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Approximately Inner $*$-Derivations of Irrational Rotation C^*-Algebras

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Abstract. Let θ be an irrational number and A_θ be the corresponding irrational rotation C^*-algebra. For any $k \in N \cup \{\infty\}$ let A_θ^k be the dense $*$-subalgebra of k-times continuously differentiable elements in A_θ with respect to the canonical action of the two dimensional torus and let $A_\theta^0 = A_\theta$. In the present paper we will show that there is an approximately inner $*$-derivation of A_θ^∞ to A_θ^∞ which is not inner if and only if θ is a non-generic irrational number.

§1. Preliminaries.

Let θ be an irrational number and A_θ be the corresponding irrational rotation C^*-algebra. Let u and v be unitary elements in A_θ with $uv = e^{2\pi i\theta}vu$ which generate A_θ. Let τ be the unique tracial state on A_θ and (π_τ, H_τ) be the cyclic representation associated with τ. We identify A_θ with $\pi_\tau(A_\theta)$. Furthermore A_θ can be identified with a dense subspace of H_τ with the L^2-norm topology. Then $\{u^m v^n\}_{m,n\in Z}$ is an orthonormal basis of H_τ. Let $t \in R \to \beta_t^{(j)}$ $(j=1, 2)$ be the one-parameter groups of automorphisms of A_θ defined by

$$\beta_t^{(1)}(u) = e^{2\pi it}u\ , \qquad \beta_t^{(1)}(v) = v$$

and

$$\beta_t^{(2)}(u) = u\ , \qquad \beta_t^{(2)}(v) = e^{2\pi it}v$$

for any $t \in R$. Let δ_1 and δ_2 be the generators of $\beta^{(1)}$ and $\beta^{(2)}$. Then by easy computation

$$\delta_1(u) = 2\pi iu\ , \qquad \delta_1(v) = 0$$

and

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$$\delta_2(u)=0\ , \qquad \delta_2(v)=2\pi iv\ .$$

Since we identify A_θ with $\pi_\tau(A_\theta)$ and τ is unique, there are one-parameter groups of unitary operators $w_t^{(j)}$ ($j=1, 2$) on H_τ such that

$$\beta_t^{(j)}(x)1=w_t^{(j)}x$$

for any $x\in A_\theta$, $t\in \boldsymbol{R}$ and $j=1, 2$. Let h_j be the anti-selfadjoint generators of $w^{(j)}$ for $j=1, 2$. Then

$$h_ju=\delta_j(u)\ , \qquad h_jv=\delta_j(v)$$

and $D(\delta_j)\subset D(h_j)$ for $j=1, 2$ where $D(\delta_j)$ and $D(h_j)$ ($j=1, 2$) denote their domains. Furthermore for any $k\in \boldsymbol{N}\cup\{\infty\}$ let A_θ^k be the dense $*$-subalgebra of k-times continuously differentiable elements in A_θ with respect to the canonical action and let $A_\theta^0=A_\theta$.

LEMMA 1. *With the above notations*

$$h_1(D(\delta_1))\perp \sum_{n\in \boldsymbol{Z}}^{\oplus} \boldsymbol{C}v^n$$

and

$$h_2(D(\delta_2))\perp \sum_{n\in \boldsymbol{Z}}^{\oplus} \boldsymbol{C}u^n\ .$$

PROOF. Let $x\in D(\delta_1)$. Then for any $n\in \boldsymbol{Z}$

$$(h_1x|v^n)=\tau(v^{-n}\delta_1(x))=\tau(\delta_1(v^{-n}x))=0$$

since τ is unique where $(\cdot|\cdot)$ is the inner product on H_τ. Hence

$$h_1(D(\delta_1))\perp \sum_{n\in \boldsymbol{Z}}^{\oplus} \boldsymbol{C}v^n\ .$$

Similarly we obtain that

$$h_2(D(\delta_2))\perp \sum_{n\in \boldsymbol{Z}}^{\oplus} \boldsymbol{C}u^n\ . \qquad \text{Q.E.D.}$$

DEFINITION. For any $x\in A_\theta$ there is a sequence $\{c_{m,n}\}\in l^2(\boldsymbol{Z}^2)$ such that

$$x=\sum_{m,n\in \boldsymbol{Z}} c_{m,n}u^mv^n$$

where the summation is considered under the L^2-norm topology. We say that $\{c_{m,n}\}$ are *the Fourier coefficients of* x.

LEMMA 2. *Let* $k\in \boldsymbol{N}$. *Let* $x\in A_\theta^k$ *and* $\{c_{m,n}\}$ *be its Fourier coefficients. Then*

$$|m|^k|c_{m,n}| \to 0 ,$$
$$|n|^k|c_{m,n}| \to 0$$

as $|m|$, $|n| \to \infty$.

PROOF. Since $x \in A_\theta^k$, $x \in D(\delta_1^k)$. Thus $x \in D(h_1^k)$. Let $\{d_{m,n}\}$ be the Fourier coefficients of $h_1^k x$. Then

$$h_1^k x = \sum d_{m,n} u^m v^n$$

where $\{d_{m,n}\} \in l^2(\boldsymbol{Z}^2)$ and the summation is considered under the L^2-norm topology. By Lemma 1, $d_{0,n}=0$ for any $n \in \boldsymbol{Z}$. Let

$$y = \sum_{m,n \in \boldsymbol{Z}, m \neq 0} \left(\frac{1}{2\pi i m}\right)^k d_{m,n} u^m v^n ,$$

where the summation is considered under the L^2-norm topology. Then by the closedness of h_1, $y \in D(h_1^k)$ and

$$h_1^k y = \sum_{m,n \in \boldsymbol{Z}, m \neq 0} d_{m,n} u^m v^n .$$

Thus $h_1^k x = h_1^k y$. Hence there is a $z \in \sum_{n \in \boldsymbol{Z}}^{\oplus} \boldsymbol{C} v^n$ such that $x = y + z$. Therefore since

$$c_{m,n} = \left(\frac{1}{2\pi i m}\right)^k d_{m,n} \qquad \text{if} \quad m \neq 0 ,$$

$$|m|^k|c_{m,n}| \to 0$$

as $|m|$, $|n| \to \infty$. Similarly

$$|n|^k|c_{m,n}| \to 0$$

as $|m|$, $|n| \to \infty$. Q.E.D.

COROLLARY 3. *Let $x \in A_\theta^\infty$ and $\{c_{m,n}\}$ be its Fourier coefficients. Then for any k, $l \in \boldsymbol{N}$*

$$|m|^k|n|^l|c_{m,n}| \to 0$$

as $|m|$, $|n| \to \infty$.

PROOF. By Lemma 2 we can easily obtain the conclusion. Q.E.D.

§2. Inner $*$-derivations of A_θ^{k+1} to A_θ^k.

Now we recall the definitions of a generic irrational number and an approximately inner $*$-derivation.

DEFINITION. Let θ be an irrational number. We say that *it is generic* if there are $C>0$ and $r>1$ such that

$$|e^{2\pi i n\theta}-1|\geqq\frac{C}{n^r}$$

for any positive integer n. That is, θ is generic if it is not *a Liouville number*.

DEFINITION. Let δ be a $*$-derivation on a dense $*$-subalgebra B of C^*-algebra A. We say it is *approximately inner* if there is a net $\{b_\nu\}$ of anti-selfadjoint elements in A such that

$$\delta(x)=\lim_{\nu\to\infty}(b_\nu x-xb_\nu)$$

for any $x\in B$. If B is countably generated as an algebra, then such a net, if it exists, may be taken to be a sequence.

PROPOSITION 4. *Let θ be a generic irrational number. If $C>0$ and $r>1$ satisfy that*

$$|e^{2\pi i n\theta}-1|\geqq\frac{C}{n^r}$$

for any positive integer n, then for any positive integer k with $k>r+2$ each approximately inner $*$-derivation δ: $A_\theta^{k+1}\to A_\theta^k$ is inner.

PROOF. We will prove the above proposition in the same way as in [2, Remark 4.3]. Since u and v are in A_θ^{k+1}, $\delta(u)$ and $\delta(v)$ are in A_θ^k. Let

$$\delta(u)=\sum_{m,n\in\boldsymbol{Z}}c_{m,n}u^m v^n\ ,$$

$$\delta(v)=\sum_{m,n\in\boldsymbol{Z}}d_{m,n}u^m v^n\ ,$$

where the summations are considered under the L^2-norm topology. Since δ is approximately inner, $\tau(u^*\delta(u))=0$. Thus $c_{1,0}=0$. Similarly $d_{0,1}=0$. And since $\delta(uv)=e^{2\pi i\theta}\delta(uv)$,

$$(1-e^{-2\pi i m\theta})c_{m+1,n}+(1-e^{-2\pi i n\theta})d_{m,n+1}=0 \tag{1}$$

for any m, $n\in\boldsymbol{Z}$. Furthermore by Lemma 2 there are $K_c>0$ and $K_d>0$ such that

$$|c_{m,n}|\leqq\frac{K_c}{|n|^k}\qquad\text{for}\quad m\in\boldsymbol{Z},\ n\in\boldsymbol{Z}-\{0\}\ ,$$

$$|d_{m,n}| \leqq \frac{K_d}{|m|^k} \qquad \text{for} \quad m \in \boldsymbol{Z} - \{0\},\ n \in \boldsymbol{Z} .$$

The derivation δ is formally implemented by an operator

$$h = \sum_{m,n} a_{m,n} u^m v^n$$

where the coefficients $\{a_{m,n}\}$ are determined by the requirements

$$c_{m+1,n} = (e^{-2\pi i n\theta} - 1) a_{m,n} , \tag{2}$$

$$d_{m,n+1} = (1 - e^{-2\pi i m\theta}) a_{m,n} . \tag{3}$$

If $m \neq 0$ and $n \neq 0$, we define

$$a_{m,n} = \frac{c_{m+1,n}}{e^{-2\pi i n\theta} - 1} .$$

Then by the equation (1), we see that the equation (3) holds. Thus

$$a_{m,n} = \frac{d_{m,n+1}}{1 - e^{-2\pi i m\theta}} .$$

Hence

$$\begin{aligned} |a_{m,n}|^2 &= \frac{|c_{m+1,n}|\,|d_{m,n+1}|}{|e^{-2\pi i n\theta} - 1|\,|1 - e^{-2\pi i m\theta}|} \\ &\leqq \frac{K_c K_d}{C^2} \frac{1}{|m|^{k-r} |n|^{k-r}} . \end{aligned}$$

If $m \neq 0$ and $n = 0$, we define

$$a_{m,0} = \frac{d_{m,1}}{1 - e^{-2\pi i m\theta}} .$$

Then by the equation (1), we see that $c_{m+1,0} = 0$. Thus the equation (2) holds and

$$\begin{aligned} |a_{m,0}| &= \left| \frac{d_{m,1}}{1 - e^{-2\pi i m\theta}} \right| \\ &\leqq \frac{K_d}{C} \frac{1}{|m|^{k-r}} . \end{aligned}$$

If $m = 0$ and $n \neq 0$, we define

$$a_{0,n} = \frac{c_{1,n}}{e^{-2\pi i n\theta} - 1} .$$

Then by the equation (1), we see that $d_{0,n+1}=0$. Thus the equation (3) holds and

$$|a_{0,n}|=\left|\frac{c_{m+1,n}}{e^{-2\pi in\theta}-1}\right|$$
$$\leqq\frac{K_c}{C}\frac{1}{|n|^{k-r}}.$$

If $m=n=0$, we define $a_{0,0}=0$. Then since $c_{1,0}=d_{0,1}=0$, the equations (2) and (3) hold. Therefore we obtain that $\{a_{m,n}\}\in l^1(\mathbf{Z}^2)$. Hence $h\in A_\theta$. Furthermore since

$$\delta(u)^*u+u^*\delta(u)=0,$$
$$\delta(v)^*v+v^*\delta(v)=0,$$

we can easily see that h is anti-selfadjoint. Q.E.D.

§3. An approximately inner *-derivation of A_θ^{k+1} to A_θ^k which is not inner.

First we will give a definition.

DEFINITION. Let θ be an irrational number and r be a positive number with $r\geqq 1$. We say that θ *is approximable by rational numbers to order* r if there is a $K(\theta)>0$, depending only on θ, such that

$$\left|\theta-\frac{p}{q}\right|<\frac{K(\theta)}{q^r}$$

is satisfied for infinitely many pairs of integers p, q with $q>0$.

By easy computation θ is approximable by rational numbers to order $r\geqq 1$ if and only if there is a $C(\theta)>0$, depending only on θ, such that

$$|e^{2\pi in\theta}-1|<\frac{C(\theta)}{n^{r-1}}$$

is satisfied for infinitely many positive integers n.

By Besicovitch [1] or Falconer [5, Theorem 8.16] we can see that for any $r\geqq 1$ there is an irrational number θ which is approximable by rational numbers to order r.

Let θ be approximable by rational numbers to order $r\geqq 3$. Let k be a positive integer with $k\leqq r$. Then there is a strictly increasing sequence $\{n_j\}_{j=1}^{\infty}$ of positive integers such that

$$|e^{2\pi i n_j\theta}-1|<\frac{C(\theta)}{n_j^{k-1}}$$

for any $j\in \boldsymbol{N}$. Let $\{a_n\}_{n\in \boldsymbol{Z}}$ be the sequence defined by

$$a_n=\begin{cases}\dfrac{1}{j}\dfrac{1}{n_j^{k-1}}\dfrac{1-e^{2\pi i n_j\theta}}{|1-e^{2\pi i n_j\theta}|} & \text{if} \quad n=n_j\\ \dfrac{1}{j}\dfrac{1}{n_j^{k-1}}\dfrac{1-e^{-2\pi i n_j\theta}}{|1-e^{-2\pi i n_j\theta}|} & \text{if} \quad n=-n_j\\ 0 & \text{elsewhere}\,.\end{cases}$$

Lemma 5. *Let θ, k, $\{n_j\}$ and $\{a_n\}$ be as above. If $k\geqq 2$, we can define a real valued function $g\in C^{k-2}(\boldsymbol{T})$ by*

$$g(t)=\sum_{n\in \boldsymbol{Z}}a_n e^{2\pi i n t}$$

where we identify $C(\boldsymbol{T})$ with the algebra of all continuous functions on $\boldsymbol{R}$ with period 1. Then it follows that $\int_0^1 g(t)dt=0$ and there is no continuous function $h\colon \boldsymbol{R}\to\boldsymbol{R}$ with period 1 satisfying that

$$g(t)=h(t)-h(t+\theta)$$

for any $t\in\boldsymbol{R}$.

Proof. We note that for any $j\in\boldsymbol{N}$

$$|a_{n_j}|=\frac{1}{jn_j^{k-1}}\leqq\frac{1}{j^k}\,.$$

Since $k\geqq 2$, $\{a_n\}\in l^1(\boldsymbol{Z})$. Hence $g\in C(\boldsymbol{T})$. By the definition of g

$$\int_0^1 g(t)=a_0=0$$

and $g(t)\in\boldsymbol{R}$ for any $t\in\boldsymbol{R}$. For any positive integer N let

$$g_N(t)=\sum_{n=-N}^{N}a_n e^{2\pi i n t}\,.$$

Then for any positive integer $l\leqq k-2$

$$\frac{d^l}{dt^l}g_N(t)=\sum_{n=-N}^{N}(2\pi i n)^l a_n e^{2\pi i n t}\,.$$

And

$$\begin{aligned}|(2\pi i n_j)^l a_{n_j}| &= \frac{(2\pi)^l}{j}\frac{1}{n_j^{k-l-1}} \\ &\leqq (2\pi)^l \frac{1}{j^{k-l}} \\ &\leqq (2\pi)^l \frac{1}{j^2} .\end{aligned}$$

Hence $\{(d^l/dt^l)g_N\}$ is a Cauchy sequence under the norm topology in $C(\boldsymbol{T})$. Therefore $g \in C^{k-2}(\boldsymbol{T})$.

Now we suppose that there is a continuous function $h: \boldsymbol{R} \to \boldsymbol{R}$ with period 1 satisfying that

$$g(t) = h(t) - h(t+\theta)$$

for any $t \in \boldsymbol{R}$. Then the Fourier series of h should be as follows:

$$\sum_{n=-\infty}^{\infty} \frac{a_n}{1-e^{2\pi i n\theta}} e^{2\pi i n t} + c$$

where c is a constant number. Since h is continuous,

$$\sum_{n=-\infty}^{\infty} \frac{a_n}{1-e^{2\pi i n\theta}}$$

is Cesàro summable. However

$$\sum_{n=-\infty}^{\infty} \frac{a_n}{1-e^{2\pi i n\theta}} = 2\sum_{j=1}^{\infty} \frac{1}{j}\frac{1}{n_j^{k-1}}\frac{1}{|1-e^{2\pi i n_j\theta}|} .$$

By the definition of $\{n_j\}$

$$\frac{1}{n_j^{k-1}}\frac{1}{|1-e^{2\pi i n_j\theta}|} > \frac{1}{n_j^{k-1}}\frac{n_j^{k-1}}{C(\theta)} = \frac{1}{C(\theta)} .$$

Since $\sum_{j=1}^{\infty} 1/j$ is not Cesàro summable, neither is $\sum_{n=-\infty}^{\infty} a_n/(1-e^{2\pi i n\theta})$. Therefore we obtain a contradiction. Q.E.D.

REMARK. Let g be as in Lemma 5. Let α be the automorphism of A_θ defined by

$$\alpha(u) = e^{2\pi i g(v)} u ,$$
$$\alpha(v) = v .$$

Then by [7] we can obtain the following facts:

$$(1) \qquad \alpha_* = \mathrm{id} \quad \text{on } K_1(A_\theta) ,$$

$$(2) \qquad \tilde{\tau}_*(K_0(A_\theta \times_\alpha \boldsymbol{Z})) = \boldsymbol{Z} + \boldsymbol{Z}\theta ,$$

$$(3) \qquad \Gamma(\alpha) = \boldsymbol{T} ,$$

where $\tilde{\tau}_*$ is the homomorphism of $K_0(A_\theta \times_\alpha \boldsymbol{Z})$ to $\boldsymbol{R}$ induced by τ and $\Gamma(\alpha)$ is the Connes spectrum of α.

Now we will introduce a new notation. For any s, $t \in \boldsymbol{R}$ let $\alpha_{(s,t)}$ be the automorphism of A_θ defined by

$$\alpha_{(s,t)}(u) = e^{2\pi i s}u , \qquad \alpha_{(s,t)}(v) = e^{2\pi i t}v .$$

Then by easy computation $\alpha_{(s,t)}(A_\theta^k) = A_\theta^k$ for any $k \in \boldsymbol{N} \cup \{\infty\}$.

Proposition 6. *Let k be an integer with $k \geqq 0$. Let θ be approximable by rational numbers to order $k+3$. Then there is an approximately inner $*$-derivation of A_θ^{k+1} to A_θ^k which is not inner.*

Proof. Let g be as in Lemma 5. Thus $g \in C^{k+1}(\boldsymbol{T})$. Let α be as in the above remark. Since $g \in C^{k+1}(\boldsymbol{T})$, $\alpha(A_\theta^l) = A_\theta^l$ for $l = 0, 1, 2, \cdots, k+1$. Hence $\alpha^{-1} \circ \delta_j \circ \alpha$ is a $*$-derivation of A_θ^{k+1} to A_θ^k for $j = 1, 2$. By Bratteli, Elliott and Jørgensen [2] there are the unique decompositions

$$\alpha^{-1} \circ \delta_1 \circ \alpha = c_{1,1}\delta_1 + c_{1,2}\delta_2 + \tilde{\delta}_1 ,$$
$$\alpha^{-1} \circ \delta_2 \circ \alpha = c_{2,1}\delta_1 + c_{2,2}\delta_2 + \tilde{\delta}_2 ,$$

where $c_{1,1}$, $c_{1,2}$, $c_{2,1}$ and $c_{2,2}$ are in $\boldsymbol{R}$ and $\tilde{\delta}_1$, $\tilde{\delta}_2$ are approximately inner $*$-derivations of A_θ^{k+1} to A_θ^k. However by the definition of α we obtain the following equations:

$$(\alpha^{-1} \circ \delta_1 \circ \alpha)(u) = 2\pi i u ,$$
$$(\alpha^{-1} \circ \delta_1 \circ \alpha)(v) = 0 ,$$
$$(\alpha^{-1} \circ \delta_2 \circ \alpha)(u) = 2\pi i g'(v) u ,$$
$$(\alpha^{-1} \circ \delta_2 \circ \alpha)(v) = 2\pi i v$$

where g' is the derivative of g. By the uniqueness of the decompositions we can see that

$$\alpha^{-1} \circ \delta_1 \circ \alpha = \delta_1 .$$

And we obtain the following equations:

$$2\pi i g'(v) u = 2\pi i c_{2,1} u + \tilde{\delta}_2(u) ,$$
$$2\pi i v = 2\pi i c_{2,2} v + \tilde{\delta}_2(v) .$$

We will show that $\tilde{\delta}_2$ is not inner. We suppose that it is inner. Then there is a selfadjoint element $a \in A_\theta$ such that

$$\tau(a)=0 ,$$
$$\tilde{\delta}_2(x)=i(ax-xa)$$

for any $x \in A_\theta^{k+1}$. Hence we get

$$au-ua=2\pi g'(v)u-2\pi c_{2,1}u ,$$

i.e.,

$$a-uau^*=2\pi g'(v)-2\pi c_{2,1} .$$

Thus

$$\tau(a-uau^*)=2\pi\tau(g'(v))-2\pi c_{2,1} .$$

Since $\tau(uau^*)=\tau(a)$ and $2\pi\tau(g'(v))=0$, $c_{2,1}=0$. Moreover

$$av-va=2\pi(1-c_{2,2})v ,$$

i.e.,

$$a-vav^*=2\pi(1-c_{2,2}) .$$

Thus

$$\tau(a-vav^*)=2\pi(1-c_{2,2}) .$$

Hence we obtain that $c_{2,2}=1$. Therefore

$$\alpha^{-1}\circ\delta_2\circ\alpha=\delta_2+ad(ia)$$

where

$$2\pi ig'(v)u=i(au-ua) ,$$
$$av-va=0 .$$

Since $av=va$, $a \in C^*(v)$ where $C^*(v)$ is the C^*-subalgebra of A_θ generated by v. Hence there is a selfadjoint element $f \in C(\boldsymbol{T})$ such that $a=f(v)$. And $\int_0^1 f(t)dt=0$ since $\tau(a)=0$. Let F be the selfadjoint element in $C(\boldsymbol{T})$ defined by

$$F(t)=\int_0^t f(s)ds$$

and let $w=e^{iF(v)}$. Then w is a unitary element in A_θ and

$$\begin{aligned} w\delta_2(w^*) &= e^{iF(v)}\delta_2(e^{-iF(v)}) \\ &= e^{iF(v)}(-iF'(v))e^{-iF(v)} \\ &= -if(v) = -ia \end{aligned}$$

where F' is the derivative of F. Therefore by easy computation

$$Ad(w)\circ\alpha^{-1}\circ\delta_1\circ\alpha\circ Ad(w^*)=\delta_1 ,$$
$$Ad(w)\circ\alpha^{-1}\circ\delta_2\circ\alpha\circ Ad(w^*)=\delta_2 .$$

Hence there are s, $t\in\boldsymbol{R}$ such that

$$\alpha\circ Ad(w^*)=\alpha_{(s,t)} ,$$

i.e.,

$$\alpha=\alpha_{(s,t)}\circ Ad(w) .$$

By Pimsner [12] we see that

$$\tilde{\tau}_*(K_0(A_\theta\times_\alpha \boldsymbol{Z}))=\boldsymbol{Z}+\boldsymbol{Z}\theta+\boldsymbol{Z}s+\boldsymbol{Z}t .$$

On the other hand by the above remark

$$\tilde{\tau}_*(K_0(A_\theta\times_\alpha \boldsymbol{Z}))=\boldsymbol{Z}+\boldsymbol{Z}\theta .$$

Thus s, $t\in\boldsymbol{Z}+\boldsymbol{Z}\theta$. Hence, since $\alpha_{(s,t)}$ is inner, so is α. However by the above remark $\Gamma(\alpha)=\boldsymbol{T}$. This is a contradiction. Therefore $\tilde{\delta}_2$ is not inner. Q.E.D.

DEFINITION. Let θ be an irrational number. We define $r(\theta)$ by

$$r(\theta)=\sup\{r\geqq 1 \mid r \text{ is a number to which } \theta \text{ is approximable by rational numbers}\} .$$

We call it *the degree of irrationality* for θ.

By Besicovitch [1] or Falconer [5, Theorem 8.16] we see that there is an irrational number θ with $r(\theta)<\infty$. And if $r(\theta)=\infty$, θ is a non-generic irrational number.

THEOREM 7. *Let θ be an irrational number and $r(\theta)$ be its degree of irrationality. If $r(\theta)>3$, then for any integer k with $r(\theta)+1<k$ each approximately inner *-derivation of A_θ^{k+1} to A_θ^k is inner and for any integer k with $0\leqq k<r(\theta)-3$ there is an approximately inner *-derivation of A_θ^{k+1} to A_θ^k which is not inner.*

PROOF. We suppose that k is an integer with $r(\theta)+1<k$. Let δ be

an approximately inner $*$-derivation of A_θ^{k+1} to A_θ^k. Then there is a real number r with $r(\theta)+1<r+1<k$ and $C>0$ satisfying that

$$|e^{2\pi in\theta}-1|\geqq\frac{C}{n^{r-1}}$$

for any positive integer n. Hence by Proposition 4 δ is inner.

Next we suppose that k is an integer with $0\leqq k<r(\theta)-3$. Then θ is approximable by rational numbers to order $k+3$. Hence by Proposition 6 there is an approximately inner $*$-derivation of A_θ^{k+1} to A_θ^k which is not inner. Q.E.D.

COROLLARY 8. *Let θ be an irrational number. Then there is an approximately inner $*$-derivation of A_θ^∞ to A_θ^∞ which is not inner if and only if θ is non-generic.*

PROOF. We suppose that θ is non-generic. By [8] there is an automorphism α of A_θ with $\alpha(A_\theta^\infty)=A_\theta^\infty$ satisfying that

(1) $\alpha_*=\mathrm{id}$ on $K_1(A_\theta)$,

(2) $\tilde{\tau}_*(K_0(A_\theta\times_\alpha \boldsymbol{Z}))=\boldsymbol{Z}+\boldsymbol{Z}\theta$,

(3) $\Gamma(\alpha)=\boldsymbol{T}$.

Then we can prove in the same way as in Proposition 6 that there is an approximately inner $*$-derivation of A_θ^∞ to A_θ^∞. And it is easy by [2, Remark 4.3] to prove the converse part. Q.E.D.

References

[1] A. S. BESICOVITCH, Sets of fractional dimensions IV: On rational approximation to real numbers, J. London Math. Soc., **9** (1934), 126-13.1

[2] O. BRATTELI, G. A. ELLIOTT and P. E. T. JØRGENSEN, Decomposition of unbounded derivations into invariant and approximately inner parts, J. Reine Angew. Math., **346** (1984), 166-193.

[3] O. BRATTELI and D. W. ROBINSON, *Operator Algebras and Quantum Statistical Mechanics I*, Springer-Verlag, 1979.

[4] G. A. ELLIOTT, The diffeomorphism group of the irrational rotation C^*-algebra, C. R. Math. Rep. Acad. Sci. Canada, **8** (1986), 329-334.

[5] K. J. FALCONER, *The Geometry of Fractal Sets*, Cambridge Univ. Press, 1985.

[6] G. H. HARDY and E. M. WRIGHT, *An Introduction to Theory of Numbers*, Oxford at the Clarendon Press, 1979.

[7] K. KODAKA, A diffeomorphism of an irrational rotation C^*-algebra by a non-generic rotation, to appear in J. Operator Theory.

[8] ———, Diffeomorphisms of irrational rotation C^*-algebras by non-generic rotations II, preprint.

[9] S. LANG, *Introduction to Diophantine Approximations*, Addison-Wesley, 1966.
[10] R. MAÑÉ, *Ergodic Theory and Differentiable Dynamics*, Springer-Verlag, 1987.
[11] G. K. PEDERSEN, *C^*-Algebras and their Automorphism Groups*, Academic Press, 1979.
[12] M. V. PIMSNER, Ranges of traces on K_0 of reduced crossed products by free groups, Lecture Notes in Math., **1132** (1983), 374–408, Springer.
[13] M. A. RIEFFEL, C^*-algebras associated with irrational rotations, Pacific J. Math., **93** (1981), 415–429.
[14] H. TAKAI, On a problem of Sakai in unbounded derivations, J. Funct. Anal., **43** (1981), 202–208.

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