\,V_{3,3}\,.$$

As we have

$$V_{1,1}=c'u_{1,1}=\frac{1}{16}\,, \qquad V_{3,3}=c'u_{3,3}=c'u_{1,1}=\frac{1}{16}\,,$$
$$V_{3,1}=c'u_{3,1}+C_{3,1}(u,\ u')=-\frac{3}{80}+\left(-\frac{1}{4}\right)C_{4,0}(u,\ u')\,,$$

we get

$$(k_1)^2((k_1)^2+(k_2)^2)=208+480C_{4,0}(u,\ u') \tag{4.4}$$

which proves that k_2 is also a constant. From this and (4.2) we get

$$(k_1)^2(k_2)^2=8\cdot\left(\frac{36}{25}\right)(1-10C_{4,0}(u,\ u'))(2+5C_{4,0}(u,\ u'))\,. \tag{4.5}$$

Differentiating (4.3) with respect to t, we get, as k_1 and k_2 are constants,

$$\begin{aligned}-((k_1)^2+(k_2)^2)k_1i_2+k_1k_2k_3i_4\\ =40\,V_0-192\,V_2+24\,V_4\,.\end{aligned}$$

Thus we have

$$\begin{aligned}&(k_1)^2((k_1)^2+(k_2)^2)^2+(k_1k_2k_3)^2\\ &\quad=1600\,V_{0,0}-15360\,V_{2,0}+36864\,V_{2,2}+1920\,V_{4,0}\\ &\qquad-9216\,V_{4,2}+576\,V_{4,4}\,.\end{aligned}$$

Then, substituting

$$V_{4,0}=c'u_{4,0}+C_{4,0}(u,\ u')=\frac{1}{40}+C_{4,0}(u,\ u')\ ,$$

$$V_{4,2}=c'u_{4,2}=-\frac{1}{24}\ ,\qquad V_{4,4}=c'u_{4,4}=\frac{1}{8}$$

into this formula, we get

$$\begin{aligned}&(k_1)^2((k_1)^2+(k_2)^2)^2+(k_1k_2k_3)^2\\&\quad=\frac{16448}{5}+8064C_{4,0}(u,\ u')\end{aligned}$$

and further

$$\begin{aligned}&(k_1)^4(k_2k_3)^2\\&\quad=\left(\frac{16448}{5}+8064C_{4,0}(u,\ u')\right)\left(\frac{68}{5}+24C_{4,0}(u,\ u')\right)\\&\qquad-(208+480C_{4,0}(u,\ u'))^2\\&\quad=8^3\cdot\left(\frac{36}{25}\right)(1-10C_{4,0}(u,\ u'))(2+5C_{4,0}(u,\ u'))\end{aligned}$$

by virtue of (4.2) and (4.4). From this result and (4.5) it becomes clear that the curvatures satisfy $(k_1)^4(k_2k_3)^2=8^2(k_1k_2)^2$, hence

$$(k_1)^2(k_3)^2=64 \tag{4.6}$$

if $k_2\neq 0$.

From (4.4) and (4.6) we get

$$(k_1)^2((k_1)^2+(k_2)^2+(k_3)^2)=272+480C_{4,0}(u,\ u')\ .$$

This and (4.2) result in

$$(k_1)^2+(k_2)^2+(k_3)^2=20\ . \tag{4.7}$$

Thus we have obtained the following theorem.

THEOREM 4.1. *Let $\Gamma=f(\gamma)$ be a geodesic of $f(S^3(1))$ where f is an isometric minimal immersion* $\in$ IMI(3, 4), *and k_1, k_2, k_3 be the curvatures of Γ when it is considered as a curve in the ambient $\boldsymbol{R}^{25}$. Then the curvatures are constants and $k_1\geqq 8^{1/2}$. If $k_2\neq 0$, the curvatures satisfy* (4.6) *and* (4.7).

REMARK. This theorem shows that $k_3=0$ can occur only if $k_2=0$.

If $k_2=0$, we get from (4.5)

$$\left(C_{4,0}(u, u')+\frac{2}{5}\right)\left(C_{4,0}(u, u')-\frac{1}{10}\right)=0 . \tag{4.8}$$

If $C_{4,0}(u, u')=-2/5$ occurs, we get $(k_1)^2=4$, contrary to $k_1 \geqq 8^{1/2}$. Hence we have

$$C_{4,0}(u, u')=\frac{1}{10} .$$

From $k_1=$constant and $k_2=0$ we see that the curve is a circle in a 2-plane of R^{25}. Thus we get the following theorem.

THEOREM 4.2. *The necessary and sufficient condition for an isometric minimal immersion $f \in \mathrm{IMI}(3, 4)$ to have a geodesic which is a circle in a 2-plane of the ambient R^{25} is that the associated tensor C is such that there exists an orthonormal pair of vectors $\{v, w\}$ in R^4 satisfying $C_{4,0}(v, w)=1/10$. The curvature is given by $k_1=4$. If $f \in \mathrm{IMI}(3, 4)$ is such that the associated tensor C satisfies $C_{4,0}(v, w)=1/10$ for some orthonormal $\{v, w\}$, then the great circle $u(t)=v\cos t+w\sin t$ of $S^3(t)$ is sent by f into a 2-plane of R^{25} and the image is a circle of radius 1/4. Besides, C belongs to $\partial L(3, 4)$.*

PROOF. Let us show that $C \in \partial L(3, 4)$. (4.5) states that, if C belongs to $L(3, 4)$, then

$$(1-10C_{4,0}(v, w))(2+5C_{4,0}(v, w)) \geqq 0$$

for every orthonormal $\{v, w\}$. Now, let an element $C \in L(3, 4)$ be such that, for some orthonormal $\{v, w\}$, $C_{4,0}(v, w)=1/10$. Then taking any $\lambda>1$, we get $\lambda C_{4,0}(v, w)>1/10$, hence

$$(1-10\lambda C_{4,0}(v, w))(2+5\lambda C_{4,0}(v, w))<0 .$$

Thus λC does not belong to $L(3, 4)$ and this proves $C \in \partial L(3, 4)$.

Geodesics which are circles exist [7]. Thus there also exists C such as stated in Theorem 4.2.

§5. Value of $C_{4,0}(v, w)$ which $C \in L(3, 4)$ can take when $\{v, w\}$ is a set of orthonormal vectors.

As X is a solution of the differential equation (4.1), there exists a set of vectors a_0, a_1, b_1, a_2, b_2 in R^{25} such that

$$
\begin{aligned}
X(t) &= a_0 + a_1 \cos 2t + b_1 \sin 2t + a_2 \cos 4t + b_2 \sin 4t \,, \\
\frac{dX}{dt} &= -2a_1 \sin 2t + 2b_1 \cos 2t - 4a_2 \sin 4t + 4b_2 \cos 4t \,, \\
\frac{d^2X}{dt^2} &= -4a_1 \cos 2t - 4b_1 \sin 2t - 16a_2 \cos 4t - 16b_2 \sin 4t \,, \\
\frac{d^3X}{dt^3} &= 8a_1 \sin 2t - 8b_1 \cos 2t + 64a_2 \sin 4t - 64b_2 \cos 4t \,, \\
\frac{d^4X}{dt^4} &= 16a_1 \cos 2t + 16b_1 \sin 2t + 256a_2 \cos 4t + 256b_2 \sin 4t \,.
\end{aligned} \tag{5.1}
$$

On the other hand, X satisfies

$$
\begin{aligned}
&\langle X, X\rangle = \frac{1}{8}\,, \quad \left\langle X, \frac{dX}{dt}\right\rangle = 0\,, \quad \left\langle \frac{dX}{dt}, \frac{dX}{dt}\right\rangle = 1\,, \\
&\left\langle X, \frac{d^2X}{dt^2}\right\rangle = -1\,, \quad \left\langle X, \frac{d^3X}{dt^3}\right\rangle = 0\,, \quad \left\langle \frac{dX}{dt}, \frac{d^2X}{dt^2}\right\rangle = 0\,, \\
&\left\langle \frac{d^2X}{dt^2}, \frac{d^2X}{dt^2}\right\rangle = (k_1)^2\,, \quad \left\langle \frac{d^2X}{dt^2}, \frac{d^3X}{dt^3}\right\rangle = 0\,.
\end{aligned}
$$

Thus we can see that a_0, a_1, b_1, a_2, b_2 are mutually orthogonal satisfying

$$
\begin{aligned}
&\langle a_0, a_0\rangle = \frac{1}{64}(-12 + (k_1)^2)\,, \\
&\langle a_1, a_1\rangle = \langle b_1, b_1\rangle = \frac{1}{48}(16 - (k_1)^2)\,, \\
&\langle a_2, a_2\rangle = \langle b_2, b_2\rangle = \frac{1}{192}(-4 + (k_1)^2)\,.
\end{aligned}
$$

This result shows that k_1 is restricted by

$$12 \leqq (k_1)^2 \leqq 16\,. \tag{5.2}$$

Now let us consider all great circles of $S^3(1)$. Then, in place of $C_{4,0}(u, u')$, we can take $C_{4,0}(v, w)$ where $\{v, w\}$ is an arbitrary set of orthonormal vectors. From (4.2) and (5.2) we get the following theorem.

THEOREM 5.1. *Let C be a point of $W(3, 4)$ in $L(3, 4)$ and $\{v, w\}$ be a set of orthonormal vectors of R^4. Then $C_{4,0}(v, w)$ satisfies*

$$-\frac{1}{15} \leqq C_{4,0}(v, w) \leqq \frac{1}{10}\,.$$

Let us cite an example from § 9 of [6]. We take a homogeneous harmonic polynomial a of degree four in the variables ξ_1, ξ_2, ξ_3

$$\begin{aligned} a &= a^{\kappa\lambda\mu\nu}\xi_\kappa\xi_\lambda\xi_\mu\xi_\nu \\ &= 5(\xi^4+\eta^4+\zeta^4)-3(\xi^2+\eta^2+\zeta^2)^2 \end{aligned}$$

where $\xi=\xi_1$, $\eta=\xi_2$, $\zeta=\xi_3$. Taking as in [3] a set $\{J_1, J_2, J_3\}$ of linear transformations acting on $\boldsymbol{R}^4$ such that

$$\begin{aligned} &J_2J_3=-J_3J_2=J_1 , \quad J_3J_1=-J_1J_3=J_2 , \\ &J_1J_2=-J_2J_1=J_3 , \quad J_1J_1=J_2J_2=J_3J_3=-1 , \end{aligned}$$

we can define an element $C=C_J^{(a)}$ of $W(3, 4)$ by

$$\begin{aligned} &C_J^{(a)}(v, v, v, v; w, w, w, w) \\ &\qquad = a^{\kappa\lambda\mu\nu}\xi_\kappa(v, w)\xi_\lambda(v, w)\xi_\mu(v, w)\xi_\nu(v, w) \end{aligned}$$

where $\xi_\kappa(v, w)=\langle J_\kappa w, v\rangle$.

Let $\{v, w\}$ be an arbitrary set of orthonormal vectors. If we put $\alpha=-\langle J_1w, v\rangle$, $\beta=-\langle J_2w, v\rangle$, $\gamma=-\langle J_3w, v\rangle$, then we have $\alpha^2+\beta^2+\gamma^2=1$ and $w=(\alpha J_1+\beta J_2+\gamma J_3)v$. Conversely, if α, β, γ satisfy $\alpha^2+\beta^2+\gamma^2=1$ and v is a unit vector, then v and $w=(\alpha J_1+\beta J_2+\gamma J_3)v$ make an orthonormal pair. Then $C=C_J^{(a)}$ satisfies

$$C_{4,0}(v, w)=5(\alpha^4+\beta^4+\gamma^4)-3 .$$

On the other hand, according to [6], $(1/20)C_J^{(a)}$ is a boundary point of $L(3, 4)$. This satisfies

$$\left(\frac{1}{20}C_J^{(a)}\right)_{4,0}(v, w)=\frac{1}{4}(\alpha^4+\beta^4+\gamma^4)-\frac{3}{20} .$$

If we put $\alpha=1$, $\beta=\gamma=0$, then we get $1/10$. If we put $\alpha=\beta=\gamma=3^{-1/2}$, then we get $-1/15$.

Let the set of orthonormal pair of vectors of $\boldsymbol{R}^4$ be denoted by OP. Then we can state the following corollary.

COROLLARY 5.2. *We have*

$$\{C_{4,0}(v, w) \mid C\in L(3, 4), \{v, w\}\in \mathrm{OP}\}=[-1/15, 1/10] .$$

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