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A Certain Factorization of Selfdual Cones Associated with Standard Forms of Injective Factors

Yasuhide MIURA

Iwate University

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Introduction.

In [4] Connes characterized the injective factors on a separable Hilbert space. Applying the reduction theory to the characterization, Choi and Effros proved the equivalence of the injectivity and the semidiscreteness of von Neumann algebras in their papers [1] and [2].

On the other hand, in the category of the ordered Hilbert space many authors gave the answers to the questions how the algebraic structure of a von Neumann algebra determines the structure of the underlying Hilbert space and how the structure of a von Neumann algebra is characterized by the related selfdual cone. Schmitt and Wittstock [11] introduced the notion of the matrix ordered standard form and constructed the von Neumann algebra by using the family of selfdual cones. Furthermore, Schmitt [13] proved the characterization of matrix ordered standard forms of injective von Neumann algebras via several properties in the category of matrix ordered spaces.

Now, Schmitt and Wittstock [12], and Tomiyama and the author [7] investigated the ordered Hilbert spaces induced by the selfdual cones arising in the standard forms of von Neumann algebras introduced by Haagerup [5] and characterized the tensor product of the selfdual cones. In §1 we shall characterize the Hilbert spaces associated with standard forms of injective factors from the point of view of the semidiscreteness and consider the approximation property of n-positive and completely positive maps of a Hilbert-Schmidt class. In §2 we shall investigate a certain factorization of selfdual cones associated with standard forms of injective factors by smooth maximal abelian subalgebras.

We refer mainly [14], [16] and [17] for standard results in the theory of operator algebras.

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Let $(\mathcal{M}, \mathcal{H}, J, \mathcal{P})$ be a standard form of a von Neumann algebra $\mathcal{M}$ where J is an isometric involution and $\mathcal{P}$ is a selfdual cone in a Hilbert space $\mathcal{H}$. Namely, the following conditions are satisfied:

(i) $J\mathcal{M}J=\mathcal{M}'$;

(ii) $JxJ=x^*$ for every x in the center of $\mathcal{M}$;

(iii) $xJxJ\mathcal{P}\subset\mathcal{P}$ for every $x\in\mathcal{M}$.

Note that J becomes an involution such that $J\xi=\xi$, $\xi\in\mathcal{P}$. For example, let M_n be an algebra of all complex $n\times n$ matrices. Suppose that M_n operates on $H_n(=M_n)$ by matrix multiplication from the left. Then (M_n, H_n, S_n, H_n^+) is a standard form where S_n is a matrix involution and H_n^+ means a set of positive matrices $M_n^+\subset M_n$. The second example is the case where $\mathcal{M}$ has a cyclic and separating vector $\xi_0\in\mathcal{H}$. Then $\mathcal{P}=[\Delta^{1/4}\mathcal{M}^+\xi_0]$ is a selfdual cone in $\mathcal{H}$ where Δ denotes the module operator with respect to ξ_0.

Let $M_n(\mathcal{H})=\mathcal{H}\otimes M_n$ be a tensor product of two Hilbert spaces $\mathcal{H}$ and M_n. For a subalgebra $\mathcal{N}\subset\mathcal{M}$, we put

$$P_n(\mathcal{N})=\left\{[\xi_{ij}]\in M_n(\mathcal{H}) \;\middle|\; \sum_{i,j=1}^n a_iJa_jJ\xi_{ij}\in\mathcal{P} \text{ for } \forall\{a_i\}\subset\mathcal{N}\right\}$$

for $n\in\mathrm{N}$. Then one easily sees that $P_n(\mathcal{N})$ is a closed convex cone in $M_n(\mathcal{H})$. In particular,

$$P_n(\mathcal{M})=\overline{\mathrm{co}}\{[a_iJa_jJ\xi]\in M_n(\mathcal{H}) \mid a_i\in\mathcal{M},\ \xi\in\mathcal{P}\}$$

and $P_n(\mathcal{M})$ is a selfdual cone in $M_n(\mathcal{H})$ where $\overline{\mathrm{co}}$ denotes the closed convex hull (cf. [7; Proposition 2.4], [11; Lemma 1.1]).

Let $(\mathcal{M}_1, \mathcal{H}_1, J_1, \mathcal{P}_1)$ and $(\mathcal{M}_2, \mathcal{H}_2, J_2, \mathcal{P}_2)$ be two standard forms. A linear (or conjugate linear) map $\phi\colon \mathcal{H}_1\to\mathcal{H}_2$ is said to be n-positive if, for $[\xi_{ij}]\in P_n(\mathcal{M}_1)$, $[\phi(\xi_{ij})]$ belongs to $P_n(\mathcal{M}_2)$. If ϕ is n-positive for all $n\in\mathrm{N}$, then ϕ is said to be completely positive. For any $\xi\in\mathcal{H}_1$, let R_ξ be a right slice map of $\mathcal{H}_1\otimes\mathcal{H}_2$ into $\mathcal{H}_2$ with respect to ξ such that

$$R_\xi(\xi'\otimes\eta')=(\xi', \xi)\eta', \qquad \xi'\in\mathcal{H}_1,\ \eta'\in\mathcal{H}_2.$$

For any $x\in\mathcal{H}_1\otimes\mathcal{H}_2$ we put

$$r(x)(\xi)=R_\xi(x), \qquad \xi\in\mathcal{H}_1.$$

Then $r(x)$ is a bounded conjugate linear map of $\mathcal{H}_1$ into $\mathcal{H}_2$, which belongs to a Hilbert-Schmidt class. Conversely, a conjugate linear map of a Hilbert-Schmidt class of $\mathcal{H}_1$ into $\mathcal{H}_2$ is written by $r(x)$ for some $x\in\mathcal{H}_1\otimes\mathcal{H}_2$. Put

$$\mathscr{P}_1 \hat{\otimes} \mathscr{P}_2 = \{x \in \mathscr{H}_1 \otimes \mathscr{H}_2 \mid r(x) \text{ is completely positive}\} .$$

Then $\mathscr{P}_1 \hat{\otimes} \mathscr{P}_2$ is a selfdual cone in $\mathscr{H}_1 \otimes \mathscr{H}_2$ and $(\mathscr{M}_1 \otimes \mathscr{M}_2, \mathscr{H}_1 \otimes \mathscr{H}_2, J_1 \otimes J_2, \mathscr{P}_1 \hat{\otimes} \mathscr{P}_2)$ is a standard form (cf. [7; Theorem 2.8], [12; Theorem 1.1]). In particular, we have

$$P_n(\mathscr{M}_1) = P_1 \hat{\otimes} M_n^+ , \qquad n \in \mathbf{N} .$$

Note that the closure of the algebraic tensor product of $\mathscr{P}_1$ and $\mathscr{P}_2$, that is, the closed convex hull of the set $\{\xi \otimes \eta \mid \xi \in \mathscr{P}_1, \eta \in \mathscr{P}_2\}$ is selfdual in $\mathscr{H}_1 \otimes \mathscr{H}_2$ if and only if either $\mathscr{M}_1$ or $\mathscr{M}_2$ is abelian (cf. [8; Theorem]).

§ 1. Characterization of the Hilbert space associated with standard forms of injective factors.

We first consider the continuity of a positive map. Let $(\mathscr{M}_1, \mathscr{H}_1, J_1, \mathscr{P}_1)$ and $(\mathscr{M}_2, \mathscr{H}_2, J_2, \mathscr{P}_2)$ be two standard forms. Suppose that ϕ is a positive map of $\mathscr{H}_1$ into $\mathscr{H}_2$. By [10; V 5.5 Theorem] a real positive linear functional on $H_1' = \mathscr{P}_1 - \mathscr{P}_1$ is continuous. Since $\mathscr{P}_2$ is a selfdual cone in $\mathscr{H}_2$, $\mathscr{P}_2$ is weakly normal (cf. [10; V 3.3 Corollary 3]. It follows that a positive linear map of $\mathscr{H}_1$ into $\mathscr{H}_2$ is continuous by [10; V 5.6]. Considering the decomposition of ϕ into selfadjoint maps, we obtain the following lemma:

LEMMA 1.1. *Keep the notations above. Any positive linear map of $\mathscr{H}_1$ into $\mathscr{H}_2$ is continuous.*

LEMMA 1.2. *Suppose that $\mathscr{M}$ is a von Neumann algebra on a Hilbert space $\mathscr{H}$ with a cyclic and separating vector ξ_0 in $\mathscr{H}$. Let J and Δ denote the canonical involution and the modular operator with respect to ξ_0. Then*

$$\begin{aligned} P_n(\mathscr{M}) &= \overline{\mathrm{co}}\{[a_i J a_j J \xi_0] \in M_n(\mathscr{H}) \mid a_i \in \mathscr{M}\} \\ &= \overline{\mathrm{co}}\{[\Delta^{1/4} a_i a_j^* \xi_0] \in M_n(\mathscr{H}) \mid a_i \in \mathscr{M}\} . \end{aligned}$$

PROOF. For each $a_i \in \mathscr{M}$ there exists a net $\{\eta_\alpha^{(i)}\}$ of the Tomita algebra in the achieved left Hilbert algebra $\mathscr{M}\xi_0$ such that $[\pi(\eta_\alpha^{(i)})]$ is bounded and converges to a_i in the strong $*$-topology. We have

$$\begin{aligned} [a_i J a_j J \xi_0] &= \lim [\pi(\eta_\alpha^{(i)}) J \pi(\eta_\alpha^{(j)}) J \xi_0] \\ &= \lim [\pi(\eta_\alpha^{(i)}) \Delta^{1/2} \pi(\eta_\alpha^{(j)})^* \xi_0] \\ &= \lim [\Delta^{1/4} A_i A_j^* \xi_0] , \end{aligned}$$

where $A_i = \Delta^{-1/4} a_i \Delta^{1/4}$. Similarly we have the converse inclusion.

LEMMA 1.3. *Let $\mathscr{M}$ be a von Neumann algebra on a Hilbert space $\mathscr{H}$ with a cyclic and separating vector ξ_0 and ϕ a vector state ω_{ξ_0}. Suppose that $\mathscr{N}$ is a von Neumann subalgebra of $\mathscr{M}$ and ε is a normal conditional expectation of $\mathscr{M}$ onto $\mathscr{N}$ satisfying $\phi\circ\varepsilon=\phi$. Let e denote the projection of $\mathscr{H}$ onto the closure $\mathscr{K}$ of $\mathscr{N}\xi_0$. Then we obtain the following statements:*

i) *e is a completely positive projection on $\mathscr{H}$ with respect to the order induced by the selfdual cone $\mathscr{P}=[\Delta^{1/4}\mathscr{M}^+\xi_0]$,*

ii) *$(\mathscr{N}, \mathscr{K}, Je, e\mathscr{P})$ is the standard form and*

$$P_n(\mathscr{N}|\mathscr{K})=(e\otimes 1_n)P_n(\mathscr{M}), \qquad n\in \mathbf{N},$$

where J and Δ denote the canonical involution and the modular operator with respect to ξ_0.

PROOF. i) Let e be a projection of $\mathscr{H}$ onto $\mathscr{K}$. If $x\in\mathscr{M}$ and $y\in\mathscr{N}$, then

$$\begin{aligned}(ex\xi_0, y\xi_0)&=(x\xi_0, y\xi_0)=\phi(y^*x)=\phi(\varepsilon(y^*x))\\ &=\phi(y^*\varepsilon(x))=(\varepsilon(x)\xi_0, y\xi_0).\end{aligned}$$

It follows that

$$e(x\xi_0)=\varepsilon(x)\xi_0, \qquad x\in\mathscr{M}.$$

By the proof of [15; Theorem] we obtain the commutativity $e\Delta=\Delta e$. For $a_i\in\mathscr{M}$, we have

$$[e\Delta^{1/4}a_ia_j^*\xi_0]=[\Delta^{1/4}ea_ia_j^*\xi_0]=[\Delta^{1/4}\varepsilon(a_ia_j^*)\xi_0].$$

Since ε is a completely positive map of $\mathscr{M}$ onto $\mathscr{N}$ we have by Lemma 1.2

$$\begin{aligned}(e\otimes 1_n)P_n(\mathscr{M})&\subset\{(\Delta^{1/4}\otimes 1_n)M_n(\mathscr{M})^+(\xi_0\otimes 1_n)\}^-\\ &=\overline{\mathrm{co}}\{[\Delta^{1/4}a_ia_j^*\xi_0]\mid a_i\in\mathscr{M}\}.\end{aligned}$$

Therefore e is completely positive.

ii) Since e is completely positive, we see that $(e\otimes 1_n)P_n(\mathscr{M})$ is a selfdual cone in $M_n(\mathscr{H})$ for $n\in\mathbf{N}$. Since $\mathscr{K}$ is invariant for $\mathscr{N}$, we put $\mathscr{N}_{\mathscr{K}}=\mathscr{N}|\mathscr{K}$. Since ξ_0 is a cyclic and separating vector for $\mathscr{N}_{\mathscr{K}}$, an achieved left Hilbert algebra $\mathscr{N}_{\mathscr{K}}\xi_0$ is contained in $\mathscr{M}\xi_0$ in the sense of a left Hilbert algebra by the proof of [15; Theorem], whose involution $J_{\mathscr{K}}$ coincides with $eJe=Je$ and the related modular operator $\Delta_{\mathscr{K}}$ coincides with $e\Delta e=\Delta e$. It follows that

$$
\begin{aligned}
(e\otimes 1_n)P_n(\mathscr{M}) &= \overline{\mathrm{co}}\{[e\Delta^{1/4}ea_ia_j^*\xi_0]\in M_n(\mathscr{H}) \mid a_i\in\mathscr{M}\}\\
&= \overline{\mathrm{co}}\{[\Delta_{\mathscr{K}}^{1/4}\varepsilon(a_ia_j^*)\xi_0]\in M_n(\mathscr{K}) \mid a_i\in\mathscr{M}\}\\
&= P_n(\mathscr{N}_{\mathscr{K}})\,, \qquad n\in\mathbf{N}\,.
\end{aligned}
$$

This completes the proof.

THEOREM 1.4. *Let $\mathscr{H}$ be a separable Hilbert space. Suppose that $(\mathscr{M}, \mathscr{H}, J, \mathscr{P})$ is a standard form of an injective factor (or semifinite injective von Neumann algebra). Then there exists an increasing sequence of completely positive projections of finite rank on $\mathscr{H}$ which converges strongly to the identity map on $\mathscr{H}$.*

PROOF. If $\mathscr{M}$ is a Krieger's factor on $\mathscr{H}$, then $\mathscr{M}$ satisfies the following condition by [3; Theorem 1]:

There exists an increasing sequence $\{\varepsilon_k\}$ of normal conditional expectations of $\mathscr{M}$ onto finite dimensional subalgebras N_k with (*)

$$\left(\bigcup_{k=1}^{\infty} N_k\right)^{-s} = \mathscr{M}\,, \qquad \text{(strong closure)}\,.$$

By using [3; Lemma 2] we can find a cyclic and separating vector ξ_0 such that

$$\phi\circ\varepsilon_k=\phi,\ k\in\mathbf{N}\,, \quad\text{and}\quad \varepsilon_k(x)\to x,\ x\in\mathscr{M}$$

in the strong topology where ϕ denotes the vector state with respect to ξ_0. Let f_k denote a projection of $\mathscr{H}$ onto $[N_k\xi_0]$. Then f_k is strongly convergent to the identity map. Thus we obtain that f_k is completely positive in the sense of the order induced by the cone $\mathscr{P}_{\xi_0}=[\Delta_{\xi_0}^{1/4}\mathscr{M}^+\xi_0]$ by Lemma 1.3. If $(\mathscr{M}, \mathscr{H}, J, \mathscr{P})$ is standard, then there exists a unitary u in $\mathscr{M}'$ such that

$$J_{\xi_0}=uJu^{-1}\,, \quad \mathscr{P}_{\xi_0}=u\mathscr{P}\,,$$

(cf. [5; Theorem 2.18]). Put $e_k=u^{-1}f_ku$, $k\in\mathbf{N}$. We have then the inclusion $(e_k\otimes 1_n)P_n(\mathscr{M})\subset P_n(\mathscr{M})$ for $k, n\in\mathbf{N}$. In fact, if $a_i\in\mathscr{M}$ $(1\leqq i\leqq n)$ and $\eta\in\mathscr{P}$ then

$$
\begin{aligned}
&[e_ka_iJa_jJ\eta]=[u^{-1}f_ka_iJ_{\xi_0}a_jJ_{\xi_0}u\eta]\\
&\qquad\in(u^{-1}\otimes 1_n)(\mathscr{P}_{\xi_0}\hat{\otimes}M_n^+)=\overline{\mathrm{co}}\{[u^{-1}b_iJ_{\xi_0}b_jJ_{\xi_0}\xi_0] \mid b_i\in\mathscr{M}\}=P_n(\mathscr{M})\,.
\end{aligned}
$$

By [4; Theorem 7.5, Theorem 7.7] an injective III$_\lambda$-factor, $0\leqq\lambda<1$, on a separable Hilbert space $\mathscr{H}$ is a Krieger's factor. And an injective III$_1$-

factor on $\mathscr{H}$ is also a Krieger's factor by [6; Corollary 2.4]. Suppose that $\mathscr{M}$ is a semifinite injective von Neumann algebra on $\mathscr{H}$. Applying the reduction theory, we obtain that

$$\mathscr{M}=\mathscr{A}_0\otimes\mathscr{R}_0\oplus\mathscr{A}_1\otimes\mathscr{R}_0\otimes B(\mathscr{K})$$

because of the unicity of a hyperfinite II_1-factor $\mathscr{R}_0$ where $\mathscr{A}_0$ and $\mathscr{A}_1$ denote abelian von Neumann algebras on some separable Hilbert spaces and $\mathscr{K}$ a separable Hilbert space. Let ϕ be a faithful normal state on $\mathscr{M}$ decomposed into normal states on each of the components. Then there exists an increasing sequence of normal conditional expectations satisfying the above property (*).

We remark that Theorem 1.4 is valid for the infinite tensor product $\mathscr{M}=\otimes_{n=1}^{\infty}M_n$ on $\mathscr{H}=\otimes_{n=1}^{\infty}H_n$ with respect to the normal product state $\phi=\otimes_{n=1}^{\infty}\omega_{\xi_n}$ on $\mathscr{M}$ where M_n is a finite dimensional von Neumann algebra on H_n with a cyclic and separating vector ξ_n, $n\in\mathbf{N}$. In fact, put $N_k=\otimes_{n=1}^{k}M_n$ and $D_k=\otimes_{n=k+1}^{\infty}M_n$ with respect to $\{\xi_n: n=k+1, k+2,\cdots\}$. We identify N_k and D_k with subalgebras $N_k\otimes I$ and $I\otimes D_k$ of $\mathscr{M}$. Put $\varepsilon_k=L_{\phi|D_k}$ where L denotes the left slice map of $\mathscr{M}=N_k\otimes D_k$ to N_k. Then ε_k enjoys the condition (*) (cf. [19; Proposition 1]).

PROPOSITION 1.5. *Let $(\mathscr{M}_1, \mathscr{H}_1, J_1, \mathscr{P}_1)$ and $(\mathscr{M}_2, \mathscr{H}_2, J_2, \mathscr{P}_2)$ be two standard forms of von Neumann algebras either of which is an injective factor (or semifinite and injective) on a separable Hilbert space. Suppose that ϕ is an n-positive (resp. a completely positive) map of $\mathscr{H}_1$ into $\mathscr{H}_2$. Then there exists an increasing sequence $\{\phi_k\}$ of n-positive (resp. completely positive) maps of a Hilbert-Schmidt class of $\mathscr{H}_1$ into $\mathscr{H}_2$ such that $\phi_k(\xi)$ converges to $\phi(\xi)$ for all ξ in $\mathscr{H}_1$.*

PROOF. If $\mathscr{M}_1$ is an injective factor or a semifinite injective von Neumann algebra, then there exists an increasing sequence $\{e_k\}$ of completely positive projections of $\mathscr{H}_1$ into $\mathscr{H}_1$ which converges strongly to the identity map on $\mathscr{H}_1$ by Theorem 1.4. By Lemma 1.1 ϕe_k is an n-positive (resp. a completely positive) map of a Hilbert-Schmidt class of $\mathscr{H}_1$ into $\mathscr{H}_2$ and $\{\phi e_k(\xi)\}$ converges to $\phi(\xi)$ for $\xi\in\mathscr{H}_1$. This completes the proof.

COROLLARY 1.6. *Let $(\mathscr{M}_1, \mathscr{H}_1, J_1, \mathscr{P}_1)$ and $(\mathscr{M}_2, \mathscr{H}_2, J_2, \mathscr{P}_2)$ be two standard von Neumann algebras where $\mathscr{H}_1$ and $\mathscr{H}_2$ are separable. Then the following conditions are equivalent:*

i) *Any n-positive map of $\mathscr{H}_1$ into $\mathscr{H}_2$ is $n+1$-positive.*

ii) *Any n-positive map of $\mathscr{H}_1$ into $\mathscr{H}_2$ is completely positive.*

iii) *Either* $\mathcal{M}_1$ *or* $\mathcal{M}_2$ *is finite of type* I *with bounded degree not more than* n.

PROOF. Suppose that either $\mathcal{M}_1$ or $\mathcal{M}_2$ is finite of type I with bounded degree not more than n and ϕ is an n-positive map of $\mathcal{H}_1$ into $\mathcal{H}_2$. By Proposition 1.5 there exists a sequence $\{\phi_k\}$ of n-positive maps of a Hilbert-Schmidt class of $\mathcal{H}_1$ into $\mathcal{H}_2$ such that $\phi_k(\xi)$ converges to $\phi(\xi)$, $\xi \in \mathcal{H}_1$. Then $J_2\phi_k$ is an n-positive conjugate linear map of a Hilbert-Schmidt class of $\mathcal{H}_1$ into $\mathcal{H}_2$ with respect to standard forms $(\mathcal{M}_1, \mathcal{H}_1, J_1, \mathcal{P}_1)$ and $(\mathcal{M}_2', \mathcal{H}_2, J_2, \mathcal{P}_2)$. For, we obtain the equality

$$P_n(\mathcal{M}_2')=\overline{\mathrm{co}}\{[J_2a_iJ_2a_jJ_2\xi] \mid \xi \in \mathcal{P}_2,\ a_i \in \mathcal{M}_2\} .$$

Hence, $J_2\phi_k$ is completely positive for each $k \in \mathbf{N}$ by [9; Theorem 6]. Therefore, $J_2\phi=\lim J_2\phi_k$ is a completely positive map of $\mathcal{H}_1$ into $\mathcal{H}_2$ with respect to $(\mathcal{M}_1, \mathcal{H}_1, J_1, \mathcal{P}_1)$ and $(\mathcal{M}_2', \mathcal{H}_2, J_2, \mathcal{P}_2)$. It follows that ϕ is a completely positive map of $\mathcal{H}_1$ into $\mathcal{H}_2$ with respect to $(\mathcal{M}_1, \mathcal{H}_1, J_1, \mathcal{P}_1)$ and $(\mathcal{M}_2, \mathcal{H}_2, J_2, \mathcal{P}_2)$. We obtain the other implications by [9; Theorem 6]. This completes the proof.

We remark that Tomiyama [20] has shown the difference of n-positivity and complete positivity in C^*-algebras, the dual of C^*-algebras and the predual of von Neumann algebras.

In this section we shall lastly consider the converse of Theorem 1.4. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{H}_n^+ \subset M_n(\mathcal{H})$ $(n \in \mathbf{N})$ a family of selfdual cones. Then $\mathcal{H}$ is called a matrix ordered Hilbert space with selfdual cones if

$$\alpha\mathcal{H}_m^+\alpha^* \subset \mathcal{H}_n^+$$

for every α in a set of $n\times m$ matrices $M_{n,m}$ (cf. [11; Definition 1.2]). Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$. Let $\mathcal{H}_n^+$ $(n \in \mathbf{N})$ be a family of selfdual cones in $M_n(\mathcal{H})$. If $(\mathcal{M}, \mathcal{H}, J, \mathcal{H}_1^+)$ is a standard form and if for every $a \in \mathcal{M}\otimes M_{n,m}$ $(m,\ n \in \mathbf{N})$

$$aJ_{n,m}aJ_{m,m}(\mathcal{H}_m^+) \subset \mathcal{H}_n^+$$

holds, then $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)$ is called a matrix ordered standard form where $J_{n,m}\colon M_{n,m}(\mathcal{H}) \to M_{m,n}(\mathcal{H})$ is defined by $[\xi_{ij}] \to [J\xi_{ji}]$. If $(\mathcal{M}, \mathcal{H}, J, \mathcal{P})$ is a standard form of a von Neumann algebra, then $(\mathcal{M}, \mathcal{H}, P_n(\mathcal{M}))$ is a matrix ordered standard form (cf. [11; Lemma 1.1]). Then $(\mathcal{H}, P_n(\mathcal{M}),$ $n \in \mathbf{N})$ is a matrix ordered Hilbert space.

THEOREM 1.7. *Let* $(\mathcal{M}, \mathcal{H}, J, \mathcal{P})$ *be a standard form of a von*

Neumann algebra. If there exists a net $\{e_i\}$ $(i\in \mathbf{I})$ of completely positive projections of finite rank on $\mathscr{H}$ converging strongly to 1 such that every completed face of $(e_i\otimes 1_n)P_n(\mathscr{M})$ $(\forall i\in\mathbf{I}, \forall n\in\mathbf{N})$ is projectable, then $\mathscr{M}$ is injective.

PROOF. One easily sees that for each i, $(e_i\mathscr{H}, (e_i\otimes 1_n)P_n(\mathscr{M}), n\in\mathbf{N})$ is a matrix ordered Hilbert space. By [11; Theorem 4.3] and [13; Theorem 3.1], $e_i\mathscr{H}$ is an Arveson space. Now let N_m be any operator system in M_m and ϕ a completely positive map of N_m into $\mathscr{H}$. By [13; Theorem 2.4] $e_i\phi$ has a completely positive extension $\Phi_i: M_m\to\mathscr{H}$. Since

$$\|\Phi_i(1)\|=\|e_i\phi(1)\|\leqq\|\phi(1)\|$$

and $\xi\leqq\eta$ $(\xi, \eta\in\mathscr{P})$ implies that $\|\xi\|\leqq\|\eta\|$, $\{\Phi_i\}$ is bounded. Then we can find a weak limit Φ of a subnet of $\{\Phi_i\}$, which is a completely positive map of M_m into $\mathscr{H}$ and $\Phi\supset\phi$. It follows that $\mathscr{H}$ is finitely injective. Using again [13; Theorem 2.4, Theorem 3.1], $\mathscr{M}$ is injective. This completes the proof.

§ 2. Factorization of the selfdual cones associated with standard forms of injective factors.

In order to consider the factorization of the cone $P_n(\mathscr{M})$ by a maximal abelian subalgebra of $\mathscr{M}$, we need a few lemmata. The next lemma is a basic result of the positive cones in the matrix algebras.

LEMMA 2.1. *Let $\mathscr{N}$ be a direct sum of matrix algebras such that*

$$\mathscr{N}=M_{m_1}\oplus\cdots\oplus M_{m_k}\,,\qquad (1\leqq m_1,\cdots, m_k<\infty)\,.$$

Put

$$\mathscr{B}=A_{m_1}\oplus\cdots\oplus A_{m_k}\,,$$

where A_{m_i} is a set of diagonal matrices in M_{m_i} for $1\leqq i\leqq k$. Then $P_n(\mathscr{N})$ $(=\mathscr{N}^+\hat{\otimes}M_n^+)$ coincides with $P_n(\mathscr{B})$.

PROOF. We first consider the case $\mathscr{N}=M_m$. It is clear that $P_n(\mathscr{N})\subset P_n(A_m)$. Conversely, we choose $[\xi_{ij}]\in M_n(M_m)$ in $P_n(A_m)$. Put

$$a_i=\begin{bmatrix}\lambda_1^{(i)} & & 0\\ & \ddots & \\ 0 & & \lambda_m^{(i)}\end{bmatrix},\qquad \xi_{ij}=\begin{bmatrix}\xi_{11}^{(i,j)} & \cdots & \xi_{1m}^{(i,j)}\\ \vdots & & \vdots\\ \xi_{m1}^{(i,j)} & \cdots & \xi_{mm}^{(i,j)}\end{bmatrix}\in M_m\,.$$

We have then,

$$\sum_{i,j=1}^{n} a_i S_m a_j S_m \xi_{ij} = \sum_{i,j=1}^{n} a_i \xi_{ij} a_j^*$$

$$= [a_1 \cdots a_n] \begin{bmatrix} \xi_{11} & \cdots & \xi_{1n} \\ \vdots & & \vdots \\ \xi_{n1} & \cdots & \xi_{nn} \end{bmatrix} \begin{bmatrix} a_1^* \\ \vdots \\ a_n^* \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} \Xi [\alpha_1^* \cdots \alpha_m^*]$$

$$= \begin{bmatrix} \alpha_1 \Xi \alpha_1^* & \cdots & \alpha_1 \Xi \alpha_m^* \\ \vdots & & \vdots \\ \alpha_m \Xi \alpha_1^* & \cdots & \alpha_m \Xi \alpha_m^* \end{bmatrix},$$

which belongs to M_m^+ where $\alpha_i \in M_{1,mn}$ and

$$\alpha_1 = [\lambda_1^{(1)} 0 \cdots 0,\ \lambda_1^{(2)} 0 \cdots 0,\ \cdots,\ \lambda_1^{(n)} 0 \cdots 0]\ ,$$
$$\cdots\cdots\cdots$$
$$\alpha_m = [0 \cdots 0 \lambda_m^{(1)},\ 0 \cdots 0 \lambda_m^{(2)},\ \cdots,\ 0 \cdots 0 \lambda_m^{(n)}]\ ,$$

and

$$\Xi = [\xi_{ij}] = \begin{bmatrix} [\xi_{pq}^{(1,1)}] & \cdots & [\xi_{pq}^{(1,n)}] \\ \vdots & & \vdots \\ [\xi_{pq}^{(n,1)}] & \cdots & [\xi_{pq}^{(n,n)}] \end{bmatrix} \in M_n(M_m)\ .$$

Since

$$(\Xi(\alpha_1^* + \cdots + \alpha_m^*),\ \alpha_1^* + \cdots + \alpha_m^*) = \sum_{i,j=1}^{m} \alpha_i \Xi \alpha_j^* \geqq 0\ ,$$

one sees that Ξ belongs to $M_{mn}^+ = P_n(M_m)$ for $m, n \in \mathbf{N}$.

In general case, since

$$P_n(A_{m_1} \oplus \cdots \oplus A_{m_k}) = P_n(A_{m_1}) \oplus \cdots \oplus P_n(A_{m_k})\ , \qquad n \in \mathbf{N}$$

we obtain the desired result.

LEMMA 2.2. *Let $(\mathscr{M}, \mathscr{H}, J, \mathscr{P})$ be a standard form of a von Neumann algebra. If $\mathscr{A}$ is a von Neumann subalgebra of $\mathscr{M}$, then*

$$P_n(\mathscr{A})' = \overline{\text{co}}\{[a_i J a_j J \xi] \in M_n(\mathscr{H}) \mid a_i \in \mathscr{A},\ \xi \in \mathscr{P}\}\ , \qquad n \in \mathbf{N}\ .$$

In particular, if $\mathscr{M}$ has a cyclic and separating vector ξ_0 in $\mathscr{H}$ and $\mathscr{P} = [\Delta^{1/4} \mathscr{M}^+ \xi_0]$, then

$$P_n(\mathscr{A})' = \overline{\mathrm{co}}\{[a_i c J a_j c J \xi_0] \in M_n(\mathscr{H}) \mid a_i \in \mathscr{A},\ c \in \mathscr{M}\}, \qquad n \in \mathbf{N}.$$

PROOF. If $[\eta_{ij}]$ belongs to $P_n(\mathscr{A})$, then for $a_i \in \mathscr{A}$ and $\xi \in \mathscr{P}$ we have

$$([\eta_{ij}], [a_i J a_j J \xi]) = \sum_{i,j=1}^{n} (\eta_{ij}, a_i J a_j J \xi)$$
$$= \sum_{i,j=1}^{n} (a_i^* J a_j^* J \eta_{ij}, \xi) \geqq 0 .$$

Hence

$$P_n(\mathscr{A}) \subset \overline{\mathrm{co}}\{[a_i J a_j J \xi] \mid a_i \in \mathscr{A},\ \xi \in \mathscr{P}\}', \qquad n \in \mathbf{N}.$$

One immediately obtains the converse inclusion. In the case where $\mathscr{M}$ has a cyclic and separating vector ξ_0, we obtain the desired equality from the fact that any element ξ of $\mathscr{P}$ belongs to the closure of the set $\{aJaJ\xi_0 \mid a \in \mathscr{M}\}$.

The next proposition shows that the positive cone in the injective von Neumann algebra $M_n(\mathscr{M})$ can be factorized by an abelian subalgebra of $\mathscr{M}$.

PROPOSITION 2.3. *Let $\mathscr{M}$ be an injective von Neumann algebra on a separable Hilbert space $\mathscr{H}$. Then there exists an abelian subalgebra $\mathscr{A}$ of $\mathscr{M}$ such that*

$$M_n(\mathscr{M})^+ = \overline{\mathrm{co}}^s\{[a_i c c^* a_j^*] \in M_n(\mathscr{M}) \mid a_i \in \mathscr{A},\ c \in \mathscr{M}\}, \qquad n \in \mathbf{N}.$$

PROOF. If $\mathscr{M}$ is an injective von Neumann algebra on a separable Hilbert space $\mathscr{H}$, then $\mathscr{M}$ is AF, that is, there exists an increasing sequence $\{N_k\}$ of finite dimensional von Neumann subalgebras of $\mathscr{M}$ satisfying

$$\mathscr{M} = \left\{\bigcup_{i=1}^{\infty} N_k\right\}^{-s}.$$

Let A_k be a maximal abelian subalgebra of N_k such that $A_k \subset A_{k+1}$, $k \in \mathbf{N}$. For a fixed $k \in \mathbf{N}$, since N_k is a finite dimensional subalgebra of $\mathscr{M}$, N_k is considered as a finite direct sum of full operator algebras on finite dimensional Hilbert spaces. Then, A_k is decomposed into a finite direct sum of maximal abelian subalgebras of each component. Hence there exists an isomorphism ρ of N_k onto a finite direct sum L of matrix algebras such that $\rho(A_k)$ is the direct sum B of diagonal matrices. Note that $P_n(L)$ $(=M_n(L)^+)$ is a selfdual cone in the Hilbert space $M_n(L)$. By

Lemma 2.1 and Lemma 2.2, we obtain that

$$M_n(L)^+=\overline{\text{co}}\{[a_i cc^* a_j^*] \mid a_i \in B,\ c \in L\}\ , \qquad n \in \mathbf{N}\ .$$

Since any element of the positive cone $M_n(N_k)$ can be written as a finite sum of the elements $[a_i a_j^*] \in M_n(N_k)$, we have

$$M_n(N_k)^+=\overline{\text{co}}\{[a_i cc^* a_j^*] \mid a_i \in A_k,\ c \in N_k\}\ .$$

Now, we put

$$\mathscr{A}=\bigcup_{k=1}^{\infty} A_k\ .$$

Since $\{A_k\}$ is an increasing sequence of abelian von Neumann subalgebras of $\mathscr{M}$, $\mathscr{A}$ is an abelian subalgebra of $\mathscr{M}$. Therefore we obtain that

$$M_n(\mathscr{M})^+=\left\{\bigcup_{k=1}^{\infty} M_n(N_k)\right\}^{-s} \subset \overline{\text{co}}^s\{[a_i cc^* a_j^*] \mid a_i \in \mathscr{A},\ c \in \mathscr{M}\}\ .$$

This completes the proof.

A maximal abelian subalgebra $\mathscr{A}$ of a von Neumann algebra $\mathscr{M}$ is said to be smooth if there exists a normal conditional expectation of $\mathscr{M}$ onto $\mathscr{A}$ (cf. [18; Definition]). We have some results of smooth maximal abelian subalgebras. Takesaki [15] proved that $\mathscr{M}$ is finite if and only if any maximal abelian subalgebra of $\mathscr{M}$ is smooth. Tomiyama [19] showed the existence of a smooth maximal abelian subalgebra of the infinite tensor product $\mathscr{M}=\bigotimes_{k=1}^{\infty} N_k$ on $\mathscr{H}=\bigotimes_{k=1}^{\infty} H_k$ with respect to the normal product state $\phi=\bigotimes_{k=1}^{\infty} \omega_{\xi_k}$ on $\mathscr{M}$ where all N_k are semifinite factors and all vector states ω_{ξ_k} on N_k are faithful.

We shall lastly give the factorization of the selfdual cone $P_n(\mathscr{M})$ related to the standard form of an injective factor $\mathscr{M}$, which also shows the existence of a smooth maximal abelian subalgebra in $\mathscr{M}$.

THEOREM 2.4. *Let $(\mathscr{M}, \mathscr{H}, J, \mathscr{P})$ be a standard form of an injective factor (or a semifinite injective von Neumann algebra) $\mathscr{M}$ where $\mathscr{H}$ is separable. Then there exists a smooth maximal abelian subalgebra $\mathscr{A}$ of $\mathscr{M}$ such that*

$$P_n(\mathscr{M})=P_n(\mathscr{A})\ , \qquad n \in \mathbf{N}\ ,$$

that is

$$P_n(\mathscr{M})=\overline{\text{co}}\{[a_i J a_j J \xi] \in M_n(\mathscr{H}) \mid a_i \in \mathscr{A},\ \xi \in \mathscr{P}\}\ , \qquad n \in \mathbf{N}\ .$$

PROOF. By the proof of Theorem 1.4, there exists an increasing sequence $\{\varepsilon_k\}$ of normal conditional expectations with respect to ϕ converging strongly to 1 such that $\varepsilon_k(\mathscr{M})=N_k$ is a finite dimensional von Neumann subalgebra of $\mathscr{M}$ and ϕ is a vector state with respect to some cyclic and separating vector ξ_0 for $\mathscr{M}$. Because of the unitary equivalence of isomorphic standard forms, we may assume that $J=J_{\xi_0}$ and $\mathscr{P}=[\Delta^{1/4}\mathscr{M}^+\xi_0]$ where Δ is a modular operator with respect to ξ_0. Put

$$\pi_k=\varepsilon_k|N_{k+1}\ ,\quad \phi_k=\phi|N_k\ ,\qquad k\in\mathbf{N}\ .$$

We have then

$$\sigma_t^{\phi}(x)=\sigma_t^{\phi_k}(x)\ ,\qquad x\in N_k,\ k\in\mathbf{N}\ .$$

Since a one-parameter automorphism group of N_k is inner, there exists a positive operator $h_k\in N_k$ such that

$$\sigma_t^{\phi_k}(x)=h_k^{it}xh_k^{-it}\ ,\qquad x\in N_k\ .$$

Now, we shall first construct the smooth maximal abelian subalgebra. Let A_1 be a maximal abelian subalgebra of N_1 containing all spectral projections of h_1. Since

$$h_1^{it}xh_1^{-it}=h_2^{it}xh_2^{-it}\ ,\qquad x\in N_1\ ,$$

$h_2^{-it}h_1^{it}$ belongs to a relative commutant of N_1 in N_2, $t\in\mathbf{R}$. Let C be an abelian subalgebra of N_2 generated by A_1 and $h_2^{-it}h_1^{it}$, $t\in\mathbf{R}$. Since h_1^{it} belongs to A_1, all spectral projections of h_2 belong to C. Let A_2 be a maximal abelian subalgebra of N_2 containing C. By induction we choose a maximal abelian subalgebra A_k of N_k, $k\in\mathbf{N}$. Since A_k is contained in the fixed point algebra of $\sigma_t^{\phi_k}$, there exists a conditional expectation σ_k of N_k onto A_k, $k\in\mathbf{N}$. For any minimal projection p in A_{k+1}, we can choose a minimal projection q in A_k such that $p\leqq q$, $k\in\mathbf{N}$. In fact, there exists a minimal projection q in A_k with $pq\neq 0$. We have then $pq\leqq p$. Since p is a 1-dimensional projection in A_{k+1}, $pq=p$ and so $p\leqq q$. Now, we have then $\pi_k(p)\leqq q$. Since q is a 1-dimensional projection in A_k, $\pi_k(p)=\lambda q$, $0\leqq\lambda\leqq 1$. Hence we have $\pi_k(A_{k+1})=A_k$. Then the following diagram is commutative:

$$\begin{array}{ccc} N_{k+1} & \xrightarrow{\sigma_{k+1}} & A_{k+1} \\ \downarrow{\scriptstyle \pi_k} & & \downarrow{\scriptstyle \pi_k} \\ N_k & \xrightarrow{\sigma_k} & A_k\ . \end{array}$$

In fact, since $\pi_k\circ\sigma_{k+1}$ and $\sigma_k\circ\pi_k$ are normal conditional expectations of N_{k+1} onto A_k, and

$$\phi(\pi_k\circ\sigma_{k+1}(x))=\phi(\sigma_{k+1}(x))=\phi(x)$$
$$=\phi(\pi_k(x))=\phi(\sigma_k\circ\pi_k(x)) , \qquad x\in N_{k+1} ,$$

that is,

$$\phi_{k+1}\circ\pi_k\circ\sigma_{k+1}=\phi_{k+1}\circ\sigma_k\circ\pi_k ,$$

we obtain the equality $\pi_k\circ\sigma_{k+1}=\sigma_k\circ\pi_k$ by the unicity of a normal conditional expectation (see the proof of [15; Theorem]). By [19; Lemma 3 and Theorem 4] there exists a projection σ of norm one of the C^*-algebra generated by N_k, $k\in\mathbf{N}$ onto the C^*-algebra generated by A_k, $k\in\mathbf{N}$, such that $\sigma\supset\sigma_k$, $k\in\mathbf{N}$, which has a normal extension. Hence, if we put

$$\mathscr{A}=\left(\bigcup_{k=1}^{\infty} A_k\right)^{-s} ,$$

$\mathscr{A}$ is a smooth maximal abelian subalgebra of $\mathscr{M}$.

For a fixed $k\in\mathbf{N}$, there exists an isomorphism Φ of N_k onto a finite direct sum L_k of matrix algebras such that $\Phi(A_k)$ is a direct sum of the diagonal algebras. By Lemma 1.3, we note that (N_k, H_k, J_k, Q_k) is a standard form where $H_k=e_k\mathscr{H}$, $J_k=Je_k$ and $Q_k=e_k\mathscr{P}$ for the completely positive projection e_k as in the proof of Theorem 1.4. Let $(L_k, K_k, S^{(k)}, L_k^+)$ mean a canonical standard form. Then there exists an isometry u of H_k onto K_k such that $\Phi(x)=uxu^{-1}$ $(x\in N_k)$, $S^{(k)}=uJ_ku^{-1}$, $L_k^+=uQ_k$. If $a_i\in A_k$ and $c\in N_k$, then

$$a_icJ_ka_jcJ_k\xi_0=u^{-1}\Phi(a_i)\Phi(c)S^{(k)}\Phi(a_j)\Phi(c)S^{(k)}\eta_0$$

for $\eta_0\in L_k^+$. By Lemma 2.1, we have

$$P_n(N_k|H_k)=P_n(A_k|H_k) , \qquad n\in\mathbf{N} .$$

Now, for each $k\in\mathbf{N}$, we have by Lemma 1.3

$$(e_k\otimes 1_n)P_n(\mathscr{M})=P_n(N_k|H_k)$$
$$=\overline{\mathrm{co}}\{[a_icJ_ka_jcJ_k\xi_0] \mid a_i\in A_k,\ c\in N_k\}$$
$$\subset\overline{\mathrm{co}}\{[a_icJa_jcJ\xi_0] \mid a_i\in\mathscr{A},\ c\in\mathscr{M}\} .$$

If k tends to ∞, we obtain the required equality. This completes the proof.

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Present Address:
Department of Mathematics
College of Humanities and Social Sciences, Iwate University
Morioka 020, Japan