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Hilbert Spaces of Analytic Functions and the Gegenbauer Polynomials

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Introduction.

Let $\boldsymbol{F}$ be the Fock type Hilbert space of analytic functions $f(z)$ of n complex variables $z=(z_1, z_2, \cdots, z_n) \in \boldsymbol{C}^n$, with the scalar product

$$(f, g)=\pi^{-n}\int_{C^n}\overline{f(z)}g(z)\exp(-|z_1|^2-\cdots-|z_n|^2)dz_1\cdots dz_n ,$$

with

$$dz_1\cdots dz_n=dx_1\cdots dx_n dy_1\cdots dy_n , \qquad z_j=x_j+iy_j ,$$

and let $\boldsymbol{H}$ be the usual Hilbert space $L^2(\boldsymbol{R}^n)$. Bargmann constructed in [1] a unitary mapping A from $\boldsymbol{H}$ to $\boldsymbol{F}$ given by an integral operator whose kernel is related in some definite sense to the Hermite polynomials. More precisely, $f=A\phi$ for $\phi \in \boldsymbol{H}$ is defined by

$$f(z)=\int_{R^n}A(z, q)\phi(q)d^nq$$

with

$$A(z, q)=\pi^{-n/4}\prod_{j=1}^{n}\exp\left\{-\frac{1}{2}(z_j^2+q_j^2)+2^{1/2}z_jq_j\right\} .$$

The purpose of the present paper is to show that similar constructions are possible for some other classical orthogonal polynomials.

§1. The arguments for the Gegenbauer polynomials.

Let λ be a positive real number. The Gegenbauer polynomials C_m^λ, $m=0, 1, 2, \cdots$, are defined as the coefficients in the expansion

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$$(1-2zq+z^2)^{-\lambda}=\sum_{m=0}^{\infty} C_m^\lambda(q)z^m\ , \qquad (-1<q<1,\ |z|<1)$$

and have the following orthogonality relation:

$$\int_{-1}^{1} C_m^\lambda(q)C_k^\lambda(q)(1-q^2)^{\lambda-1/2}dq=\begin{cases} 0 & (m\neq k) \\ \dfrac{\pi\Gamma(m+2\lambda)}{2^{2\lambda-1}(m+\lambda)m!\,[\Gamma(\lambda)]^2} & (m=k)\ . \end{cases}$$

Let ϕ_m^λ stand for the normalization of C_m^λ with respect to $K_\lambda=L^2((-1, 1), (1-q^2)^{\lambda-1/2})$, with the scalar product

$$(\phi, \psi)_\lambda=\int_{-1}^{1} \overline{\phi(q)}\psi(q)(1-q^2)^{\lambda-1/2}dq\ .$$

$\{\phi_m^\lambda \mid m=0, 1, 2, \cdots\}$ is a complete orthonormal system in K_λ since C_m^λ is a polynomial of degree m.

The Hilbert space F_λ consists of analytic functions f of one complex variable on B, the unit disk, $|z|<1$. The inner product is given by

$$\langle f, g\rangle_\lambda=\int_B \overline{f(z)}g(z)\rho_\lambda(|z|^2)dxdy \qquad (z=x+iy)$$

where

$$\rho_\lambda(t)=\begin{cases} \dfrac{1}{\Gamma(2\lambda-1)}t^{\lambda-1}\displaystyle\int_t^1 s^{-\lambda}(1-s)^{2\lambda-2}ds & (\lambda>1/2) \\ t^{\lambda-1}\left\{\dfrac{\Gamma(1-\lambda)}{\Gamma(\lambda)}-\dfrac{1}{\Gamma(2\lambda-1)}\displaystyle\int_0^t s^{-\lambda}(1-s)^{2\lambda-2}ds\right\} & (0<\lambda\leqq 1/2)\ . \end{cases}$$

The next equality is easily computed for $\lambda>1/2$.

$$\int_B |z|^{2m}\rho_\lambda(|z|^2)dxdy=\frac{\pi m!}{(m+\lambda)\Gamma(m+2\lambda)}\ . \tag{1}$$

From the analyticity with respect to λ, this equality holds also for $0<\lambda\leqq 1/2$.

PROPOSITION 1. *Suppose that f is an element of F_λ with the power series expansion*

$$f(z)=\sum_{m=0}^{\infty}\alpha_m z^m\ .$$

Then

$$\|f\|_\lambda^2=\langle f, f\rangle_\lambda=\sum_{m=0}^{\infty}\frac{\pi m!}{(m+\lambda)\Gamma(m+2\lambda)}|\alpha_m|^2 .$$

PROOF. For $0<\sigma<1$,

$$\int_{|z|\leqq\sigma}|f(z)|^2\rho_\lambda(|z|^2)dxdy$$
$$=\sum_{m=0}^{\infty}|\alpha_m|^2\int_{|z|\leqq\sigma}|z|^{2m}\rho_\lambda(|z|^2)dxdy\leqq\|f\|_\lambda^2 ,$$

which implies our assertion immediately. q.e.d.

COROLLARY. *If $f\in \boldsymbol{F}_\lambda$, then*

$$|f(z)|\leqq\|f\|_\lambda(h_\lambda(|z|^2))^{1/2} ,$$

where

$$h_\lambda(\xi)=\sum_{m=0}^{\infty}\frac{(m+\lambda)\Gamma(m+2\lambda)}{\pi m!}\xi^m .$$

The preceding corollary asserts that the strong convergence in $\boldsymbol{F}_\lambda$ implies the uniform convergence on every compact subset of B. Hence $\boldsymbol{F}_\lambda$ is a Hilbert space and from Proposition 1 we see that a complete orthonormal system in $\boldsymbol{F}_\lambda$ is given by the functions

$$u_m^\lambda(z)=\left(\frac{(m+\lambda)\Gamma(m+2\lambda)}{\pi m!}\right)^{1/2}z^m .$$

The main theorem is

THEOREM 1. *A unitary operator, $f=A_\lambda\phi$, of $\boldsymbol{K}_\lambda$* onto *$\boldsymbol{F}_\lambda$ is defined by*

$$f(z)=\int_{-1}^{1}A_\lambda(z, q)\phi(q)(1-q^2)^{\lambda-1/2}dq ,$$

where

$$A_\lambda(z, q)=\frac{2^{\lambda-1/2}\Gamma(\lambda)\lambda}{\pi}\,\frac{1-z^2}{(1-2zq+z^2)^{\lambda+1}} .$$

PROOF. It is easy to see that

$$A_\lambda(z, q)=\frac{2^{\lambda-1/2}\Gamma(\lambda)}{\pi}\sum_{m=0}^{\infty}(m+\lambda)C_m^\lambda(q)z^m .$$

(i.e., $A_\lambda(z, q)$ can be regarded as a generating function for the Gegenbauer polynomials.) Therefore

$$A_\lambda(z, q)=\sum_{m=0}^{\infty} \phi_m^\lambda(q)u_m^\lambda(z) . \tag{2}$$

We can consider that the right-hand side is the Fourier expansion for $A_\lambda(z, q)$ as a function of q, because $\sum_{m=0}^{\infty} |u_m^\lambda(z)|^2<\infty$ for $z \in B$.

Let $\phi \in \boldsymbol{K}_\lambda$, then

$$\begin{aligned}(A_\lambda\phi)(z)&=\Big(\sum_{m=0}^{\infty} \overline{u_m^\lambda(z)}\phi_m^\lambda,\ \phi\Big)_\lambda \\ &=\sum_{m=0}^{\infty} (\phi_m^\lambda, \phi)_\lambda u_m^\lambda(z) .\end{aligned}$$

Hence, $\|A_\lambda\phi\|_\lambda^2=\sum_{m=0}^{\infty} |(\phi_m^\lambda, \phi)_\lambda|^2$, i.e.,

$$\|A_\lambda\phi\|_\lambda=\|\phi\|_\lambda . \tag{3}$$

Substituting ϕ_m^λ for ϕ, we obtain

$$u_m^\lambda=A_\lambda\phi_m^\lambda . \tag{4}$$

It follows from (3), (4) that A_λ is a unitary operator of $\boldsymbol{K}_\lambda$ onto $\boldsymbol{F}_\lambda$.

q.e.d.

The inverse operator A_λ^{-1}, which exists by Theorem 1, cannot be expressed as an integral operator like A_λ. But we have

PROPOSITION 2. *If $f \in \boldsymbol{F}_\lambda$, then*

$$(A_\lambda^{-1}f)(q)=\underset{\sigma\to 1}{\text{l.i.m.}}\int_{|z|\leq\sigma} \overline{A_\lambda(z, q)}f(z)\rho_\lambda(|z|^2)dxdy ,$$

or, more precisely

$$\lim_{\sigma\to 1}\int_{-1}^{1}\left|(A_\lambda^{-1}f)(q)-\int_{|z|\leq\sigma}\overline{A_\lambda(z, q)}f(z)\rho_\lambda(|z|^2)dxdy\right|^2(1-q^2)^{\lambda-1/2}dq=0 .$$

PROOF. Let f be an element of $\boldsymbol{F}_\lambda$ with the power series expansion

$$f(z)=\sum_{m=0}^{\infty} \alpha_m u_m^\lambda(z) .$$

Using (2), for $0<\sigma<1$, we obtain

$$\begin{aligned}&\int_{|z|\leq\sigma}\overline{A_\lambda(z, q)}f(z)\rho_\lambda(|z|^2)dxdy \\ &\quad=\sum_{m=0}^{\infty} \phi_m^\lambda(q)\alpha_m\int_{|z|\leq\sigma}|u_m^\lambda(z)|^2\rho_\lambda(|z|^2)dxdy .\end{aligned}$$

The right-hand side can be regarded as the Fourier expansion for the left-hand side. Hence,

$$\begin{aligned}\int_{-1}^{1}\left|(A_\lambda^{-1}f)(q)-\int_{|z|\leqq\sigma}\overline{A_\lambda(z,\ q)}f(z)\rho_\lambda(|z|^2)dxdy\right|^2(1-q^2)^{\lambda-1/2}dq\\=\sum_{m=0}^{\infty}|\alpha_m-\alpha_m M_m^\lambda(\sigma)|^2\\=\sum_{m=0}^{\infty}|1-M_m^\lambda(\sigma)|^2|\alpha_m|^2\ ,\end{aligned}$$

where

$$M_m^\lambda(\sigma)=\int_{|z|\leqq\sigma}|u_m^\lambda(z)|^2\rho_\lambda(|z|^2)dxdy\ .$$

Therefore, we obtain the assertion. q.e.d.

REMARK. The Gegenbauer polynomials for $\lambda=1/2$ coincide with the Legendre polynomials P_m, $m=0, 1, 2, \cdots$. Hence if we put $\lambda=1/2$ in Theorem 1, we obtain the desired result for the Legendre polynomials. In particular, we remark that in this case $\rho_{1/2}(t)=t^{-1/2}$.

§ 2. Some application to spherical harmonics.

We consider some application of Theorem 1 to the Funk-Hecke formula.

For a fixed $n\geqq 3$, let S^{n-1} denote the surface of the unit sphere in R^n, $d\sigma$ the element of surface area on S^{n-1} and ω_n the total surface area. Let finally H_k be the space of all spherical harmonics of order k on S^{n-1}.

Now the Funk-Hecke formula is given in the following theorem.

THEOREM 2 (Funk-Hecke). *Suppose that ϕ is an element of $L^1((-1,1),$ $(1-q^2)^{(n-3)/2})$. Then, for $S_k\in H_k$ and $\omega\in S^{n-1}$,*

$$\begin{aligned}\int_{S^{n-1}}\phi((\omega,\ \tau))S_k(\tau)d\sigma(\tau)\\=\omega_{n-1}S_k(\omega)\int_{-1}^{1}\phi(q)P(k,\ q)(1-q^2)^{(n-3)/2}dq\ ,\end{aligned}$$

where

$$P(k,\ q)=\frac{k!\,\Gamma(n-2)}{\Gamma(k+n-2)}C_k^{(n-2)/2}(q)\ .$$

First of all, we let r be an element of the open interval $(-1, 1)$ and set $\phi(q)=A_{(n-2)/2}(r, q)$ in the Funk-Hecke formula. Using the formula (4) $u_m^\lambda=A_\lambda\phi_m^\lambda$ with $\lambda=(n-2)/2$, we then obtain the following:

$$\int_{S^{n-1}} A_{(n-2)/2}(r, (\omega, \tau))S_k(\tau)d\sigma(\tau)$$
$$=2^{-(n-3)/2}\omega_{n-1}\frac{\Gamma(n-2)}{\Gamma((n-2)/2)}r^k S_k(\omega) . \tag{5}$$

On the other hand, the integral kernel $A_{(n-2)/2}(r, q)$ and the Poisson kernel

$$p_r(q)=\frac{1-r^2}{(1-2rq+r^2)^{n/2}}$$

differ only by a constant. So we obtain

$$r^k S_k(\omega)=\omega_n^{-1}\int_{S^{n-1}} p_r((\omega, \tau))S_k(\tau)d\sigma(\tau) . \tag{6}$$

This equation is well-known as the Poisson integral. Thus we conclude that the equation (6) is only a special case of the Funk-Hecke formula.

Conversely, it is possible to obtain the Funk-Hecke formula from the equation (6). Since the element of surface area $d\sigma$ is invariant under the orthogonal group $O(n)$, we obtain that

$$\int_{S^{n-1}} \phi_m^\lambda((\omega, \tau))\phi_l^\lambda((\omega, \tau))d\sigma(\tau)$$
$$=\omega_{n-1}\int_{-1}^{1} \phi_m^\lambda(q)\phi_l^\lambda(q)(1-q^2)^{\lambda-1/2}dq$$
$$=\omega_{n-1}\delta_{ml}$$

with $\lambda=(n-2)/2$ and $\omega\in S^{n-1}$. Therefore, for $S_k\in \boldsymbol{H}_k$,

$$\sum_{m=0}^{\infty} u_m^\lambda(r)\int_{S^{n-1}} \phi_m^\lambda((\omega, \tau))S_k(\tau)d\sigma(\tau)$$
$$=\int_{S^{n-1}} \sum_{m=0}^{\infty} u_m^\lambda(r)\phi_m^\lambda((\omega, \tau))S_k(\tau)d\sigma(\tau)$$
$$=\int_{S^{n-1}} A_\lambda(r, (\omega, \tau))S_k(\tau)d\sigma(\tau)$$
$$=2^{-(\lambda-1/2)}\omega_{n-1}\frac{\Gamma(2\lambda)}{\Gamma(\lambda)}r^k S_k(\omega)$$

with $\lambda=(n-2)/2$ and $-1<r<1$. This implies

$$u_m^\lambda(r)\int_{S^{n-1}}\phi_m^\lambda((\omega,\tau))S_k(\tau)d\sigma(\tau)$$
$$=\delta_{mk}2^{-(\lambda-1/2)}\omega_{n-1}\frac{\Gamma(2\lambda)}{\Gamma(\lambda)}r^k S_k(\omega)\ ,$$

which is equivalent to the following:

$$\int_{S^{n-1}}\phi_m^\lambda((\omega,\tau))S_k(\tau)d\sigma(\tau)$$
$$=\omega_{n-1}S_k(\omega)\int_{-1}^{1}\phi_m^\lambda(q)P(k,q)(1-q^2)^{\lambda-1/2}dq\ .$$

Hence, we obtain the Funk-Hecke formula for ϕ_m^λ with $\lambda=(n-2)/2$. It follows from this result that the Funk-Hecke formula is true for any ϕ.

References

[1] V. BARGMANN, On a Hilbert space of analytic functions and an associated integral transform. Part I, Comm. Pure Appl. Math., **14** (1961), 187–214.

[2] A. ERDELYI, W. MAGNUS, F. OBERHETTINGER and F. G. TRICOMI, *Higher Transcendental Functions, Vol. 2*, McGraw-Hill, 1953.

[3] C. MÜLLER, *Spherical Harmonics*, Lecture Notes in Math., **17**, Springer-Verlag, 1966.

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