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Time Reversal of Random Walks in R^d

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Introduction.

The purpose of this paper is to give an extension, to a higher dimensional case, of the result [3] concerning time reversal of random walks.

Suppose we are given a pseudo-order $\lhd$ in R^d such that $x \lhd y$ implies $x+z \lhd y+z$ for any $z \in R^d$. We write $x \blacktriangleleft y$ if $x \lhd y$ and $x \neq y$, and put $K = \{x \in R^d : 0 \lhd x\}$. Then $x \lhd y$ if and only if $y - x \in K$. The set K contains 0 and satisfies

$$(1) \qquad x+y \in K \quad \text{if} \quad x, y \in K .$$

Throughout the paper we assume that the set K is infinite and Borel. Given a random walk $S_n = \sum_{k=1}^{n} X_k$ in R^d, we define a random time τ by

$$(2) \qquad \tau = \min\{n \geqq 1 : S_n \blacktriangleleft S_k \text{ for } 0 \leqq \forall k \leqq n-1\} ,$$

and assume that $\tau < \infty$ a.s. One more assumption, which is technical and might probably be removed, is that the random walk is countably valued, namely, if Γ denotes the (countable) set of x such that $P\{X_k = x\} > 0$ then

$$(3) \qquad P\{X_k \in \Gamma\} = 1 .$$

Next we consider the time reversal

$$(4) \qquad (0,\ S_{\tau-1} - S_\tau,\ S_{\tau-2} - S_\tau, \cdots, S_1 - S_\tau,\ -S_\tau)$$

and regard this as a (finite length) path-valued random variable. Taking independent copies w_k, $k \geqq 1$, of (4), we define a process $\{W_n, n \geqq 0\}$ by (1.1). Then our main result is that $\{W_n, n \geqq 0\}$ is a Markov chain with transition function $\hat{p}_\xi(x, y)$ given by (1.3). The result of [3] is a special

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case of the present result obtained by taking $K=[0, \infty)$ in $\boldsymbol{R}^1$.

In analogy with the ladder variables discussed in Chapter XII of [1] τ may be called the first descending strict ladder time. We also consider time reversal defined in terms of the first descending weak ladder time and, when S_n is a simple symmetric random walk on $\boldsymbol{Z}^2$, we clarify its relation to a theorem of Pitman type.

§ 1. The main results.

Given i.i.d. random variables with values in $\boldsymbol{R}^d$ and satisfying (3), we put

$$p(x, y)=P\{x+X_1=y\}=p(0, y-x) ,$$
$$\hat{p}(x, y)=P\{x-X_1=y\}=p(y, x)=p(0, x-y) ,$$
$$\hat{\Gamma}_x=\{x-z \ : \ z\in\Gamma\}=\{y \ : \ \hat{p}(x, y)>0\} ,$$

and consider the random walk:

$$S_n=X_1+\cdots+X_n \ (n\geqq 1) , \qquad S_0=0 .$$

The first descending strict ladder time τ defined by (2) is assumed to be finite a.s.

Let $K_0=K\setminus\{0\}$. Let $l\geqq 1$ be an integer and let $x_k\in K_0$, $1\leqq k\leqq l$. When $l\geqq 2$, a sequence $(x_1, \cdots, x_l)$ is said to be admissible if, for any $k=1, 2, \cdots, l-1$, there exists j such that $k<j\leqq l$ and $x_k \not\blacktriangleleft x_j$ [1]. When $l=1$, any sequence (x_1) of length 1 is said to be admissible. Note that the admissibility implies that all the x_k's are in K_0.

We consider the (countable) space

$$\mathscr{W}=\left\{\begin{array}{ll} w=(x_0, x_1, \cdots, x_l): & \\ \quad \text{(i)} & l\geqq 1 , \\ \quad \text{(ii)} & x_0=0,\ p(x_k, x_{k-1})>0 \ (1\leqq \forall k\leqq l) , \\ \quad \text{(iii)} & \text{the sequence } (x_1, \cdots, x_l) \text{ is admissible} \end{array}\right\} .$$

Note that the condition (iii) implies that $x_k\in K_0$ $(1\leqq\forall k\leqq l)$. Let

$$(w_1, w_2, \cdots)\in\mathscr{W}^\infty=\mathscr{W}\times\mathscr{W}\times\cdots ,$$

and writing $w_k=(w_k(0), w_k(1), \cdots, w_k(l_k))$, $k\geqq 1$, let us define $W=\{W_n, n\geqq 0\}$ by

1) $x \not\blacktriangleleft y$ means that $x \blacktriangleleft y$ does not hold.

$$
(1.1) \qquad W_n = \begin{cases} w_1(n) & \text{for } 0 \leqq n \leqq l_1 , \\ w_1(l_1) + w_2(n - l_1) & \text{for } l_1 < n \leqq l_1 + l_2 , \\ \vdots & \vdots \\ \sum_{j=1}^{k-1} w_j(l_j) + w_k\left(n - \sum_{j=1}^{k-1} l_j\right) & \text{for } \sum_{j=1}^{k-1} l_j < n \leqq \sum_{j=1}^{k} l_j , \\ \vdots & \vdots \end{cases} .
$$

We also write $W = w_1 w_2 \cdots$ for simplicity. We thus defined a map

$$
\varphi : \mathscr{W}^{\infty} \longrightarrow \boldsymbol{W} \qquad (\varphi(w_1, w_2, \cdots) = w_1 w_2 \cdots)
$$

where $\boldsymbol{W} = \{W : \{0, 1, \cdots\} \to \boldsymbol{R}^d\}$. Since $\mathscr{W}$ is a Polish space (a countable space with the discrete topology), $\mathscr{W}^{\infty}$ is also a Polish space. $\boldsymbol{W}$ is likewise a Polish space. It will not be hard to prove that φ is a Borel injection and hence the image $\boldsymbol{W}_0 = \varphi(\mathscr{W}^{\infty})$ is a Borel subset of $\boldsymbol{W}$. We denote by μ the probability distribution of the random variable (4) which, as is easily seen, takes values in $\mathscr{W}$ and by $\boldsymbol{P}$ the image measure (on $\boldsymbol{W}_0$) of $\mu^{\infty} = \mu \otimes \mu \otimes \cdots$ under the map φ. We thus have a stochastic process $\{W_n, n \geqq 0\}$ defined on the probability space $(\boldsymbol{W}_0, \boldsymbol{P})$.

For each $x \in K_0$ let $\mathscr{W}_{x,1} = \{(x)\}$, let for $l \geqq 2$

$$
\mathscr{W}_{x,l} = \left\{ \begin{array}{l} (x_1, \cdots, x_l) : \\ \quad \text{i)} \;\; x_1 = x \text{ and } p(x_k, x_{k-1}) > 0 \; (2 \leqq \forall k \leqq l) , \\ \quad \text{ii)} \;\; \text{the sequence } (x_1, \cdots, x_l) \text{ is admissible} \end{array} \right\} ,
$$

and put

$$
\mathscr{W}_x = \bigcup_{l=1}^{\infty} \mathscr{W}_{x,l} .
$$

For each $w = (x_1, \cdots, x_l) \in \mathscr{W}_x$ define $\mu'(w)$ by

$$
(1.2) \qquad \mu'(w) = \begin{cases} \prod_{k=1}^{l-1} p(x_{k+1}, x_k) = \prod_{k=1}^{l-1} \hat{p}(x_k, x_{k+1}) & \text{if } l \geqq 2 , \\ 1 & \text{if } l = 1 , \end{cases}
$$

and then define $\xi(x)$, $x \in K$, by

$$
\xi(x) = \begin{cases} \sum_{w \in \mathscr{W}_x} \mu'(w) & \text{for } x \in K_0 , \\ 1 & \text{for } x = 0 . \end{cases}
$$

Next put

$$\boldsymbol{K}_0=\{0\}\ ,\qquad \boldsymbol{K}_n=\bigcup_{x\in \boldsymbol{K}_{n-1}}(K_0\cap\hat{\Gamma}_x)\quad (n\geqq 1)\ ,$$

$$\boldsymbol{K}=\bigcup_{n=0}^{\infty}\boldsymbol{K}_n\ ,$$

and finally define $\hat{p}_\xi(x, y)$, x, $y\in\boldsymbol{K}$, by

$$\hat{p}_\xi(x, y)=\frac{1}{\xi(x)}\hat{p}(x, y)\xi(y)1_{K_0}(y)\ , \tag{1.3}$$

where 1_{K_0} is the indicator function of K_0. Then $\boldsymbol{K}$ is a countable subset of K and it will be proved that $\xi(x)$ is finite for any $x\in\boldsymbol{K}$ and that $\hat{p}_\xi(x, y)$ is a Markov transition function on $\boldsymbol{K}$ (Lemma 3). Now our first main result is as follows:

THEOREM 1. *$\{W_n,\ n\geqq 0,\ \boldsymbol{P}\}$ is a Markov chain on $\boldsymbol{K}$ with (one-step) transition function $\hat{p}_\xi(x, y)$.*

We next give another expression of $\xi(x)$ in the special case where K is given by (1.4) below. Let $a_1, \cdots, a_m\in\boldsymbol{R}^d$ be given and assume that they are linearly independent, so $1\leqq m\leqq d$. Let

$$K=\{x\in\boldsymbol{R}^d:\ \langle x, a_j\rangle\geqq 0,\ 1\leqq\forall j\leqq m\}\ , \tag{1.4}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\boldsymbol{R}^d$. For each $b\in\partial K$ (the boundary of K) put

$$J(b)=\{j\ :\ \langle b, a_j\rangle=0\}\ ,$$
V=the vector space ($\subset\boldsymbol{R}^d$) spanned by $\{a_j,\ 1\leqq j\leqq m\}$,
$$V_b=\{x\in V:\ \langle x, a_j\rangle\leqq 0 \text{ for } \forall j\in J(b) \text{ and } \langle x, a_k\rangle=0 \text{ for } \forall k\notin J(b)\}\ ,$$
$$H[b]=\{b+x\ :\ x\in V_b\}\ .$$

Take a copy K^* of K, let $\theta: K^*\to K$ be the natural identification map and put $K_0^*=K^*\setminus\{0^*\}$ where $0^*=\theta^{-1}(0)$. We denote by $\tilde{K}$ the disjoint sum $K\cup K^*$ and then introduce a transition function on $\tilde{K}$ as follows:

$$\tilde{p}(\tilde{x}, \tilde{y})=\begin{cases} p(\tilde{x}, \tilde{y}) & \text{if } \tilde{x}\in K,\ \tilde{y}\in K\ ,\\ \displaystyle\sum_{z\in H[\theta(\tilde{y})]\setminus\{\theta(\tilde{y})\}} p(\tilde{x}, z) & \text{if } \tilde{x}\in K,\ \tilde{y}\in\partial K^*\ ,\\ p(\theta(\tilde{x}), \theta(\tilde{y})) & \text{if } \tilde{x}\in K_0^*,\ \tilde{y}\in K^*\setminus\partial K^*\ ,\\ \displaystyle\sum_{z\in H[\theta(\tilde{y})]} p(\theta(\tilde{x}), z) & \text{if } \tilde{x}\in K_0^*,\ \tilde{y}\in\partial K^*\ ,\\ 1 & \text{if } \tilde{x}=\tilde{y}=0^*\ ,\\ 0 & \text{otherwise}\ . \end{cases} \tag{1.5}$$

Let $\{\tilde{Y}_n, n\geqq 0\}$ be a Markov chain on $\tilde{K}$ starting at 0 and with (one-step) transition function $\tilde{p}(\tilde{x}, \tilde{y})$. Then $\tilde{Y}_n \in K$ for $0\leqq n<T$, $\tilde{Y}_n \in K^*$ for $n\geqq T$ where

$$T=\min\{n\geqq 1 : \tilde{Y}_n \in K^*\} \qquad (\min \varnothing=\infty) ,$$

and 0^* is a trap. We put

$$Y_n=\begin{cases}\tilde{Y}_n & \text{if} \quad 0\leqq n<T ,\\ \theta(\tilde{Y}_n) & \text{if} \quad n\geqq T .\end{cases}$$

Then our second main result is the following:

THEOREM 2. *If K is given by* (1.4), *then for each $x\in K_0$,*

$$\xi(x)=E\{\sum_{0\leqq n<T} \mathbf{1}_A(Y_n)\}+E\{\sum_{n\geqq T} \mathbf{1}_B(Y_n)\} , \tag{1.6}$$

where

$$A=\{y : 0\lhd y \blacktriangleleft x\} , \qquad B=\{y : 0\blacktriangleleft y\lhd x\} .$$

§2. Proof of Theorem 1.

Before proving Theorem 1 we prepare three lemmas.

LEMMA 1. (i) *For any $w=(x_0, x_1, \cdots, x_l)\in \mathscr{W}$ we have*

$$\mu(w)=\hat{p}(0, x_1)\hat{p}(x_1, x_2)\cdots \hat{p}(x_{l-1}, x_l) . \tag{2.1}$$

(ii) *If $w\in\mathscr{W}$ is expressed as $w=(0, w')$ with $w'\in \mathscr{W}_x$, then*

$$\mu(w)=\hat{p}(0, x)\mu'(w') . \tag{2.2}$$

PROOF. Recalling the meaning of the phrase "$w=(x_0, x_1, \cdots, x_l)\in\mathscr{W}$", we see that the event $\{\tau=l, S_{l-k}-S_l=x_k \ (1\leqq \forall k\leqq l)\}$ is the same as the event $\{S_k=x_{l-k}-x_l \ (1\leqq \forall k\leqq l)\}$. Therefore

$$\begin{aligned}\mu(w)&=P\{\tau=l, S_{l-k}-S_l=x_k \ (1\leqq \forall k\leqq l)\}\\ &=P\{S_k=x_{l-k}-x_l \ (1\leqq \forall k\leqq l)\}\\ &=p(0, x_{l-1}-x_l)p(x_{l-1}-x_l, x_{l-2}-x_l)\cdots p(x_1-x_l, -x_l)\\ &=p(x_l, x_{l-1})p(x_{l-1}, x_{l-2})\cdots p(x_1, 0)\\ &=\text{the right hand side of (2.1)} .\end{aligned}$$

The identity (2.2) immediately follows from (2.1) and the definition of μ'.

LEMMA 2. *For any $x\in K$*

$$\sum_{y\in K_0}\hat{p}(x, y)\xi(y)=\sum_{y\in K_0\cap\hat{\Gamma}_x}\hat{p}(x, y)\xi(y)=\xi(x) . \tag{2.3}$$

PROOF. Let $x=0$. Then the left hand side of (2.3) equals

$$\sum_{y\in K_0\cap\hat{\Gamma}_0}\hat{p}(0, y)\xi(y)=\sum_{y\in K_0\cap\hat{\Gamma}_0}\sum_{w\in\mathscr{W}_y}\hat{p}(0, y)\mu'(w)=\mu(\mathscr{W})=1 .$$

Next, let $x\in K_0$. Then the left hand side of (2.3) equals

$$\begin{aligned}\sum_{y\in K_0\cap\hat{\Gamma}_x}\hat{p}(x, y)\xi(y)&=\sum_{y\in K_0\cap\hat{\Gamma}_x}\sum_{w\in\mathscr{W}_y}\hat{p}(x, y)\mu'(w)\\&=\sum_{y\in K_x\cap\hat{\Gamma}_x}\sum_{w\in\mathscr{W}_y}\hat{p}(x, y)\mu'(w)\\&\quad+\sum_{y\in(K_0\setminus K_x)\cap\hat{\Gamma}_x}\sum_{w\in\mathscr{W}_y}\hat{p}(x, y)\mu'(w) ,\end{aligned}$$

where $K_x=\{y\in\boldsymbol{R}^d : y-x\in K_0\}$. Note that $x\in K_0$ implies $K_x\subset K_0$, or more generally, $x\in K_y$ implies $K_x\subset K_y$. Let $y\in K_x$ $(x\in K_0)$ and put

$$\mathscr{W}_y'=\{w=(y_1, \cdots, y_l)\in\mathscr{W}_y : y_k\in K_x \text{ for } 1\leqq\forall k\leqq l\} .$$

Then

$$\sum_{w\in\mathscr{W}_y}\hat{p}(x, y)\mu'(w)=\sum_{w\in\mathscr{W}_y'}\hat{p}(x, y)\mu'(w)+\sum_{w\in\mathscr{W}_y\setminus\mathscr{W}_y'}\hat{p}(x, y)\mu'(w) ,$$

and consequently

$$\begin{aligned}\sum_{y\in K_0\cap\hat{\Gamma}_x}\hat{p}(x, y)\xi(y)&=\sum_{y\in K_x\cap\hat{\Gamma}_x}\sum_{w\in\mathscr{W}_y'}\hat{p}(x, y)\mu'(w)\\&\quad+\sum_{y\in K_x\cap\hat{\Gamma}_x}\sum_{w\in\mathscr{W}_y\setminus\mathscr{W}_y'}\hat{p}(x, y)\mu'(w)\\&\quad+\sum_{y\in(K_0\setminus K_x)\cap\hat{\Gamma}_x}\sum_{w\in\mathscr{W}_y}\hat{p}(x, y)\mu'(w) .\end{aligned}$$

The first term of the right hand side of the above equals 1 while the second term plus the third term yields

$$\sum_{l=2}^{\infty}\sum_{w\in\mathscr{W}_{x,l}}\mu'(w)=\xi(x)-1 .$$

This completes the proof of Lemma 2.

LEMMA 3. *$\xi(x)$ is finite for any $x\in\boldsymbol{K}$ and $\hat{p}_\xi(x, y)$ is a Markov transition function on $\boldsymbol{K}$, i.e.,*

$$\sum_{y\in\boldsymbol{K}}\hat{p}_\xi(x, y)=\sum_{y\in K_0\cap\hat{\Gamma}_x}\hat{p}_\xi(x, y)=1 , \qquad x\in\boldsymbol{K} . \tag{2.4}$$

PROOF. Recall that the identity (2.3) was proved without using the finiteness of $\xi(x)$. Putting $x=0$ in (2.3), we see that $\xi(x)<\infty$ for any

$x \in \boldsymbol{K}_1$ and so inductively $\xi(x) < \infty$ for any $x \in \boldsymbol{K}_n$, $n \geqq 1$. Therefore $\xi(x) < \infty$ for any $x \in \boldsymbol{K}$. The identity (2.4) follows immediately from (2.3) and the fact that $x \in \boldsymbol{K}$ implies $K_0 \cap \hat{\Gamma}_x \subset \boldsymbol{K}$.

PROOF OF THEOREM 1. Given a_n, $0 \leqq n \leqq m$ $(m \geqq 1)$, satisfying

$$\begin{cases} a_0 = 0 \,, & a_n \in K_0 \ (1 \leqq \forall n \leqq m) \\ \hat{p}(a_{n-1}, a_n) > 0 & \text{for} \quad 1 \leqq \forall n \leqq m \,, \end{cases} \tag{2.5}$$

we are going to compute $\boldsymbol{P}\{\Lambda\}$ where

$$\Lambda = \{W \in \boldsymbol{W}_0 : W_n = a_n \ (0 \leqq \forall n \leqq m)\} \,.$$

We note that any element W of $\boldsymbol{W}_0$ admits a unique representation

$$W = w_1 w_2 \cdots \tag{2.6}$$

where $w_k = (w_k(0), w_k(1), \cdots, w_k(l_k)) \in \mathscr{W}$, $k \geqq 1$. For W of the form (2.6) we call n $(\in \boldsymbol{N})$ a ladder time of W if $n = l_1 + \cdots + l_n$ for some $k \geqq 1$. Then n $(\in \boldsymbol{N})$ is a ladder time of W if and only if

$$W_n \blacktriangleleft W_{n'} \qquad \text{for all} \quad n' > n \,. \tag{2.7}$$

For each $w \in \boldsymbol{W}_0$ we put

$$L = L(W) = \min\{n \geqq m : n \text{ is a ladder time of } W\} \,,$$

and consider the event

$$\Lambda_l = \Lambda \cap \{L = m + l - 1\} \,, \qquad l \geqq 1 \,.$$

For typographical convenience we write a instead of a_m and put

$$\Lambda^w = \left\{ W \in \boldsymbol{W}_0 : \begin{array}{l} W_n = a_n \ (0 \leqq \forall n \leqq m) \,, \\ W_n = x_{n-m+1} \ (m \leqq \forall n \leqq m + l - 1) \\ L(W) = m + l - 1 \end{array} \right\}$$

for each $w = (x_1, \cdots, x_l) \in \mathscr{W}_{a,l}$, $l \geqq 1$. Then

$$\Lambda = \bigcup_{l=1}^{\infty} \Lambda_l = \bigcup_{l=1}^{\infty} \bigcup_{w \in \mathscr{W}_{a,l}} \Lambda^w = \bigcup_{w \in \mathscr{W}_a} \Lambda^w \,.$$

Making use of Lemma 1 we can compute $\boldsymbol{P}\{\Lambda^w\}$ for each $w = (x_1, \cdots, x_l) \in \mathscr{W}_a$. The result is

$$\boldsymbol{P}\{\Lambda^w\} = \left\{ \prod_{n=1}^{m} \hat{p}(a_{n-1}, a_n) \right\} \mu'(w) \,.$$

Therefore we have

$$\begin{aligned} P\{\Lambda\} &= \sum_{w \in \mathscr{W}_a} \left\{\prod_{n=1}^{m} \hat{p}(a_{n-1}, a_n)\right\} \mu'(w) \\ &= \left\{\prod_{n=1}^{m} \hat{p}(a_{n-1}, a_n)\right\} \xi(a_m) \\ &= \prod_{n=1}^{m} \hat{p}_\xi(a_{n-1}, a_n) . \end{aligned}$$

This means that $\{W_n, n \geqq 0, P\}$ is a Markov chain with transition function $\hat{p}_\xi(x, y)$.

§ 3. Proof of Theorem 2.

We need some preliminaries. Let V denote the vector space ($\subset \boldsymbol{R}^d$) spanned by $a_1, \cdots, a_m$. Put $K(x) = K_x \cup \{x\} = \{y \in \boldsymbol{R}^d : x \lhd y\}$, $x \in \boldsymbol{R}^d$. For a finite set F in $\boldsymbol{R}^d$ we define vertex F as the unique maximum element (with respect to $\lhd$) of the set $\{x \in V : K(x) \supset F\}$. It is easy to see that vertex F can be given by

$$\text{vertex } F = \sum_{j=1}^{m} c_j a_j , \tag{3.1a}$$

where $c_1, \cdots, c_m$ are determined uniquely by

$$\sum_{j=1}^{m} \langle a_j, a_k \rangle c_j = \min_{a \in F} \langle a, a_k \rangle , \qquad 1 \leqq k \leqq m . \tag{3.1b}$$

Making use of the representation (3.1), one can easily verify the following assertions.

1°. For any $b \in \boldsymbol{R}^d$

$$\text{vertex}[F \cup \{b\}] = \text{vertex } F + \text{vertex}\{0, b - \text{vertex } F\} . \tag{3.2}$$

2°. (i) For each $b \in K \setminus \partial K$

$$x - \text{vertex}\{0, x\} = b \iff x = b . \tag{3.3}$$

(ii) For each $b \in \partial K$

$$x - \text{vertex}\{0, x\} = b \iff x \in H[b] . \tag{3.4}$$

3°. (i) If $b_1, b_2 \in \partial K$ and $b_1 \neq b_2$, then $H[b_1] \cap H[b_2] = \varnothing$.

(ii) $\bigcup_{b \in \partial K} H[b] = (K \setminus \partial K)^c$.

To proceed, suppose we are given $w = (x_1, \cdots, x_l) \in \mathscr{W}_{x,l}$, $x \in K_0$, $l \geqq 2$. Since w is admissible, we have

$$(3.5) \qquad \text{for any } k\ (1\leqq k<l) \text{ there exists } j \text{ such that } k<j\leqq l \text{ and } x_k \blacktriangleleft x_j\ .$$

We next define $(\hat{x}_0, \hat{x}_1, \cdots, \hat{x}_{l-1})$ by

$$(3.6) \qquad \hat{x}_k = x_{l-k}\ , \qquad 0\leqq k\leqq l-1\ .$$

Then (3.5) is equivalent to

$$(3.7) \qquad \text{for any } k\ (1\leqq k<l) \text{ there exists } j \text{ such that } 0\leqq j<k \text{ and } \hat{x}_k \blacktriangleleft \hat{x}_j\ .$$

We now define $\hat{T}=\hat{T}(\hat{x}_0, \hat{x}_1, \cdots, \hat{x}_{l-1})$ by

$$(3.8) \qquad \hat{T}=\max\{k : \hat{x}_0, \hat{x}_1, \cdots, \hat{x}_k \in K(\hat{x}_0)\}+1\ .$$

LEMMA 4. *The condition* (3.7) (*and so* (3.5)) *is equivalent to the following* (3.9):

$$(3.9) \qquad \text{vertex}\{\hat{x}_0, \hat{x}_1, \cdots, \hat{x}_k\} \neq \hat{x}_k \qquad \textit{if} \quad T\leqq k<l\ .$$

PROOF. Suppose (3.7) holds. To prove that (3.9) holds it is enough to show that

$$(3.10) \qquad \text{vertex}\{\hat{x}_0, \hat{x}_1, \cdots, \hat{x}_k\} = \hat{x}_k$$

implies $\hat{x}_0, \hat{x}_1, \cdots, \hat{x}_k \in K(\hat{x}_0)$. So suppose (3.10) holds. Then

$$(3.11) \qquad \hat{x}_0, \hat{x}_1, \cdots, \hat{x}_{k-1} \in K(\hat{x}_k)\ ,$$

or equivalently

$$(3.12) \qquad \hat{x}_k \lhd \hat{x}_0, \hat{x}_1, \cdots, \hat{x}_{k-1}\ .$$

On the other hand, it follows from (3.7) that there exists j such that $0\leqq j<k$ and $\hat{x}_k \blacktriangleleft \hat{x}_j$. (3.12) then implies $\hat{x}_k=\hat{x}_j$. If $j>0$, then by (3.12) $\hat{x}_j \lhd \hat{x}_0, \hat{x}_1, \cdots, \hat{x}_{j-1}$, so by a similar argument we see that there exists j' $(0\leqq j'<j)$ such that $\hat{x}_k=\hat{x}_j=\hat{x}_{j'}$. Repeating this argument we see that $\hat{x}_k=\hat{x}_0$ and this combined with (3.11) implies $\hat{x}_0, \hat{x}_1, \cdots, \hat{x}_k \in K(\hat{x}_0)$. The converse implication (3.9)$\Rightarrow$(3.7) is easily verified.

If we define $(x_0', x_1', \cdots, x_{l-1}')$ by

$$(3.13) \qquad x_k' = \hat{x}_k - \text{vertex}\{\hat{x}_0, \hat{x}_1, \cdots, \hat{x}_k\}\ , \qquad 0\leqq j<l\ ,$$

then $x_k' \in K$, and using (3.2) we see that

(3.14) $$x_k' = x_{k-1}' + \hat{x}_k - \hat{x}_{k-1} - \text{vertex}\{0, x_{k-1}' + \hat{x}_k - \hat{x}_{k-1}\}\ , \qquad 1 \leqq k < l\ .$$

We finally define $(\tilde{x}_0, \tilde{x}_1, \cdots, \tilde{x}_{l-1})$ by

(3.15) $$\tilde{x}_k = \begin{cases} x_k' & \text{if } 0 \leqq k < \hat{T}\ , \\ \theta^{-1}(x_k') & \text{if } \hat{T} \leqq k < l\ . \end{cases}$$

Then $(\tilde{x}_0, \tilde{x}_1, \cdots, \tilde{x}_{l-1})$ is a sequence in $\tilde{K}$ with the following property:

(3.16)
(a) $\tilde{x}_0 = 0$,
(b) $0 \lhd \tilde{x}_{l-1} \blacktriangleleft x$ if $\tilde{T} = \infty$,
(c) $\tilde{x}_k \in K_0^*$ for any k such that $\tilde{T} \leqq k < l$,
(d) if $\tilde{T} < l$, then $\tilde{x}_{\tilde{T}} \in \partial K^*$ and $0 \blacktriangleleft \theta(\tilde{x}_{l-1}) \lhd x$.

Here $\tilde{T} = \tilde{T}(\tilde{x}_0, \tilde{x}_1, \cdots, \tilde{x}_{l-1})$ is defined by

(3.17) $$\tilde{T} = \min\{k : \tilde{x}_k \in K^*\} \qquad (\min \varnothing = \infty)\ .$$

Let $\tilde{p}(\tilde{x}, \tilde{y})$ be defined by (1.5). Then the equality $\sum_{\tilde{y} \in \tilde{K}} \tilde{p}(\tilde{x}, \tilde{y}) = 1$ is a consequence of 3°. For each $x \in K_0$ and $l \geqq 2$ we denote by $\tilde{\mathscr{W}}_{x,l}$ the space of all sequences $(\tilde{x}_0, \tilde{x}_1, \cdots, \tilde{x}_{l-1})$ in $\tilde{K}$ with the property (3.16) and $\tilde{p}(\tilde{x}_{k-1}, \tilde{x}_k) > 0$, $1 \leqq k < l$. We then define a map $\varphi_{x,l} : \mathscr{W}_{x,l} \to \tilde{\mathscr{W}}_{x,l}$ by $\varphi_{x,l}(w) = \tilde{w}$ where $\tilde{w} = (\tilde{x}_0, \tilde{x}_1, \cdots, \tilde{x}_{l-1})$ is determined by $w = (x_1, \cdots, x_l) \in \mathscr{W}_{x,l}$ via (3.6), (3.13) and (3.15).

LEMMA 5. *Let $x \in K_0$ and $l \geqq 2$. Then for each $\tilde{w} = (\tilde{x}_0, \tilde{x}_1, \cdots, \tilde{x}_{l-1}) \in \tilde{\mathscr{W}}_{x,l}$ we have*

$$\sum_{w \in \varphi_{x,l}^{-1}(\tilde{w})} \mu'(w) = \prod_{k=1}^{l-1} \tilde{p}(\tilde{x}_{k-1}, \tilde{x}_k)\ .$$

PROOF. We denote by $\hat{\mathscr{W}}_{x,l}$ the space of all sequences $\hat{w} = (\hat{x}_0, \hat{x}_1, \cdots, \hat{x}_{l-1})$ in K_0 satisfying (i) $\hat{x}_{l-1} = x$, (ii) $p(\hat{x}_{k-1}, \hat{x}_k) > 0$ for $1 \leqq \forall k < l$ and (iii) (3.9) holds. Define a map $\hat{\varphi}_{x,l} : \hat{\mathscr{W}}_{x,l} \to \tilde{\mathscr{W}}_{x,l}$ by $\hat{\varphi}_{x,l}(\hat{w}) = \tilde{w}$ where $\hat{w} = (\hat{x}_0, \hat{x}_1 \cdots, \hat{x}_{l-1})$ and $\tilde{w} = (\tilde{x}_0, \tilde{x}_1, \cdots, \tilde{x}_{l-1})$ is determined by (3.13) and (3.15). Since the map:

$$(x_1, \cdots, x_l) \in \mathscr{W}_{x,l} \longrightarrow \hat{w} = (x_l, x_{l-1}, \cdots, x_1) \in \hat{\mathscr{W}}_{x,l}$$

is a bijection, we have

(3.18) $$\sum_{w \in \varphi_{x,l}^{-1}(\tilde{w})} \mu'(w) = \sum_{\hat{w} \in \hat{\varphi}_{x,l}^{-1}(\tilde{w})} \prod_{k=1}^{l-1} p(\hat{x}_{k-1}, \hat{x}_k)\ .$$

We next denote by $\hat{\mathscr{W}}(\tilde{w})$ the space of all sequences $(\hat{x}_0, \hat{x}_1, \cdots, \hat{x}_{l-1})$ with the following properties (3.19) and (3.20).

(3.19) $\hat{x}_{l-1}=x$.

(3.20a) If $1\leqq k<T$, then $\hat{x}_k-\hat{x}_{k-1}=\tilde{x}_k-\tilde{x}_{k-1}$.

(3.20b) If $1\leqq k=T<l$, then $\hat{x}_k-\hat{x}_{k-1}=y-\tilde{x}_{k-1}$ for some $y\in H[\theta(\tilde{x}_k)]\setminus\{\theta(\tilde{x}_k)\}$ with $p(\tilde{x}_{k-1}, y)>0$.

(3.20c) If $T<k<l$, then

(i) $\hat{x}_k-\hat{x}_{k-1}=\theta(\tilde{x}_k)-\theta(\tilde{x}_{k-1})$ provided $\tilde{x}_k\notin\partial K^*$,

(ii) $\hat{x}_k-\hat{x}_{k-1}=y-\theta(\tilde{x}_{k-1})$ for some $y\in H[\theta(\tilde{x}_k)]$ with $p(\theta(\tilde{x}_{k-1}), y)>0$ provided $\tilde{x}_k\in\partial K^*$.

We are going to prove that $\hat{\varphi}_{x,l}^{-1}(\tilde{w})=\hat{\mathscr{W}}(\tilde{w})$. Notice first that $\hat{\varphi}_{x,l}^{-1}(\tilde{w})$ consists of all sequences $(\hat{x}_0, \hat{x}_1, \cdots, \hat{x}_{l-1})\in\hat{\mathscr{W}}_{x,l}$ satisfying (3.14) where $(x_0', x_1', \cdots x_{l-1}')$ is determined by (3.15) with $\hat{T}$ replaced by $\tilde{T}$. Then, regarding (3.14) as an equation with unknown $x_k'+\hat{x}_k-\hat{x}_{k-1}$ and making use of 2°, we can see that any $(\hat{x}_0, \hat{x}_1, \cdots, \hat{x}_{l-1})$ in $\hat{\varphi}_{x,l}^{-1}(\tilde{w})$ satisfies (3.20). Clearly it also satisfies (3.19). Conversely, we assume $\hat{w}=(\hat{x}_0, \hat{x}_1, \cdots, \hat{x}_{l-1})\in\hat{\mathscr{W}}_{x,l}(\tilde{w})$ and prove $\hat{w}\in\hat{\varphi}_{x,l}^{-1}(\tilde{w})$. The nontrivial part of this proof is to show

(i) $\hat{x}_k\in K_0$, $0\leqq k<l$,

(ii) (3.14) holds where x_k' is determined by (3.15) with $\hat{T}$ replaced by $\tilde{T}$.

To prove (i), rewrite (3.20) in the form

$$
(3.21)\quad \hat{x}_{k-1}-\tilde{\theta}(\tilde{x}_{k-1}) =\begin{cases}\hat{x}_k-\tilde{x}_k & \text{if } 1\leqq k<\tilde{T}\,,\\ \{\hat{x}_k-\theta(\tilde{x}_k)\}+\{\theta(\tilde{x}_k)-y\} & \text{if } 1\leqq k=\tilde{T}<l\,,\\ \hat{x}_k-\theta(\tilde{x}_k) & \text{if } \tilde{T}<k<l \text{ and } \tilde{x}_k\in\partial K^*\,,\\ \{\hat{x}_k-\theta(\tilde{x}_k)\}+\{\theta(\tilde{x}_k)-y\} & \text{if } T<k<l \text{ and } \tilde{x}_k\in\partial K^*\,,\end{cases}
$$

where $\tilde{\theta}(\tilde{x})=\tilde{x}$ or $\theta(\tilde{x})$ according as $\tilde{x}\in K$ or $\tilde{x}\in K^*$. Then, making use of (3.21) and also the fact that $\hat{x}_{l-1}-\theta(\tilde{x}_{l-1})=x-\theta(\tilde{x}_{l-1})\in K$ if $T<l$ and $\hat{x}_{l-1}-\tilde{x}_{l-1}\in K_0$ if $T=\infty$, one can prove that $\hat{x}_k-\tilde{\theta}(\tilde{x}_k)\in K$ for $T\leqq k<l$ and $\hat{x}_k-\tilde{x}_k\in K_0$ for $1\leqq k<T$ (induction), from which (i) follows. (ii) can also be proved by using (3.20).

Once $\hat{\varphi}_{x,l}^{-1}(\tilde{w})=\hat{\mathscr{W}}(\tilde{w})$ is proved, (3.18) yields

$$\sum_{w\in\varphi_{x,l}^{-1}(\tilde{w})}\mu'(w)=\sum_{\hat{w}\in\hat{\mathscr{W}}(\tilde{w})}\prod_{k=1}^{l-1}p(\hat{x}_{k-1},\hat{x}_k)$$

$$=\prod_{k=1}^{l-1}\tilde{p}(\tilde{x}_{k-1},\tilde{x}_k)\qquad(\text{use (1.5)})\ ,$$

so the lemma was proved.

We can now complete the proof of Theorem 2. By Lemma 5 we have

$$\xi(x)=1+\sum_{l=2}^{\infty}\sum_{\tilde{w}}\prod_{k=1}^{l-1}\tilde{p}(\tilde{x}_{k-1},\tilde{x}_k)\ ,\qquad x\in K_0\ ,\tag{3.22}$$

where the second summation is taken over all $\tilde{w}$ in $\tilde{\mathscr{W}}_{x,l}$. But this is another expression of (1.6) as can be seen as follows. The right hand side of (1.6) equals

$$\sum_{n=0}^{\infty}E\{1_{(n,\infty]}(T)1_A(Y_n)\}+\sum_{n=1}^{\infty}E\{1_{(0,n]}(T)1_B(Y_n)\}$$

$$=1+\sum_{n=1}^{\infty}[P\{0\lhd Y_n\blacktriangleleft x,\ T>n\}+P\{0\blacktriangleleft Y_n\lhd x,\ T\leqq n\}]$$

$$=\text{the right hand side of (3.22)}\ .$$

§4. Time reversal defined in terms of descending weak ladder times.

We define the first descending weak ladder time σ by

$$\sigma=\min\{n\geqq 1\ :\ S_n\lhd S_k\ \text{for}\ 0\leqq\forall k\leqq n-1\}\ ,\tag{4.1}$$

and assume $\sigma<\infty$ a.s. throughout this section. Then we can define a process $\{V_n,\ n\geqq 0\}$ exactly in the same way as we defined $\{W_n,\ n\geqq 0\}$ but with the replacement of τ by σ. This section is concerned with the Markovian property of $\{V_n,\ n\geqq 0\}$. The result will be stated without proof, since the proof is similar to that for $\{W_n,\ n\geqq 0\}$.

As in §1 we introduce the weak admissibility. Given $x_k\in K$, $1\leqq k\leqq l$ $(l\geqq 2)$, a sequence $(x_1,\cdots,x_l)$ is said to be weakly admissible if, for any $k=1,2,\cdots,l-1$, there exists j such that $k<j\leqq l$ and $x_k\not\!\!\lhd x_j$. When $l=1$, any sequence (x_1) in K (of length 1) is said to be weakly admissible. Note that if $x_1=0$ and if $(x_1,\cdots,x_l)$ is weakly admissible, then $l=1$. Put

$$\mathscr{V}=\left\{\begin{array}{ll} v=(x_0, x_1, \cdots, x_l): & \\ \text{(i)} & l\geqq 1\,, \\ \text{(ii)} & x_0=0,\ p(x_k, x_{k-1})>0 \text{ for } 1\leqq \forall k\leqq l\,, \\ \text{(iii)} & \text{the sequence } (x_1, \cdots, x_l) \text{ is weakly admissible} \end{array}\right\}.$$

For $(v_1, v_2, \cdots)\in\mathscr{V}^\infty=\mathscr{V}\times\mathscr{V}\times\cdots$ we consider

$$V=v_1v_2\cdots\,,$$

which is short for $V=\{V_n, n\geqq 0\}$ with V_n defined in a way similar to (1.1). Let $\psi:\mathscr{V}^\infty\to\boldsymbol{W}$ be defined by $\psi(v_1, v_2, \cdots)=v_1v_2\cdots$ where $\boldsymbol{W}=\{W:\{0, 1, \cdots\}\to\boldsymbol{R}^d\}$ as before, and put $\boldsymbol{V}=\psi(\mathscr{V}^\infty)$. We denote by ν the probability distribution of

$$\{0, S_{\sigma-1}-S_\sigma, S_{\sigma-2}-S_\sigma, \cdots, S_1-S_\sigma, -S_\sigma)$$

which, as is easily seen, is a random variable taking values in $\mathscr{V}$, and by $\boldsymbol{Q}$ the image measure (on $\boldsymbol{V}$) of $\nu^\infty=\nu\otimes\nu\otimes\cdots$ under the map ψ. We thus have a stochastic process $\{V_n, n\geqq 0\}$ defined on the probability space $(\boldsymbol{V}, \boldsymbol{Q})$.

For each $x\in K$ let $\mathscr{V}_{x,1}=\{(x)\}$, let for $l\geqq 2$

$$\mathscr{V}_{x,l}=\left\{\begin{array}{ll} ((x_1, \cdots x_l): & \\ \text{i)} & x_1=x \text{ and } p(x_k, x_{k-1})>0 \text{ for } 2\leqq\forall k\leqq l\,, \\ \text{ii)} & \text{the sequence } (x_1, \cdots, x_l) \text{ is weakly admissible} \end{array}\right\},$$

and put

$$\mathscr{V}_x=\bigcup_{l=1}^\infty \mathscr{V}_{x,l}\,.$$

Note that $\mathscr{V}_{0,l}=\varnothing$ for $l\geqq 2$. For each $v=(x_1, \cdots, x_l)\in\mathscr{V}_x$ define $\nu'(v)$ by

$$\nu'(v)=\begin{cases}\prod_{k=1}^{l-1} p(x_{k+1}, x_k)=\prod_{k=1}^{l-1}\hat{p}(x_k, x_{k+1}) & \text{if } l\geqq 2\,, \\ 1 & \text{if } l=1\,,\end{cases}$$

and then define $\eta(x)$, $x\in K$, by

$$\eta(x)=\begin{cases}\sum_{v\in\mathscr{V}_x}\nu'(v) & \text{for } x\in K_0\,, \\ 1 & \text{for } x=0\,.\end{cases}$$

Next define $\hat{p}_\eta(x, y)$, $x, y\in\boldsymbol{K}$, by

$$\hat{p}_\eta(x, y)=\frac{1}{\eta(x)}\hat{p}(x, y)\eta(y)1_K(y) .$$

Then it can be proved that $\eta(x)<\infty$ for any $x\in \boldsymbol{K}$ and that $\hat{p}_\eta(x, y)$ is a Markov transition function on $\boldsymbol{K}$ (remember that $\boldsymbol{K}$ was defined in § 1).

THEOREM 3. *$\{V_n\ n\geqq 0, Q\}$ is a Markov chain on $\boldsymbol{K}$ with (one-step) transition function $\hat{p}_\eta(x, y)$.*

Next we give another expression of $\eta(x)$ in the special case where K is given by (1.4) with linearly independent $a_1, \cdots, a_m$. If we define $p'(x, y)$, x, $y\in K$, by

$$p'(x, y)=\begin{cases} p(x, y) & \text{if } x\in K,\ y\in K\setminus\partial K ,\\ \sum_{z\in H[y]} p(x, z) & \text{if } x\in K,\ y\in\partial K ,\end{cases}$$

then $p'(x, y)$ is a Markov transition function on K. Let $\{Y_n', n\geqq 0\}$ be a Markov chain on K starting at 0 and with (one-step) transition function $p'(x, y)$. Also let $T'=\min\{n\geqq 1 : Y_n'=0\}$. Then the following theorem can be proved.

THEOREM 4. *If K is given by (1.4), then for any $x\in K_0$*

$$\eta(x)=1+\sum_{n=1}^{\infty}P\{0\blacktriangleleft Y_n'\lhd x,\ T'>n\} .$$

Finally we specialize the situation as follows: K is the 2-dimensional quadrant consisting of the points of $\boldsymbol{Z}^2$ with nonnegative coordinates and $\{S_n, n\geqq 0\}$ is a simple symmetric random walk on $\boldsymbol{Z}^2$ (namely, the i.i.d. random variables defining S_n are assumed to satisfy $P\{X_k=e_i\}=P\{X_k=-e_i\}=1/4$, $i=1, 2$, where e_i denotes the unit vector in $\boldsymbol{R}^2$ whose i-th coordinate is 1).

In this case $p'(x, y)$, x, $y\in K$, is given by

$$p'(x, y)=\begin{cases} 1/4 & \text{if } x\in K,\ y=x\pm e_i ,\\ 1/4 & \text{if } 0\neq x=y\in\partial K ,\\ 1/2 & \text{if } x=y=0 ,\\ 0 & \text{otherwise} ,\end{cases}$$

and $\eta(x)=(x'+1)(x''+1)$ where x' and x'' are the first and the second coordinates of x, respectively, because

$$\eta(x)-1=\sum_{n=1}^{\infty}\sum_{y}P\{Y_n'=y,\ T'>n\}$$
$$=\sum_{y}\pi(y)=(x'+1)(x''+1)-1\ ;$$

in the above $\sum_y$ is taken over all $y\in \boldsymbol{Z}^2$ such that $0 \blacktriangleleft y \triangleleft x$ and $\pi(y)$ denotes the probability that the p'-chain starting at y hits 0, which is equal to 1.

On the other hand, let $S_n^{\sharp}=S_n-2\min\{S_k:\ 0\leqq k\leqq n\}$ where the minimum is taken coordinatewise. Then it is known that $\{S_n^{\sharp},\ n\geqq 0\}$ is a Markov chain with transition function $\hat{p}_{\eta}(x, y)$ (see [2]). Thus $\{V, n\geqq 0, \boldsymbol{Q}\}$ and $\{S_n^{\sharp},\ n\geqq 0\}$ are equivalent.

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