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Spherical Hypersurfaces with Low Type Quadric Representation

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§0. Introduction.

Let S^{m-1} be the unit hypersphere of the Euclidean m-space E^m centered at the origin, embedded in the standard way. Then any submanifold of S^{m-1} we call *spherical*. If M^n is a compact spherical submanifold minimal in S^{m-1} then it is well known that the Euclidean coordinate functions restricted to M are eigenfunctions of the Laplacian Δ on M^n with the same eigenvalue n (see [25]). Spectral behavior of a spherical submanifold can be also nicely related to the second standard immersion ι of S^{m-1}. For example, A. Ros [20] studied compact minimal spherical submanifolds via the second standard immersion. He obtained characterization of those that are described by means of two different eigenvalues of Δ, i.e. those which are of 2-type via ι (for precise definition of k-type submanifolds see the next section). He showed that such submanifolds are Einstein and mass-symmetric via ι. Then M. Barros and B. Y. Chen [2] obtained generalization of Ros' characterization for spherical submanifolds which are mass-symmetric and of 2-type via ι. They also classified hypersurfaces of S^{m-1} which are mass-symmetric and of 2-type via the second standard immersion.

Let $x: M^n \to S^{m-1} \subset E^m$ be an isometric immersion and regard $x = (x_1, \cdots, x_m)^t$ as a column matrix. We define the map $\tilde{x} = \iota \circ x = xx^t$ from M into the set of $m \times m$ symmetric matrices and call it the *quadric representation* of M because the coordinates of $\tilde{x}$ depend on the coordinates of x in a quadratic manner. Studying submanifolds with finite type quadric representation amounts to studying spectral behavior of the product of coordinate functions $x_i \cdot x_j$.

In this paper we extend the above result of Barros and Chen giving the classification of spherical hypersurfaces with 2-type quadric representation without assuming mass-symmetry a priori (Th. 3.1). We also

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undertake the study of spherical hypersurfaces with 3-type quadric representation. We give a characterization and some classification results for minimal hypersurfaces of S^{m-1} which are mass-symmetric and of 3-type via ι (Theorems 4.1 and 4.2). As a byproduct, we obtain a new characterization of the minimal Cartan hypersurface $SO(3)/\boldsymbol{Z}_2\times\boldsymbol{Z}_2$ in S^4 in terms of its spectral behavior.

§ 1. Preliminaries.

Notation. Let us fix the notation (standard facts from submanifold theory can be found in [7]). Let $x: M^n\to E^m$ be an isometric immersion of a compact connected n-dimensional Riemannian manifold into a Euclidean m-space. Suppose that $e_1, e_2, \cdots, e_n, e_{n+1}, \cdots, e_m$ are local orthonormal vector fields along M such that the first n vectors are tangent to M and the remaining $m-n$ vectors normal to M. Let $\langle\ ,\ \rangle$ and $\bar{\nabla}$ be the Euclidean metric and connection on E^m, and denote by ∇, h, D, A_ξ respectively, the induced (Levi Civita) connection, second fundamental form of M, connection in the normal bundle $T^\perp M$ and the Weingarten endomorphism relative to the normal direction ξ. They are defined by the following equations:

$$\bar{\nabla}_X Y=\nabla_X Y+h(X, Y)\ , \qquad \bar{\nabla}_X\xi=-A_\xi X+D_X\xi$$

for $X, Y\in TM$ and $\xi\in T^\perp M$, where first terms on the right hand sides of these equations are tangent to M and the second terms are normal to M. Instead of A_{e_r} we write A_r for short. The connection forms ω_i^j and the mean curvature vector H of M in E^m are defined by $\nabla_{e_k}e_i=\sum_j\omega_i^j(e_k)e_j$, $H=(1/n)\sum_r(\operatorname{tr}A_r)e_r$. In this setting indices $i, j, k, \cdots$ will always range from 1 to n, and indices $r, s, \cdots$ from $n+1$ to m. If we choose $e_{n+1}//H$ then $H=\alpha e_{n+1}$ for some function α called the mean curvature of x. As usual, Δ denotes the Laplacian of M acting on smooth functions in $C^\infty(M)$, i.e. $\Delta f=\sum_{i=1}^n[(\nabla_{e_i}e_i)f-e_i(e_if)]$. For a differentiable function f, ∇f denotes its gradient and for an endomorphism field S on M we define $\operatorname{tr}(\nabla S)=\sum_k(\nabla_{e_k}S)e_k$. All manifolds will be assumed smooth, compact (without boundary) and connected, and all immersions smooth.

Finite type submanifolds. Let $\operatorname{Spec}(M)=\{0=\lambda_0<\lambda_1<\cdots<\lambda_k<\cdots\uparrow\infty\}$ be the spectrum of Δ. If we extend Δ to act on E^m-valued functions on M in a natural fashion (componentwise), then for an isometric immersion $x: M\to E^m$ we have the following spectral decomposition (in L^2-sense)

$$x=x_0+\sum_{t=1}^\infty x_t\ ,\quad \Delta x_t=\lambda_t x_t\ ,\quad x_t: M\to E^m\ , \tag{1.1}$$

where x_0 is constant vector (equal to the center of mass of M in E^m). Submanifold M is said to be of *finite type* if the spectral decomposition of x consists of only finitely many nonzero terms. Moreover, M is said to be of k-type if there are exactly k nonconstant functions $x_{t_1}, x_{t_2}, \cdots, x_{t_k}$ in the above decomposition. In that case the collection of indices $[t_1, t_2, \cdots, t_k]$ is called the *order* of the immersion. If the immersion x is of k-type then from (1.1) by taking successive Laplacians and eliminating x_t's one obtains

$$P(\Delta)(x-x_0)=0\ , \qquad \text{where} \quad P(T)=\prod_{i=1}^{k}(T-\lambda_i)\ . \tag{1.2}$$

If M is immersed into a hypersphere $S_c^{m-1}(r)$, the immersion is said to be *mass-symmetric* if the center of mass of M coincides with the center c of the hypersphere. The notion of a finite type submanifold was first introduced by B. Y. Chen and the first results were collected in [8].

Second standard immersion of S^{m-1}. On the Euclidean space E^m we have the canonical inner product $\langle\ ,\ \rangle$ given by $\langle u, v\rangle = u^t\cdot v$, where $u, v\in E^m$ are regarded as column matrices and u^t is the transpose of u. Then the unit hypersphere centered at the origin is defined as $S^{m-1}=\{x\in E^m \mid \langle x, x\rangle =1\}$.

Let $SM(m)=\{P\in GL(m; R) \mid P^t=P\}$ be the space of real symmetric $m\times m$ matrices. Since a generic symmetric matrix $P\in SM(m)$ has $m(m+1)/2$ independent entries, $SM(m)$ can be regarded as Euclidean space of dimension $N=m(m+1)/2$. Moreover, the canonical Euclidean metric on E^N is given by

$$\tilde{g}(P, Q)=\frac{1}{2}\mathrm{tr}(PQ)\ , \qquad P, Q\in SM(m)\ . \tag{1.3}$$

We denote by $\tilde{\nabla}$ the Euclidean connection on $SM(m)$. Define the map $*$ from $E^m\times E^m$ into $SM(m)$ by $X*Y=XY^t+YX^t$ for column vectors X, Y in E^m. Then $*$ is bilinear, $X*Y=Y*X$, $\tilde{g}(X*Y, Z*W)=\langle X, W\rangle\langle Y, Z\rangle+\langle Y, W\rangle\langle X, Z\rangle$ and $\tilde{\nabla}_X(Y*Z)=(\bar{\nabla}_X Y)*Z+Y*(\bar{\nabla}_X Z)$ for vector fields X, Y and Z. If M is a submanifold of E^m and X and Y are vector fields along M then the following product formula for the Laplacian holds:

$$\Delta(X*Y)=(\Delta X)*Y+X*(\Delta Y)-2\sum_i(\bar{\nabla}_{e_i}X)*(\bar{\nabla}_{e_i}Y)\ . \tag{1.4}$$

Consider now the mapping $\iota: S^{m-1}\to SM(m)$ defined by $\iota(u)=uu^t$ where $u\in S^{m-1}\subset E^m$ is a column vector in E^m of unit length. Then ι is an isometric immersion, in fact the second standard immersion of

S^{m-1}. The image $\iota(S^{m-1})$ is a real projective space which is immersed minimally in a hypersphere $S^{N-1}_{I/m}(r)$ of $SM(m)$ centered at I/m with radius $r=((m-1)/2m)^{1/2}$. $\iota(S^{m-1})$ is called a Veronese submanifold (see [22], [20], [8]).

The tangent and the normal space of $\iota(S^{m-1})$ are given respectively by

$$T_{\iota(u)}S^{m-1}=\{P\in SM(m)\mid P\iota(u)+\iota(u)P=P\}\ , \tag{1.5}$$

$$T^{\perp}_{\iota(u)}S^{m-1}=\{P\in SM(m)\mid P\iota(u)=\iota(u)P\}\ . \tag{1.6}$$

From (1.6) we see that both I and $\iota(u)$ are normal to S^{m-1} via ι, and also, for any tangent vector X to the sphere, XX^t is normal to S^{m-1}. We prove the following lemma that will be used later.

LEMMA 1.1. *For standard hypersphere $u:S^{m-1}\to E^m$, let ι be the second standard immersion $\iota:S^{m-1}\to SM(m)$ by $\iota(u)=uu^t$. If $e_1,\cdots,e_{m-1}$ is a local orthonormal frame of tangent vectors to S^{m-1} then $I=uu^t+\sum_{i=1}^{m-1}e_ie_i^t$, where I is the $m\times m$ identity matrix.*

PROOF. Consider the following $\frac{1}{2}(m^2-m+2)$ matrices: uu^t, $e_ke_k^t$ $(1\leqq k\leqq m-1)$, $e_ie_j^t+e_je_i^t$ $(1\leqq i<j\leqq m-1)$. By (1.6) they all belong to the normal space $T^{\perp}_uS^{m-1}$, and these vectors are linearly independent (they are mutually orthogonal). On the other hand,

$$\dim T^{\perp}_uS^{m-1}=\dim SM(m)-\dim T_uS^{m-1}=\frac{m(m+1)}{2}-m+1=\frac{m^2-m+2}{2}\ .$$

We conclude therefore, that $T^{\perp}_uS^{m-1}=\mathrm{Span}\{uu^t,\ e_ke_k^t,\ e_i*e_j\}$. In particular, $I=a(uu^t)+\sum_k b_k(e_ke_k^t)+\sum_{i<j}c_{ij}e_i*e_j$. Using (1.3), it is easy to see that $c_{ij}=0$ and $a=b_k=1$ for every k, proving the lemma.

Spherical isoparametric hypersurfaces. A spherical hypersurface is called *isoparametric* if its principal curvatures (and their respective multiplicities) are constant [6]. Every isoparametric hypersurface belongs to entire family of parallel hypersurfaces, each of which has constant principal curvatures. In each isoparametric family of spherical hypersurfaces there is a unique one which is minimal in the sphere (see [3], [16], [17]). E. Cartan classified isoparametric hypersurfaces of S^{n+1} with two distinct principal curvatures as standard products of two spheres [3], and he found that those with three distinct principal curvatures are precisely the tubes of constant radius over the standard embeddings of FP^2 for $F=\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{Q}$ (quaternions), $\boldsymbol{O}$ (Cayley numbers) in S^4, S^7, S^{13}, S^{25} respectively [4]. Isoparametric spherical hypersurfaces with three principal curvatures are all homogeneous. They are identified as $SO(3)/\boldsymbol{Z}_2\times\boldsymbol{Z}_2$,

$SU(3)/T^2$, $Sp(3)/Sp(1)^3$, $F_4/Spin(8)$ of dimensions 3, 6, 12, 24 respectively (see [12], [24]). The minimal hypersurface of the type $SO(3)/\boldsymbol{Z}_2\times\boldsymbol{Z}_2$ in S^4 is called the *Cartan hypersurface.*

In [5] Cartan gave examples of two families of isoparametric hypersurfaces in S^5 and S^9 with four distinct principal curvatures of the same multiplicity (respectively 1 and 2). The isoparametric family M_t^4 in S^5 has particularly nice representation by the map

$$S^1\times S_{3,2}\longrightarrow S^5\subset E^6$$

given by

$$(\theta,\ (x,\ y))\longrightarrow e^{i\theta}(\cos t\,x+i\sin t\,y) \tag{1.7}$$

where $S_{3,2}$ denotes Stiefel manifold of orthonormal pairs of vectors in E^3 and S^1 is the unit circle. More precisely each isoparametric hypersurface $M_t^4\subset S^5$ with four principal curvatures is the image of the map (1.7) which doubly covers M_t^4. The minimal one is obtained when $t=\pi/8$ (see [17]). Nomizu used this map to construct infinite family of isoparametric hypersurfaces M_t^{2n} in S^{2n+1} with four principal curvatures of multiplicities 1, $n-1$, 1 and $n-1$. Takagi has shown in [23] that any isoparametric hypersurface with four curvatures such that the multiplicity of one curvature is 1 is congruent to the example M_t^{2n} of Nomizu for some t.

All examples of isoparametric spherical hypersurfaces known by Cartan are homogeneous. In particular, isoparametric hypersurfaces with four principal curvatures of the same multiplicity 1 or 2 mentioned above are $SO(2)\times SO(3)/\boldsymbol{Z}_2$, respectively $Sp(2)/T^2$. The following theorem of H. F. Münzner [14] is the major result in the theory.

THEOREM 1.1. (a) *The number* ν *of distinct principal curvatures of an isoparametric hypersurfaces satisfies* $\nu=1$, 2, 3, 4 *or* 6.

(b) *If* $k_1>k_2>\cdots>k_\nu$ *are distinct principal curvatures of an isoparametric spherical hypersurface with respective multiplicities* m_1, $m_2,\cdots,m_\nu$, *then*

$$k_i=\cot\theta_i\ ,\qquad 0<\theta_1<\cdots<\theta_\nu<\pi$$

where

$$\theta_i=\theta_1+\frac{i-1}{\nu}\pi\ ,\qquad 1\leqq i\leqq\nu\ ,\quad \textit{with}\quad \theta_1<\frac{\pi}{\nu}\ ,$$

and the multiplicities satisfy $m_i=m_{i+2}$ (*subscripts* $\mathrm{mod}\,\nu$).

As a consequence, there are at most two different multiplicities m_1,

m_2 for principal curvatures and if ν is 3 then all multiplicities are equal. Using the results of [1], [23], [24], [19] it is not difficult to completely classify isoparametric hypersurfaces with four curvatures of the same multiplicity m. Namely, from [1] we know $m\in\{1, 2\}$. If $m=1$, then the results of Takagi [23] and Takagi and Takahashi [24] classify such hypersurface as $SO(2)\times SO(3)/Z_2$. If $m=2$, by the result of Ozeki and Takeuchi [19] the hypersurface is homogeneous and therefore according to the list in [24] must be $Sp(2)/T^2$. Therefore these hypersurfaces are exactly those two found by Cartan in [5].

Next, we give the list (taken from [12]) of all isoparametric hypersurfaces in sphere with three or four distinct principal curvatures of the same multiplicity. As remarked by Hsiang and Lawson, homogeneous isoparametric hypersurfaces in sphere arise from isotropy representations of the corresponding symmetric spaces of rank 2. For our hypersurfaces, their isometry groups G, actions ψ, principal isotropy groups H, common multiplicity of principal curvatures m and dimension n are given as follows (first four examples in the table have three curvatures, remaining two have four).

Isoparametric hypersurfaces in sphere with three or four principal curvatures of the same multiplicity

G	Ψ	H	m	n
$SO(3)$	$S^2\rho_3-\theta$	$Z_2\times Z_2$	1	3
$SU(3)$	$\mathrm{Ad}_{SU(3)}$	T^2	2	6
$Sp(3)$	$\wedge^2\nu_3-\theta$	$Sp(1)^3$	4	12
F_4	ϕ_1	$Spin(8)$	8	24
$SO(2)\times SO(3)$	$\rho_2\otimes\rho_3$	Z_2	1	4
$Sp(2)$	Ad	T^2	2	8

§2. Computation of $\Delta^2\tilde{x}$ and $\Delta^3\tilde{x}$.

Let $x: M^n\to E^m$ be an isometric immersion. In this section we derive formulas for the iterated Laplacians $\Delta^2\tilde{x}$ and $\Delta^3\tilde{x}$ of the quadric representation $\tilde{x}=xx^t$. Since $\Delta x=-nH$ and $\bar{\nabla}_{e_i}x=e_i$, we compute $\Delta\tilde{x}=\frac{1}{2}\Delta(x*x)$ using product formula (1.4). We get

$$\Delta\tilde{x}=-nH*x-\sum_i e_i*e_i\,. \tag{2.1}$$

To find $\Delta^2\tilde{x}$ we first find $-\sum_i\Delta(e_i*e_i)$ and then $\Delta(H*x)$. We can assume that at a given point p we have $(\nabla_{e_k}e_i)(p)=0$ (normal coordinate system). Then the Laplacian becomes $\Delta f=-\sum_k e_k e_k f$ at point p, so we have first

$$\sum_i \tilde{\nabla}_{e_k}(e_i * e_i)=2\sum_i h(e_k, e_i)*e_i$$

and then at p

$$\begin{aligned}(2.2)\qquad -\sum_i \Delta(e_i*e_i) &= \sum_{k,i} \tilde{\nabla}_{e_k}\tilde{\nabla}_{e_k}(e_i*e_i)\\ &= -2\sum_{k,i}[A_{h(e_k,e_i)}e_k]*e_i+2\sum_{k,i}[D_{e_k}h(e_k, e_i)]*e_i+2\sum_{k,i}h(e_k, e_i)*h(e_k, e_i)\ .\end{aligned}$$

Now we compute each sum separately. Since $h(e_k, e_i)=\sum_r \langle h(e_k, e_i), e_r\rangle e_r$, we obtain

$$\sum_{k,i} h(e_k, e_i)*h(e_k, e_i)=\sum_{i,k,r,s}\langle A_r e_i, e_k\rangle\langle A_s e_i, e_k\rangle e_r*e_s=\sum_{r,s}\operatorname{tr}(A_rA_s)e_r*e_s\ ,$$

$$\sum_{k,i}[A_{h(e_k,e_i)}e_k]*e_i=\sum_{i,k,r}\langle A_r e_k, e_i\rangle(A_re_k)*e_i=\sum_{k,r}(A_re_k)*(A_re_k)\ ,$$

and, by Codazzi equation,

$$\sum_{k,i}[D_{e_k}h(e_k, e_i)]*e_i=n\sum_i(D_{e_i}H)*e_i\ .$$

Substituting these formulas into (2.2) and putting it together we see that at point p the following equation holds:

$$(2.3)\qquad -\sum_i \Delta(e_i*e_i)=2n\sum_i(D_{e_i}H)*e_i+2\sum_{r,s}\operatorname{tr}(A_rA_s)e_r*e_s-2\sum_{k,r}(A_re_k)*(A_re_k)\ .$$

Neither left hand side nor right hand side of (2.3) depend on the adapted frame chosen, so the formula is true for any (local) frame at any point of M.

Next we compute $\Delta(H*x)$ using product formula for the Laplacian

$$(2.4)\qquad \Delta(H*x)=(\Delta H)*x-nH*H+2\sum_i(A_He_i)*e_i-2\sum_i(D_{e_i}H)*e_i\ .$$

Combining (2.1), (2.3) and (2.4) we finally obtain the following formula for $\Delta^2\tilde{x}$:

$$\begin{aligned}(2.5)\qquad \Delta^2\tilde{x}= &-n(\Delta H)*x+n^2H*H-2n\sum_i(A_He_i)*e_i+4n\sum_i(D_{e_i}H)*e_i\\ &+2\sum_{r,s}\operatorname{tr}(A_rA_s)e_r*e_s-2\sum_{r,k}(A_re_k)*(A_re_k)\ .\end{aligned}$$

The right hand side of (2.5) is independent of the chosen point and an adapted frame at that point. If M^n is a hypersurface of S^{n+1} in E^{n+2} then the Laplacian of the mean curvature vector H of M^n in E^m can be computed as [8, p. 273]

$$\Delta H = (\Delta\alpha')\xi + \operatorname{tr}(\bar{\nabla} A_H) + (\|A_\xi\|^2 + n)H' - n\alpha^2 x \,, \tag{2.6}$$

where, as usual, symbols with $'$ denote objects and quantities relative to the immersion of M^n into hypersphere S^{n+1}. The mean curvature of M^n in E^{n+2} is denoted by α and the one in S^{n+1} by α'. They are related via $\alpha^2 = \alpha'^2 + 1$. ξ is a local unit normal vector field of M in S^{n+1} such that $\xi // H'$, hence $H' = \alpha'\xi$ (H' is the mean curvature vector of M in S^{n+1} and $H = H' - x$). $\operatorname{tr}(\bar{\nabla} A_H)$ is defined by

$$\operatorname{tr}(\bar{\nabla} A_H) = \sum_{i=1}^{n} (\nabla_{e_i} A_H) e_i + \sum_{i=1}^{n} A_{D_{e_i} H} e_i$$

where $e_1, \cdots, e_n$ is a local orthonormal frame of tangent vectors of M^n. For a spherical hypersurface we have by [9]

$$\operatorname{tr}(\bar{\nabla} A_H) = n\alpha'\nabla\alpha' + 2A(\nabla\alpha') \,. \tag{2.7}$$

Putting this back into (2.6) and combining with (2.5) we obtain for the spherical hypersurface

$$\begin{aligned} \Delta^2 \tilde{x} = {} & -n[\Delta\alpha' + \alpha'(\|A\|^2 + 3n + 4)]\xi * x - n W * x \\ & + n(n\alpha^2 + n + 2) x * x + (n^2\alpha'^2 + 2\|A\|^2)\xi * \xi \\ & + 4n\xi * (\nabla\alpha') - 2(n+1)\sum_i e_i * e_i \\ & - 2n\alpha' \sum_i (Ae_i) * e_i - 2\sum_i (Ae_i) * (Ae_i) \,, \end{aligned} \tag{2.8}$$

where $A = A_\xi$, $W = \operatorname{tr}(\bar{\nabla} A_H) = n\alpha'\nabla\alpha' + 2A(\nabla\alpha')$ and $\|A\|^2 = \operatorname{tr} A^2$ (cf. [2]). Assume that M^n has constant mean curvature α' in S^{n+1}. Then the calculations from before give

$$-\Delta \sum_i e_i * e_i = 2n x * x + 2\|A\|^2 \xi * \xi - 4n\alpha' x * \xi - 2\sum_i (e_i * e_i + Ae_i * Ae_i) \,, \tag{2.9}$$

and since $\Delta\xi = \|A\|^2\xi - (\operatorname{tr} A)x$, we also obtain

$$\Delta(\xi * \xi) = 2\|A\|^2 \xi * \xi - 2n\alpha' x * \xi - 2\sum_k (Ae_k) * (Ae_k) \,, \tag{2.10}$$

$$\Delta(x * \xi) = -n\alpha' x * x - n\alpha' \xi * \xi + (n + \|A\|^2) x * \xi + 2\sum_k (Ae_k) * e_k \,. \tag{2.11}$$

Next, we outline deduction of formulas for $-\Delta \sum_i (Ae_i) * e_i$ and $-\Delta \sum_i (Ae_i) * (Ae_i)$ leaving out the details. We do computations at a point p, where we assume $(\nabla_{e_k} e_i)(p) = 0$ for every k and i. We always write $h(X, Y) = \langle AX, Y\rangle\xi - \langle X, Y\rangle x$.

$$
\begin{aligned}
-\Delta \sum_i (Ae_i)*e_i &= \sum_{i,k} \tilde{\nabla}_{e_k}\tilde{\nabla}_{e_k}(Ae_i*e_i) \\
&= \sum_{i,k} \tilde{\nabla}_{e_k}\{[\nabla_{e_k}(Ae_i)+h(e_k, Ae_i)]*e_i + Ae_i*[\nabla_{e_k}e_i + h(e_k, e_i)]\} \\
&= \sum_{i,k} \{\tilde{\nabla}_{e_k}[\nabla_{e_k}(Ae_i)*e_i] + \tilde{\nabla}_{e_k}(Ae_i*\nabla_{e_k}e_i)\} + 2\sum_k \tilde{\nabla}_{e_k}(\xi*A^2e_k - x*Ae_k) \\
&= \sum_{i,k} [\nabla_{e_k}\nabla_{e_k}(Ae_i)*e_i + Ae_i*\nabla_{e_k}\nabla_{e_k}e_i] - 2\sum_i [Ae_i*e_i + Ae_i*A^2e_i] \\
&\quad + 2\sum_k [\xi*(\nabla_{e_k}A)(Ae_k) + \xi*\nabla_{e_k}(A^2e_k) - 2x*\nabla_{e_k}(Ae_k)] \\
&\quad + 2(\mathrm{tr}\, A)x*x - 4(\mathrm{tr}\, A^2)x*\xi + 2(\mathrm{tr}\, A^3)\xi*\xi \,.
\end{aligned}
$$

However, at p we have

$$
\begin{aligned}
&\sum_k \nabla_{e_k}(Ae_k) = \mathrm{tr}(\nabla A) = \nabla(\mathrm{tr}\, A) = 0 \,, \\
&\sum_k [(\nabla_{e_k}A)(Ae_k) + \nabla_{e_k}(A^2e_k)] = 2\,\mathrm{tr}(\nabla A^2) \,, \qquad \text{and} \\
&\sum_{i,k} [\nabla_{e_k}\nabla_{e_k}(Ae_i)*e_i + Ae_i*\nabla_{e_k}\nabla_{e_k}e_i] = -\sum_i (\Delta A)e_i*e_i \,,
\end{aligned}
$$

where $\Delta A = \sum_{i=1}^n [\nabla_{\nabla_{e_i}e_i}A - \nabla_{e_i}(\nabla_{e_i}A)]$ is the trace Laplacian of the shape operator. Therefore, we obtain

$$
\begin{aligned}
(2.12)\qquad -\Delta \sum_i (Ae_i)*e_i &= -\sum_i [(\Delta A)e_i*e_i + 2Ae_i*e_i + 2Ae_i*A^2e_i] \\
&\quad + 2(\mathrm{tr}\, A)x*x - 4(\mathrm{tr}\, A^2)x*\xi + 2(\mathrm{tr}\, A^3)\xi*\xi + 4\xi*\mathrm{tr}(\nabla A^2) \,.
\end{aligned}
$$

In a similar fashion one obtains

$$
\begin{aligned}
(2.13)\qquad -\Delta \sum_i (Ae_i)*(Ae_i) &= -2\sum_i [((\Delta A)e_i)*(Ae_i) + Ae_i*Ae_i + A^2e_i*A^2e_i] \\
&\quad + 2(\mathrm{tr}\, A^2)x*x - 4(\mathrm{tr}\, A^3)x*\xi + 2(\mathrm{tr}\, A^4)\xi*\xi \\
&\quad + 4\xi*\mathrm{tr}(\nabla A^3) - 4x*\mathrm{tr}(\nabla A^2) \\
&\quad + 2\sum_{i,k} (\nabla_{e_k}A)e_i*(\nabla_{e_k}A)e_i \,.
\end{aligned}
$$

One of the results of K. Nomizu and B. Smyth in [18] is computation of ΔA for spherical hypersurface with $\mathrm{tr}\, A = \mathrm{const.}$ Namely,

$$(2.14)\qquad \Delta A = (\mathrm{tr}\, A^2 - n)A + (\mathrm{tr}\, A)I - (\mathrm{tr}\, A)A^2 \,.$$

Also, using the Codazzi equation we have

$$(2.15)\qquad \mathrm{tr}(\nabla A^2) = \frac{1}{2}\nabla(\mathrm{tr}\, A^2) + A(\nabla(\mathrm{tr}\, A)) \,,$$

$$(2.16)\qquad \mathrm{tr}(\nabla A^3) = \frac{1}{3}\nabla(\mathrm{tr}\, A^3) + \frac{1}{2}A(\nabla(\mathrm{tr}\, A^2)) + A^2(\nabla(\mathrm{tr}\, A)) \,,$$

where in our case $\nabla(\operatorname{tr} A)=0$.

Now taking Laplacian of (2.8) and using (1.4), (2.1), (2.9-2.16) we get the following expression for $\Delta^3\tilde{x}$ for a spherical hypersurface with constant mean curvature:

$$
\begin{aligned}
(2.17)\qquad \Delta^3\tilde{x}=&[(4+(\operatorname{tr} A)^2)\|A\|^2+(\operatorname{tr} A)^2(5n+8)+4n(n+1)^2]\,x*x\\
&+\{2\Delta\|A\|^2+\|A\|^2[4\|A\|^2+3(\operatorname{tr} A)^2+4n+4]\\
&\qquad+(\operatorname{tr} A)^2(3n+4)+4(\operatorname{tr} A)(\operatorname{tr} A^3)+4\operatorname{tr} A^4\}\xi*\xi\\
&-\{8\operatorname{tr} A^3+(\operatorname{tr} A)[\Delta\|A\|^2+\|A\|^2(\|A\|^2+4n+16)\\
&\qquad+4(\operatorname{tr} A)^2+7n^2+16n+8]\}\,x*\xi\\
&-2x*[(\operatorname{tr} A)A(\nabla\|A\|^2)+2\nabla\|A\|^2]\\
&+\xi*\left[\frac{8}{3}\nabla(\operatorname{tr} A^3)+12A(\nabla\|A\|^2)+6(\operatorname{tr} A)\nabla\|A\|^2\right]\\
&-4[(\operatorname{tr} A)^2+(n+1)^2]\sum_i e_i*e_i-4(\operatorname{tr} A)(\|A\|^2+n+4)\sum_i e_i*Ae_i\\
&-8(1+\|A\|^2)\sum_i Ae_i*Ae_i-4\sum_i A^2e_i*A^2e_i\\
&+4\sum_{i,k}(\nabla_{e_k}A)e_i*(\nabla_{e_k}A)e_i\ .
\end{aligned}
$$

We remark that each sum in the formulas (2.12), (2.13) and (2.17) is independent of the particular frame $\{e_i\}$ chosen.

§ 3. Spherical hypersurfaces with 2-type quadric representation.

Suppose that $\tilde{x}=xx^t$ is of 2-type. Then we have $\tilde{x}=\tilde{x}_0+\tilde{x}_p+\tilde{x}_q$, where $\tilde{x}_0=\text{const}$, $\Delta\tilde{x}_p=\lambda_p\tilde{x}_p$, $\Delta\tilde{x}_q=\lambda_q\tilde{x}_q$ and hence

$$
(3.1)\qquad \Delta^2\tilde{x}-(\lambda_p+\lambda_q)\Delta\tilde{x}+\lambda_p\lambda_q(\tilde{x}-\tilde{x}_0)=0\ .
$$

In order to eliminate constant vector $\tilde{x}_0$ from this equation we find the directional derivative $\tilde{\nabla}_X$ of (3.1) with respect to an arbitrary tangent vector field X and compute different components of such expression. Let $Q(\tilde{x})=\Delta^2\tilde{x}-(\lambda_p+\lambda_q)\Delta\tilde{x}+\lambda_p\lambda_q\tilde{x}$. Then from (3.1), finding respectively $\xi*\xi$ and $x*x$ component of $\tilde{\nabla}_X[Q(\tilde{x})]$ we get

$$
\begin{aligned}
0&=\tilde{g}(\tilde{\nabla}_X[Q(\tilde{x})],\ \xi*\xi)\\
&=X\tilde{g}(Q(\tilde{x}),\ \xi*\xi)+2\tilde{g}(Q(\tilde{x}),\ AX*\xi)\\
&=X(2n^2\alpha'^2+4\|A\|^2)+8n\langle\nabla\alpha',\ AX\rangle\\
&=\langle X,\ 4n^2\alpha'\nabla\alpha'+4\nabla\|A\|^2+8nA(\nabla\alpha')\rangle\ ,
\end{aligned}
$$

$$\begin{aligned}0&=\tilde{g}(\tilde{\nabla}_X[Q(\tilde{x})],\ x*x)\\&=X\tilde{g}(Q(\tilde{x}),\ x*x)-2\tilde{g}(Q(\tilde{x}),\ X*x)\\&=2nX(n\alpha^2+n+2)+2n\tilde{g}(W*x,\ X*x)\\&=\langle X,\ 6n^2\alpha'\nabla\alpha'+4nA(\nabla\alpha')\rangle\ .\end{aligned}$$

Therefore,

$$A(\nabla\alpha')=-\frac{3}{2}n\alpha'\nabla\alpha'\ , \tag{3.2}$$

$$n^2\alpha'\nabla\alpha'+\nabla\|A\|^2+2nA(\nabla\alpha')=0\ . \tag{3.3}$$

These two equations imply

$$\nabla\|A\|^2=2n^2\alpha'\nabla\alpha'\ ,\qquad W=\mathrm{tr}(\bar{\nabla}A_H)=-2n\alpha'\nabla\alpha'\ . \tag{3.4}$$

Let $U=\{p\in M\ |\ \nabla(\alpha')^2\neq 0 \text{ at } p\}$. Then U is an open (possibly empty) subset of M, and on U we obviously have also $\alpha'\neq 0$ and $\nabla\alpha'\neq 0$. If U is nonempty, then by (3.2) we see that $\nabla\alpha'$ is an eigenvector of the shape operator A on U with eigenvalue $-\frac{3}{2}n\alpha'$. On U we choose unit tangent vector e_1 to be in the direction of $\nabla\alpha'$, i.e. $e_1=\nabla\alpha'/\|\nabla\alpha'\|$. We find e_1*e_1 component of $\tilde{\nabla}_X[Q(\tilde{x})]$ on U setting first $X=\nabla\alpha'$. Combining (2.1), (2.8) and (3.1) we get the following:

$$\begin{aligned}0&=\tilde{g}(\tilde{\nabla}_{\nabla\alpha'}[Q(\tilde{x})],\ e_1*e_1)\\&=(\nabla\alpha')\tilde{g}(Q(\tilde{x}),\ e_1*e_1)-2\tilde{g}(Q(\tilde{x}),\ h(\nabla\alpha',\ e_1)*e_1)\\&=-3n^2(\nabla\alpha')(\alpha'^2)+\tilde{g}(Q(\tilde{x}),\ 3n\alpha'\xi*\nabla\alpha'+2x*\nabla\alpha')\\&=10n^2\alpha'\|\nabla\alpha'\|^2\ .\end{aligned}$$

From this we conclude $\alpha'=0$ or $\nabla\alpha'=0$ at any point of U. However, this is a contradiction, and hence U must be empty. This means that $\nabla(\alpha')^2=0$ everywhere on M, i.e. $\alpha'=\text{const.}$ Therefore, a hypersurface of S^{n+1} with 2-type quadric representation must have constant mean curvature α' in sphere.

We are ready now to prove the following classification result.

THEOREM 3.1. *Let $x:M^n\to S^{n+1}$ be an isometric immersion of a compact n-dimensional Riemannian manifold M into S^{n+1} ($n\geqq 2$). Then $\tilde{x}=xx^t$ is of 2-type if and only if either*

(1) *M is a small hypersphere of S^{n+1} of radius $r<1$, or*

(2) *$M=S^p(r_1)\times S^{n-p}(r_2)$, with the following possibilities for the radii r_1 and r_2:*

i) $$r_1^2=\frac{p+1}{n+2}\,,\quad r_2^2=\frac{n-p+1}{n+2}\,;$$

ii) $$r_1^2=\frac{p+2}{n+2}\,,\quad r_2^2=\frac{n-p}{n+2}\,;$$

iii) $$r_1^2=\frac{p}{n+2}\,,\quad r_2^2=\frac{n-p+2}{n+2}\,.$$

The immersions in (1) *and* (2) *are given in a natural way.*

PROOF. If M is one of the submanifolds described in (1) and (2), then M is of 2-type via the second standard immersion of the sphere as shown in [2]. Conversely, let us assume that for a spherical hypersurface $x:M^n\to S^{n+1}$ the quadric representation $\tilde{x}$ is of two type. Then (3.1) holds, and from the above we see that the mean curvature α' of x is constant. In that case $\nabla\alpha'=W=\nabla\|A\|^2=0$.

Let e_k, $k=1, 2, \cdots, n$ be the local orthonormal vector fields of principal directions of A and let μ_k be the corresponding principal curvatures. Then, by a similar computation as before, from $\tilde{g}(\tilde{\nabla}_{e_k}[Q(\tilde{x})], x*e_k)=0$ we obtain for every k

$$\begin{aligned}(3.5)\qquad 0=&[2(n^2\alpha^2+n^2+4n+2)-2(n+1)(\lambda_p+\lambda_q)+\lambda_p\lambda_q]\\&+n\alpha'(\|A\|^2+3n+8-\lambda_p-\lambda_q)\mu_k+4\mu_k^2\,.\end{aligned}$$

This is a quadratic equation in μ_k with constant coefficients which do not depend on k. We conclude, therefore, that each principal curvature is constant and that there are at most two distinct principal curvatures. If M has only one principal curvature, i.e. if it is umbilical, then M is a small hypersphere of S^{n+1}. If M has two distinct (constant) principal curvatures then M is the standard product of two spheres, $M=S^p(r_1)\times S^{n-p}(r_2)$ with $r_1^2+r_2^2=1$ (see [3] or [21]). Then, according to [2] (Lemma 3), such product will be of 2-type via $\tilde{x}$ if and only if the radii satisfy precisely those three possibilities listed in (2).

Theorem 3.1 is a generalization of a result of M. Barros and B. Y. Chen, who obtained similar result assuming, in addition, M to be mass-symmetric (cf. [2]).

§4. Minimal spherical hypersurfaces which are of 3-type and mass-symmetric via $\tilde{x}$.

Here we concentrate on minimal spherical hypersurfaces, even though the investigation can be carried out for spherical hypersurfaces with constant mean curvature in quite an analogous way (with a bit more involved calculations). We have the following characterization of minimal spherical hypersurfaces with 3-type quadric representation.

THEOREM 4.1. *Let $x: M^n \to S^{n+1}$ be an isometric immersion of a compact manifold M^n as a minimal hypersurface of S^{n+1}. If $\tilde{x}$ is mass-symmetric and of 3-type then*
(1) $\operatorname{tr} A = \operatorname{tr} A^3 = 0$,
(2) $\operatorname{tr} A^2$ *and* $\operatorname{tr} A^4$ *are constant,*
(3) $\operatorname{tr}(\nabla_X A)^2 = \langle A^2X, A^2X\rangle + p\langle AX, AX\rangle + q\langle X, X\rangle$,
for every tangent vector $X \in TM$, where p and q are constants (depending on the order of M, $\operatorname{tr} A^2$ and $\operatorname{tr} A^4$). Conversely, if (1), (2) and (3) hold then M is mass-symmetric and of 1-, 2-, or 3-type via $\tilde{x}$.

PROOF. Suppose that M^n is mass-symmetric and of 3-type via $\tilde{x}$ so that $\tilde{x}_0 = I/(n+2)$ and

$$\Delta^3\tilde{x} + a\Delta^2\tilde{x} + b\Delta\tilde{x} + c\left(\tilde{x} - \frac{I}{n+2}\right) = 0 , \tag{4.1}$$

where a, b and c are constants. Recall that from Lemma 1.1 we also have

$$2I = x*x + \xi*\xi + \sum_i e_i*e_i . \tag{4.2}$$

We use formulas (2.17) (with $\operatorname{tr} A = 0$), (2.8), (2.1) and (4.2) to find different components of (4.1). Namely, from $x*\xi$ component of (4.1) we get $\operatorname{tr} A^3 = 0$, and $x*x$ and $\xi*\xi$ components give respectively

$$8[n(n+1)^2 + \operatorname{tr} A^2] + 4an(n+1) + 2bn + c\frac{n+1}{n+2} = 0 , \tag{4.3}$$

$$8[(\operatorname{tr} A^2)^2 + (n+1)(\operatorname{tr} A^2) + \operatorname{tr} A^4] + 4a(\operatorname{tr} A^2) - c\frac{1}{n+2} = 0 . \tag{4.4}$$

Therefore, $\operatorname{tr} A^2$ and $\operatorname{tr} A^4$ are constant, and (2.17) simplifies to

$$\begin{aligned}\Delta^3\tilde{x} = {} & 4[n(n+1)^2 + \operatorname{tr} A^2]x*x + 4[(\operatorname{tr} A^2)^2 + (n+1)(\operatorname{tr} A^2) + \operatorname{tr} A^4]\xi*\xi \\ & - 4(n+1)^2\sum_i e_i*e_i - 8(1+\operatorname{tr} A^2)\sum_i Ae_i*Ae_i \\ & - 4\sum_i A^2e_i*A^2e_i + 4\sum_{i,k} (\nabla_{e_k}A)e_i*(\nabla_{e_k}A)e_i .\end{aligned} \tag{4.5}$$

Next, we find $X*Y$ component of (4.1) for arbitrary pair X, Y of tangent vector fields on M. Observe first that

$$\sum_{i,k} \langle (\nabla_{e_k} A)e_i, X\rangle\langle(\nabla_{e_k}A)e_i, Y\rangle = \sum_{i,k} \langle e_i, (\nabla_{e_k}A)X\rangle\langle e_i, (\nabla_{e_k}A)Y\rangle\rangle$$
$$= \sum_k \langle(\nabla_X A)e_k, (\nabla_Y A)e_k\rangle\rangle$$
$$= \mathrm{tr}(\nabla_X A)\circ(\nabla_Y A)\ ,$$

by the Codazzi equation and symmetry of the operator $\nabla_{e_k} A$.

Now applying $\tilde{g}(-, X*Y)$ to (4.1) and taking into account (2.1), (2.8) and (4.5) we get

$$-8(n+1)^2\langle X, Y\rangle - 16(1+\mathrm{tr}\, A^2)\langle AX, AY\rangle - 8\langle A^2X, A^2Y\rangle$$
$$+8\,\mathrm{tr}(\nabla_X A)\circ(\nabla_Y A) - 4a\langle AX, AY\rangle - 4a(n+1)\langle X, Y\rangle$$
$$-2b\langle X, Y\rangle - c\frac{1}{n+2}\langle X, Y\rangle = 0\ ,$$

from where

$$\mathrm{tr}(\nabla_X A)\circ(\nabla_Y A) = \langle A^2X, A^2Y\rangle + p\langle AX, AY\rangle + q\langle X, Y\rangle\ , \tag{4.6}$$

where p and q are constants given by

$$p = \frac{a}{2} + 2(1+\mathrm{tr}\, A^2)\ , \tag{4.7}$$

$$q = (n+1)^2 + \frac{a}{2}(n+1) + \frac{b}{4} + \frac{c}{8(n+2)}\ . \tag{4.8}$$

It is easy to see that (4.6) is equivalent, by linearization, to

$$\mathrm{tr}(\nabla_X A)^2 = \langle A^2X, A^2X\rangle + p\langle AX, AX\rangle + q\langle X, X\rangle\ , \tag{4.9}$$

for any $X \in TM$. Therefore, we proved necessity of the conditions (1), (2), (3).

Conversely, given (1), (2) and (3) we have to show that we can find constants a, b and c so that (4.1) holds. That boils down to solving the system of the following four equations (4.3), (4.4), (4.7) and (4.8) for a, b, c. This system of four linear equations in three unknowns can be uniquely solved if the eliminant is zero, i.e. if

$$\mathrm{tr}\, A^4 + p\,\mathrm{tr}\, A^2 + qn + (n - \mathrm{tr}\, A^2)\mathrm{tr}\, A^2 = 0\ .$$

But this formula is always satisfied under our conditions (1)-(3), by virtue of

$$0 = \frac{1}{2}\Delta(\mathrm{tr}\, A^2) = \|\nabla A\|^2 - \mathrm{tr}(\Delta A)A$$

(cf. [18, p. 369], also formula (4.24)). Therefore $P(\Delta)(\tilde{x}-\tilde{x}_0)=0$, where $P(t)=t^3+at^2+bt+c$. Note that M need not be exactly of 3-type, i.e. can be of 1- or 2-type, for example if there is a factor P' of P of degree 1 or 2 so that $P'(\Delta)(\tilde{x}-\tilde{x}_0)=0$.

If M is only assumed to have constant mean curvature in S^{n+1} then using (2.1), (2.8), (2.17) and (4.1) we can prove the following theorem in the same way we proved the preceding one.

THEOREM 4.1.a. *Let $x: M^n \to S^{n+1}$ be a compact constant mean curvature hypersurface of S^{n+1}. If $\tilde{x}$ is mass-symmetric and of 3-type then*
(1) *$\operatorname{tr} A^k$ is constant for $k=1, 2, 3, 4$,*
(2) *$\operatorname{tr}(\nabla_X A)^2=\langle A^2X, A^2X\rangle+p\langle AX, AX\rangle+q\langle AX, X\rangle+r\langle X, X\rangle$,*
for every tangent vector field $X\in TM$, where p, q and r are constants.

Every minimal isoparametric hypersurface with three different principal curvatures has quadric representation of 3-type as seen from the following lemma.

LEMMA 4.1. *If $M^n\subset S^{n+1}$ is a compact minimal isoparametric spherical hypersurface with exactly three distinct principal curvatures, then M^n is mass-symmetric and of 3-type via $\tilde{x}$.*

PROOF. From the Gauss equation we obtain the following for principal directions e_i, e_k and corresponding curvatures λ_i, λ_k $(i\neq k)$:

$$
\begin{aligned}
(4.10)\qquad R(e_i, e_k, e_k, e_i) &= 1+\lambda_i\lambda_k \\
&= e_i(\omega_k^i(e_k))-e_k(\omega_k^i(e_i)) \\
&\quad +\sum_j \omega_k^j(e_k)\omega_j^i(e_i)-\sum_j \omega_k^j(e_i)\omega_j^i(e_k) \\
&\quad -\sum_j \omega_k^j(e_i)\omega_k^i(e_j)+\sum_j \omega_i^j(e_k)\omega_k^i(e_j)\ .
\end{aligned}
$$

For an isoparametric hypersurface, the Codazzi equation $(\nabla_{e_i}A)e_k=(\nabla_{e_k}A)e_i$ is equivalent to the following:

$$
(4.11)\qquad (\lambda_k-\lambda_j)\omega_k^j(e_i)=(\lambda_i-\lambda_j)\omega_i^j(e_k)\ , \qquad \text{for every}\quad i, j, k\ ,
$$

and hence

$$
(4.12)\qquad \omega_i^j(e_k)=0\ , \qquad \text{for}\quad \lambda_k=\lambda_j\neq\lambda_i\ .
$$

Therefore, if $\lambda_i\neq\lambda_k$ formula (4.10) reduces to

$$
(4.13)\qquad 1+\lambda_i\lambda_k=-\sum_j \omega_k^j(e_i)\omega_j^i(e_k)-\sum_j \omega_k^j(e_i)\omega_k^i(e_j)+\sum_j \omega_i^j(e_k)\omega_k^i(e_j)\ .
$$

All four minimal isoparametric spherical hypersurfaces with three distinct principal curvatures have curvatures equal to $-\sqrt{3}$, 0 and $\sqrt{3}$, and the common multiplicity m satisfies $m \in \{1, 2, 4, 8\}$, so that $\operatorname{tr} A = \operatorname{tr} A^3 = 0$. In order to prove that these hypersurfaces are of 3-type and mass-symmetric it is enough to check condition (3) of Theorem 4.1, which can be also written as

$$\operatorname{tr}(\nabla_{e_i} A)^2 = \lambda_i^4 + p\lambda_i^2 + q \,, \tag{4.14}$$

where e_i is a principal direction, λ_i corresponding principal curvature, and p and q constants. We transform $\operatorname{tr}(\nabla_{e_i} A)^2$ as

$$\operatorname{tr}(\nabla_{e_i} A)^2 = \sum_{k,j} (\lambda_k - \lambda_j)^2 [\omega_k^j(e_i)]^2 \,. \tag{4.15}$$

Let $e_1, \cdots, e_m$ be the set of principal directions that correspond to $-\sqrt{3}$ eigenvalue, $e_{m+1}, \cdots, e_{2m}$ the set of principal directions that correspond to 0 eigenvalue, and $e_{2m+1}, \cdots, e_{3m}$ those corresponding to $\sqrt{3}$ eigenvalue. We use the boldface type to denote the following set of indices

$$\mathbf{1} = \{1, \cdots, m\} \,, \quad \mathbf{2} = \{m+1, \cdots, 2m\} \quad \text{and} \quad \mathbf{3} = \{2m+1, \cdots, 3m\} \,.$$

Let $i \in \mathbf{1}$, $k \in \mathbf{2}$ be any two indices so that e_i, e_k are two principal directions corresponding to the curvatures $-\sqrt{3}$, 0 respectively. Then from (4.13) using (4.12) we obtain

$$1 = 1 + \lambda_i \lambda_k = -\sum_{j \in \mathbf{3}} \omega_k^j(e_i)\omega_j^i(e_k) - \sum_{j \in \mathbf{3}} \omega_k^j(e_i)\omega_k^i(e_j) + \sum_{j \in \mathbf{3}} \omega_i^j(e_k)\omega_k^i(e_j) \,. \tag{4.16}$$

From the Codazzi equation (4.11) we get for $j \in \mathbf{3}$

$$\sqrt{3}\,\omega_j^k(e_i) = -\sqrt{3}\,\omega_i^k(e_j) \,, \quad 2\sqrt{3}\,\omega_j^i(e_k) = \sqrt{3}\,\omega_k^i(e_j) \,,$$

so that

$$\omega_k^j(e_i) = \omega_i^k(e_j) \,, \quad \omega_i^j(e_k) = \frac{1}{2}\omega_i^k(e_j) \,. \tag{4.17}$$

Now in (4.16) we express everything in terms of $\omega_i^k(e_j)$ using (4.17) and simplify to get

$$\sum_{j \in \mathbf{3}} [\omega_i^k(e_j)]^2 = 1 \,, \qquad \text{for every} \quad i \in \mathbf{1},\ k \in \mathbf{2} \,. \tag{4.18}$$

By a similar computation, using expressions for $1 + \lambda_i \lambda_k$, where $i \in \mathbf{2}$, $k \in \mathbf{3}$ and, respectively, $i \in \mathbf{1}$, $k \in \mathbf{3}$, we obtain

$$\sum_{j\in 1}[\omega_i^k(e_j)]^2=1\,, \qquad \text{for every} \quad i\in 2,\ k\in 3\,, \tag{4.19}$$

$$\sum_{j\in 2}[\omega_i^k(e_j)]^2=\frac{1}{4}\,, \qquad \text{for every} \quad i\in 1,\ k\in 3\,. \tag{4.20}$$

Next, we compute $\mathrm{tr}(\nabla_{e_i}A)^2$ from (4.15) to get for $i\in 1$:

$$\begin{aligned}\mathrm{tr}(\nabla_{e_i}A)^2&=12\sum_{\substack{j\in 2\\ k\in 3}}[\omega_i^k(e_j)]^2+3\sum_{\substack{j\in 3\\ k\in 2}}[\omega_i^k(e_j)]^2\\ &=3m+12\frac{m}{4}=6m\,.\end{aligned}$$

Similar computation can be carried out for $i\in 2$ and $i\in 3$, yielding the same result, so

$$\mathrm{tr}(\nabla_{e_i}A)^2=6m\,, \qquad \text{for every} \quad i=1,\cdots,3m\,. \tag{4.21}$$

Therefore, we see that (4.14) is satisfied with $p=-3$ and $q=6m$. We conclude that all minimal isoparametric spherical hypersurfaces with three curvatures are of 3-type via $\tilde{x}$.

As a matter of fact we can show that

$$\Delta^3\tilde{x}+a\Delta^2\tilde{x}+b\Delta\tilde{x}+c\Big(\tilde{x}-\frac{I}{3m+2}\Big)=0$$

is satisfied for $a=-(10+24m)$, $b=4[(3m+1)(15m+6)-2]$ and $c=-48m\times(3m+1)(3m+2)$, so that the three eigenvalues of the Laplacian arising from the decomposition $\tilde{x}=\tilde{x}_0+\tilde{x}_p+\tilde{x}_q+\tilde{x}_r$ are $\lambda_p=6m$, $\lambda_q=2(3m+1)$ and $\lambda_r=4(3m+2)$. The spectrum of the Cartan hypersurface was computed in [15], from which we determine its order via $\tilde{x}$ to be [2, 3, 8].

EXAMPLE 4.1. **Minimal isoparametric hypersurface in S^5 with 4 principal curvatures.** As discussed in the Preliminaries, there is only one (up to isometries) minimal isoparametric hypersurface M^4 in S^5 with four curvatures which is the image of the following map

$$\begin{aligned}&S^1\times S_{3,2}\longrightarrow S^5\subset E^6\,,\\ &(\theta,(x,y))\longrightarrow z=e^{i\theta}(\cos t\,x+i\sin t\,y)\,,\end{aligned} \tag{4.22}$$

for $t=\pi/8$. In general, (4.22) defines the isoparametric family studied by Cartan [5] and Nomizu [16], [17]. To parametrize the Stiefel manifold $S_{3,2}$ choose x to be an arbitrary vector of the sphere S^2, i.e. $x=(\cos\alpha\cos\beta,\ \cos\alpha\sin\beta,\ \sin\alpha)$, and choose vectors u and v of S^2 that span the plane perpendicular to x, e.g. $u=(-\sin\beta,\ \cos\beta,\ 0)$ and $v=u\times x=(\sin\alpha\cos\beta,\ \sin\alpha\sin\beta,\ -\cos\alpha)$. For any vector $y\perp x$, $y=\cos\phi\ u+\sin\phi\ v$

so $y=(-\sin\beta\cos\phi+\sin\alpha\cos\beta\sin\phi,\ \cos\beta\cos\phi+\sin\alpha\sin\beta\sin\phi,\ -\cos\alpha\sin\phi)$. Denote $r=\cos t$ and $s=\sin t$. Then from (4.22) and the consideration above we have the following parametrization of M^4:

$$
\begin{aligned}
&z_1=r\cos\theta\cos\alpha\cos\beta-s\sin\theta(-\sin\beta\cos\phi+\sin\alpha\cos\beta\sin\phi)\ ,\\
&z_2=r\cos\theta\cos\alpha\sin\beta-s\sin\theta(\cos\beta\cos\phi+\sin\alpha\sin\beta\sin\phi)\ ,\\
&z_3=r\cos\theta\sin\alpha+s\sin\theta\cos\alpha\sin\phi\ ,\\
&z_4=r\sin\theta\cos\alpha\cos\beta+s\cos\theta(-\sin\beta\cos\phi+\sin\alpha\cos\beta\sin\phi)\ ,\\
&z_5=r\sin\theta\cos\alpha\sin\beta+s\cos\theta(\cos\beta\cos\phi+\sin\alpha\sin\beta\sin\phi)\ ,\\
&z_6=r\sin\theta\sin\alpha-s\cos\theta\cos\alpha\sin\phi\ .
\end{aligned}
\tag{4.23}
$$

We differentiate $z=(z_1,\cdots,z_6)$ to get basis vector fields $\partial_1=\partial/\partial\theta$, $\partial_2=\partial/\partial\alpha$, $\partial_3=\partial/\partial\beta$, $\partial_4=\partial/\partial\phi$. We compute components of the metric tensor $g_{ij}=\langle\partial_i,\partial_j\rangle$ to get the following matrix $G=(g_{ij})$.

$$
G=\begin{pmatrix}
1 & 2rs\sin\phi & -2rs\cos\alpha\cos\phi & 0\\
2rs\sin\phi & r^2+s^2\sin^2\phi & -s^2\cos\alpha\cos\phi\sin\phi & 0\\
-2rs\cos\alpha\cos\phi & -s^2\cos\alpha\cos\phi\sin\phi & s^2+\cos^2\alpha\,(r^2-s^2\sin^2\phi) & -s^2\sin\alpha\\
0 & 0 & -s^2\sin\alpha & s^2
\end{pmatrix}.
$$

The determinant of this matrix is computed to be $\det G=r^2s^2(1-4r^2s^2)\cos^2\alpha$. One can also compute the inverse matrix G^{-1} and the Christoffel's symbols (see [10]). We want to find the shape operator A of the hypersurface and the basis of principal directions. First it turns out that the unit normal ξ is obtained by differentiating z with respect to t, i.e. take $\xi=-\partial/\partial t$. For every $i, j=1, 2, 3, 4$ we can compute $\langle A(\partial_i),\partial_j\rangle=-\langle\bar\nabla_{\partial_i}\xi,\partial_j\rangle$ and find the matrix of A in the basis $\{\partial_i\}$. We get

$$
\begin{aligned}
A(\partial_1)&=\frac{r^2-s^2}{1-4r^2s^2}\left\{-2rs\partial_1+\sin\phi\,\partial_2-\frac{\cos\phi}{\cos\alpha}\partial_3-\tan\alpha\cos\phi\,\partial_4\right\}\ ,\\
A(\partial_2)&=\frac{r^2-s^2}{1-4r^2s^2}\left\{\sin\phi\,\partial_1+\frac{s}{r}(\cos^2\phi-2r^2)\partial_2\right.\\
&\qquad\left.+\frac{s}{r}\,\frac{\sin\phi\cos\phi}{\cos\alpha}\partial_3+\frac{s}{r}\tan\alpha\sin\phi\cos\phi\,\partial_4\right\}\ ,\\
A(\partial_3)&=\frac{r^2-s^2}{1-4r^2s^2}\left\{-\cos\alpha\cos\phi\,\partial_1+\frac{s}{r}\cos\alpha\sin\phi\cos\phi\,\partial_2\right.\\
&\qquad\left.+\frac{s}{r}(\sin^2\phi-2r^2)\partial_3-\frac{1}{rs}\sin\alpha(r^2-s^2\sin^2\phi)\partial_4\right\}\ ,\\
A(\partial_4)&=\frac{r}{s}\partial_4\ .
\end{aligned}
$$

Minimal hypersurface in the family (4.22) is obtained when $t=\pi/8$. In that case $r=\cos\pi/8=\sqrt{2+\sqrt{2}}\Big/2$, $s=\sin\pi/8=\sqrt{2-\sqrt{2}}\Big/2$. Principal curvatures of minimal M^4 are given as follows (Th. 1.1):

$$k_1=\sqrt{2}+1\ ,\quad k_2=-\sqrt{2}-1\ ,\quad k_3=\sqrt{2}-1\ ,\quad k_4=1-\sqrt{2}\ .$$

Next we find the orthonormal basis of principal directions by diagonalizing matrix of A in the basis $\{\partial_i\}$. We get the following principal directions corresponding respectively to the curvatures k_1, k_2, k_3, k_4:

$$e_1=\sqrt{4+2\sqrt{2}}\,\frac{\partial}{\partial\phi}\ ,$$

$$e_2=\frac{\sqrt{4+2\sqrt{2}}}{2}\left\{\frac{\partial}{\partial\theta}-\sin\phi\,\frac{\partial}{\partial\alpha}+\frac{\cos\phi}{\cos\alpha}\,\frac{\partial}{\partial\beta}+\tan\alpha\,\cos\phi\,\frac{\partial}{\partial\phi}\right\}\ ,$$

$$e_3=\frac{\sqrt{4-2\sqrt{2}}}{2}\left\{-\frac{\partial}{\partial\theta}-\sin\phi\,\frac{\partial}{\partial\alpha}+\frac{\cos\phi}{\cos\alpha}\,\frac{\partial}{\partial\beta}+\tan\alpha\,\cos\phi\,\frac{\partial}{\partial\phi}\right\}\ ,$$

$$e_4=\sqrt{4-2\sqrt{2}}\left\{\cos\phi\,\frac{\partial}{\partial\alpha}+\frac{\sin\phi}{\cos\alpha}\,\frac{\partial}{\partial\beta}+\tan\alpha\,\sin\phi\,\frac{\partial}{\partial\phi}\right\}\ .$$

To check if M^4 is of 3-type via $\tilde{x}$ or not we find connection coefficients with respect to the basis $\{e_i\}$. For example, we compute

$$\omega_1^2(e_3)=\langle\nabla_{e_3}e_1,\,e_2\rangle=0\ ,\quad \omega_1^2(e_4)=\sqrt{2-\sqrt{2}}\ ,\quad \cdots\ \text{etc}\ .$$

But combining the equations of Gauss, Codazzi and condition (3) of Theorem 4.1, it follows that in order that M^4 be mass-symmetric and of 3-type via $\tilde{x}$ we must have

$$[\omega_1^2(e_3)]^2=[\omega_1^2(e_4)]^2=\frac{2-\sqrt{2}}{2}\quad\text{and}\quad[\omega_3^4(e_1)]^2=[\omega_3^4(e_2)]^2=\frac{2+\sqrt{2}}{2}\ .$$

Therefore, M^4 is not mass-symmetric and of 3-type via $\tilde{x}$.

We now prove the following characterization of the Cartan hypersurface.

THEOREM 4.2. *Let $x:M^n\to S^{n+1}$ be a compact minimal hypersurface of S^{n+1} of dimension $n\leqq 5$. Then $\tilde{x}$ is mass-symmetric and of 3-type if and only if $n=3$ and $M^3=SO(3)/\boldsymbol{Z}_2\times\boldsymbol{Z}_2$ is the Cartan hypersurface.*

PROOF. From Lemma 4.1, we know that the Cartan hypersurface is mass-symmetric and of 3-type via $\tilde{x}$. Conversely, suppose that $\tilde{x}$ is mass-symmetric and of 3-type. We will show that M^n is necessarily isoparametric. From the computation carried out before that is already clear for $n \leqq 4$. If we compute $\Delta(\operatorname{tr} A^m)$ we obtain

$$\Delta(\operatorname{tr} A^m) = m(\operatorname{tr} A^2 - n)(\operatorname{tr} A^m) - \sum_i \sum_{j \neq k} \operatorname{tr}(A \circ \cdots \circ A \circ \overset{j}{\nabla_{e_i}} A \circ A \circ \cdots A \circ \overset{k}{\nabla_{e_i}} A \circ \cdots \circ A) . \tag{4.24}$$

In particular, for $m=3$ we have

$$\Delta(\operatorname{tr} A^3) = 3(\operatorname{tr} A^2 - n)(\operatorname{tr} A^3) - 6 \sum_i \operatorname{tr}[(\nabla_{e_i} A)^2 \circ A] . \tag{4.25}$$

Since $\operatorname{tr} A^3 = 0$ by Theorem 4.1, we will have ($\{e_i\}$ is chosen to be the basis of principal directions)

$$\begin{aligned}
0 &= \sum_i \operatorname{tr}[(\nabla_{e_i} A)^2 \circ A] \\
&= \sum_{i,k} \langle (\nabla_{e_i} A)^2 A e_k, e_k \rangle \\
&= \sum_{i,k} \langle (\nabla_{e_i} A)^2 (\lambda_k e_k), e_k \rangle \\
&= \sum_{i,k} \langle (\nabla_{e_i} A)(\lambda_k e_k), (\nabla_{e_i} A) e_k \rangle , \quad \text{since } \nabla_{e_i} A \text{ is symmetric} \\
&= \sum_{i,k} \lambda_k \langle (\nabla_{e_i} A) e_k, (\nabla_{e_i} A) e_k \rangle , \quad \text{since } \nabla_{e_i} A \text{ is a tensor} \\
&= \sum_{i,k} \lambda_k \langle (\nabla_{e_k} A) e_i, (\nabla_{e_k} A) e_i \rangle , \quad \text{by the Codazzi equation} \\
&= \sum_k \lambda_k \operatorname{tr}(\nabla_{e_k} A)^2 \\
&= \sum_k \lambda_k (\lambda_k^4 + p \lambda_k^2 + q) , \quad \text{by condition (3) of Th. 4.1} \\
&= \operatorname{tr} A^5 + p \operatorname{tr} A^3 + q \operatorname{tr} A \\
&= \operatorname{tr} A^5 .
\end{aligned}$$

Therefore, conditions (1)-(3) of Theorem 4.1 imply also $\operatorname{tr} A^5 = 0$. We conclude that for $n \leqq 5$ the hypersurface M has to be isoparametric. If M has only one curvature it has to be umbilical in S^{n+1} and therefore (since it is minimal) great hypersphere which is of 1-type via $\tilde{x}$. If M has two distinct principal curvatures and is minimal it must be Clifford minimal hypersurface $M = M_{p,n-p} = S^p(\sqrt{p/n}) \times S^{n-p}(\sqrt{(n-p)/n})$. But the product of spheres that satisfies the conditions of our Theorem 4.1 must be of 2-type as can be seen from the following argument.

Suppose λ_1 and λ_2 are the two principal curvatures of multiplicities

m_1 and m_2 respectively. Then $\operatorname{tr} A = \operatorname{tr} A^3 = 0$ implies $m_1\lambda_1 + m_2\lambda_2 = m_1\lambda_1^3 + m_2\lambda_2^3 = 0$. Also, we have $1+\lambda_1\lambda_2=0$. Using this to eliminate m_1, m_2, and λ_2 we obtain $\lambda_1^2=\lambda_1^6=(n-p)/p$. Thus, $p=n-p=n/2$, $\lambda_1=\pm 1$ and $\lambda_2=\mp 1$. So, n has to be even, $p=n-p$, $p/n=1/2$ and $S^p(\sqrt{p/n})\times S^{n-p}(\sqrt{(n-p)/n})=S^p(\sqrt{1/2})\times S^p(\sqrt{1/2})$. This hypersurface is mass-symmetric and of 2-type by Lemma 3 of [2]. If M has three curvatures, then according to the classification of Cartan M is the Cartan hypersurface which indeed is mass-symmetric and of 3-type via $\tilde{x}$. If M has four principal curvatures, then the result of Takagi [23] classifies such hypersurface as the one considered in Example 4.1 which is not of 3-type via $\tilde{x}$. Finally, M cannot have five principal curvatures by the result of Münzner (Theorem 1.1(a)). This completes the proof of the theorem.

REMARK. The proof above does not a priori exclude the case $n=1$. Actually, if $n=1$ there are no minimal curves in S^2 which are of 3-type in $SM(3)$ via $\tilde{x}$ because such a curve is automatically a great circle of S^2 (totally geodesic), and therefore of 1-type via $\tilde{x}$.

Theorem 4.2 gives a new characterization of the Cartan hypersurface in terms of the spectrum of its Laplacian. For other characterizations see [13] and the references there.

As seen in Lemma 4.1, in dimensions greater than 5 there are other examples of spherical hypersurfaces which are of 3-type and mass-symmetric via $\tilde{x}$.

LEMMA 4.2. *If $M^n\subset S^{n+1}$ is a compact minimal isoparametric hypersurface which is mass-symmetric and of 3-type via $\tilde{x}$, then M^n can possibly have only* 3, 4 *or* 6 *distinct principal curvatures of the same multiplicity.*

PROOF. First, we saw before, from the proof of Theorem 4.2, that if $\nu=1$ or 2 then $\tilde{x}$ is not of 3-type, and if $\nu=3$ then $\tilde{x}$ is of 3-type. If there are six distinct principal curvatures, then by [1] the curvatures k_i have the same multiplicity ($m=1$ or 2) and they can be computed using Theorem 1.1(b) with $\theta=\pi/12$ (giving the minimal hypersurface) to be

$$2+\sqrt{3}\,,\quad 1\,,\quad 2-\sqrt{3}\,,\quad -(2-\sqrt{3})\,,\quad -1\,,\quad -(2+\sqrt{3})\,.$$

We see that these hypersurfaces satisfy conditions (1) and (2) of Theorem 4.1 and to determine if they are of 3-type and mass-symmetric via $\tilde{x}$ one needs to check the condition (3). It is likely (but still not known) that all isoparametric spherical hypersurfaces with six curvatures are homogeneous. That is proved when $m=1$ [11], classifying such hypersurface as $G_2/SO(4)$, but not yet for $m=2$.

If $\nu=4$, then

$$k_1=\cot\theta\ ,\quad k_2=\cot\left(\theta+\frac{\pi}{4}\right),\quad k_3=\cot\left(\theta+\frac{\pi}{2}\right),\quad k_4=\cot\left(\theta+\frac{3\pi}{4}\right),$$

and there are at most two different multiplicities m_1 (of k_1 and k_3) and m_2 (of k_2 and k_4). Then from $\operatorname{tr} A=0$ and $\operatorname{tr} A^3=0$ we get respectively

$$m_1\frac{\cos 2\theta}{\sin 2\theta}-m_2\frac{\sin 2\theta}{\cos 2\theta}=0\ ,\quad \text{i.e.}\quad \tan^2 2\theta=\frac{m_1}{m_2}\ ,$$

$$m_1\frac{\cos 2\theta(4-\sin^2 2\theta)}{\sin^3 2\theta}-m_2\frac{\sin 2\theta(3+\sin^2 2\theta)}{\cos^3 2\theta}=0\ ,$$

from where

$$\frac{m_1}{m_2}=\tan^4 2\theta\frac{3+\sin^2 2\theta}{4-\sin^2 2\theta}\ .$$

Let $r=m_1/m_2$. Then from these two equations we get

$$r=r^2\frac{3+\sin^2 2\theta}{4-\sin^2 2\theta}\ ,$$

which implies

$$\sin^2 2\theta=\frac{4-3r}{r+1}\ ,\qquad \text{hence}\quad r=\tan^2 2\theta=\frac{4-3r}{4r-3}\ .$$

From the last relation we have $r=1$, i.e. $m_1=m_2$ so multiplicities of all four curvatures are equal. We also get $\theta=\pi/8$, and four curvatures to be $k_1=\sqrt{2}+1$, $k_2=\sqrt{2}-1$, $k_3=1-\sqrt{2}$, $k_4=-\sqrt{2}-1$. Therefore, as argued in the Preliminaries, the common multiplicity of curvatures is 1 or 2. If the common multiplicity is 1 then M^4 has to be the hypersurface considered in Example 4.1 which is not of 3-type via $\tilde{x}$. If the common multiplicity is 2, then M^8 is minimal homogeneous hypersurface in S^9 of type $Sp(2)/T^2$, and again one needs to compute connection coefficients and check the condition (3) of Th. 4.1 to see if it is of 3-type.

REMARKS. 1. From the above we know three eigenvalues of any isoparametric spherical hypersurface with three curvatures. Even though the spectrum of the Cartan hypersurface is known, not much information is available about eigenvalues of the Laplacian for other isoparametric hypersurfaces with three curvatures. It is known, however, that any minimal spherical hypersurface with $\operatorname{tr} A^2=\text{const}$ has n, $\operatorname{tr} A^2$ and $n+\operatorname{tr} A^2$ as three eigenvalues of the Laplacian (cf. [15]).

2. In order to check which minimal isoparametric spherical hypersurfaces (or at least homogeneous ones) with four or six principal curvatures are mass-symmetric and of 3-type via $\tilde{x}$, one has to check the condition (3) of Theorem 4.1. That can be done (for homogeneous ones) by the methods of [24], considering the action of the Lie group $K=Sp(2)$ or G_2 on the Euclidean space $\boldsymbol{m}$ arising from the Cartan decomposition $\boldsymbol{g}=\boldsymbol{k}\oplus\boldsymbol{m}$ of the corresponding orthogonal symmetric Lie algebra $(\boldsymbol{g}, \boldsymbol{k}, \sigma)$, but the computations involved are rather long. First, one has to choose a point $P\in\boldsymbol{a}$ (a 2-dimensional abelian subspace of $\boldsymbol{m}$) so that the orbit of P under the adjoint action of K is minimal in sphere. That requires some manipulation with the roots of the Lie algebra determined by $\boldsymbol{a}$. Second, one needs to find the principal directions for the shape operator and compute the connection coefficients. The shape operator of an orbit hypersurface is given by $AX=-[Y, \xi]$, where ξ is the unit normal to the hypersurface in sphere (ξ is perpendicular to P, and ξ and P span $\boldsymbol{a}$), and $Y\in\boldsymbol{k}$ is such a vector so that $X=Y_P^*=[Y, P]$ (cf. [24]).

3. Also, it would be important to resolve if any minimal spherical hypersurface which is of 3-type and mass-symmetric via $\tilde{x}$ is necessarily isoparametric and if only such submanifolds are those four examples with three curvatures.

Techniques used in this paper can be modified to study hypersurfaces of a projective space which are of low type via the first standard embedding of a projective space. The author hopes to pursue this idea in a future paper.

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