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The Left Cells and Their W-Graphs of Weyl Group of Type F_4

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§1. Introduction.

Let (W, S) be a finite Coxeter system. In [5], Kazhdan and Lusztig constructed W-graphs. They are obtained from the left cells of W and give the representations of Hecke algebra $\mathscr{H}$ corresponding to W. In this paper, we assume (W, S) has type F_4. The Kazhdan-Lusztig polynomials $P_{y,w}$ for (W, S) of type F_4 are already calculated (see [13]). In Section 2, we recall the definition of $P_{y,w}$ and several relations on W. The main results of this paper appear in Section 3. Using some data of $P_{y,w}$, we determine the left cells and two-sided cells. The left and two-sided cells are explicitly constructed from certain easily described subsets of W (Theorem 3.1), and we describe the natural W-graph corresponding to each left cell (Theorem 3.2). After describing each W-graph, we discuss some relations between the Duflo involutions and the conjugate classes in W by examining each case (Proposition 3.6).

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§2. Preliminaries.

Let (W, S) be an arbitrary Coxeter system, with the Bruhat order "$\leqq$" and the length function $l: W \to N$. Let $\mathscr{A} = Z[q^{1/2}, q^{-1/2}]$ be the ring of Laurent polynomials in $q^{1/2}$ where $q^{1/2}$ is an indeterminate and let $\mathscr{H}$ be the Hecke algebra of (W, S) over $\mathscr{A}$ with standard basis $\{T_w \mid w \in W\}$. In [5], Kazhdan and Lusztig defined the special basis $\{C_w \mid w \in W\}$ for $\mathscr{H}$, given by

$$C_w = \sum_{y \leqq w} (-1)^{l(w)-l(y)} q^{l(w)/2-l(y)} P_{y,w}(q^{-1}) T_y ,$$

where $P_{y,w} \in \boldsymbol{Z}[q]$ is a polynomial in q of degree $\leqq (l(w)-l(y)-1)/2$ for

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$y \leqq w$, and $P_{w,w}=1$.

In the same paper, Kazhdan and Lusztig defined several relations on W. For any $w \in W$, set $\mathscr{L}(w)=\{s \in S \mid sw<w\}$ and $\mathscr{R}(w)=\{s \in S \mid ws<w\}$.

For y, $w \in W$,

$y \prec w$ if $y<w$, $l(w)-l(y)$ is odd and $P_{y,w}=\mu(y, w)q^{(l(w)-l(y)-1)/2}+$lower powers of q where $\mu(y, w)$ is a non-zero integer,

$y - w$ if $y \prec w$ or $w \prec y$,

$y \leqq_{\mathrm{L}} w$ if there exists a sequence $y=x_0, x_1, x_2, \cdots, x_n=w$ such that for each i $(1 \leqq i \leqq n)$, $x_{i-1}-x_i$ and $\mathscr{L}(x_{i-1}) \nsubseteq \mathscr{L}(x_i)$,

$y \leqq_{\mathrm{LR}} w$ if there exists a sequence $y=x_0, x_1, x_2, \cdots, x_n=w$ such that for each i $(1 \leqq i \leqq n)$, we have either $x_{i-1} \leqq_{\mathrm{L}} x_i$ or $x_{i-1}^{-1} \leqq_{\mathrm{L}} x_i^{-1}$.

Let $\sim_{\mathrm{L}}$ be the equivalence relation associated with the preorder $\leqq_{\mathrm{L}}$; thus $y \sim_{\mathrm{L}} w$ means $y \leqq_{\mathrm{L}} w \leqq_{\mathrm{L}} y$. The corresponding equivalence classes are called the left cells of W. A right cell of W is a set of form $\{w \in W \mid w^{-1} \in \Gamma\}$ where Γ is a left cell. Let $\sim_{\mathrm{LR}}$ be the equivalence relation associated with the preorder $\leqq_{\mathrm{LR}}$; thus $y \sim_{\mathrm{LR}} w$ means $y \leqq_{\mathrm{LR}} w \leqq_{\mathrm{LR}} y$. The corresponding equivalence classes are called the two-sided cells of W.

We can calculate the Kazhdan-Lusztig polynomials $P_{y,w}$ which are defined by the following formula (see [4, 2.2c]);

For any $w \in W$, if $sw<w$, then

$$P_{y,w}=q^{1-c}P_{sy,sw}+q^{c}P_{y,sw}-\sum_{\substack{y \leqq z \prec sw \\ sz<z}} \mu(z, sw)q^{(l(w)-l(z))/2}P_{y,z}$$

where $c=1$ if $s \in \mathscr{L}(y)$, $c=0$ if $s \notin \mathscr{L}(y)$.

Also we have an algorithm for the calculation of $P_{y,w}$ (see [4], and also [12]). The calculation of $P_{y,w}$ for (W, S) of type F_4 was carried out on a computer (see [13]).

§ 3. Main results.

Hereafter we assume that (W, S) is of type F_4 whose generators are given by the set of $S=\{s_1, s_2, s_3, s_4\}$ satisfying the following relations;

$$1^2=2^2=3^2=4^2=e\ ,$$
$$(12)^3=(34)^3=(23)^4=e\ ,$$
$$13=31,\ 14=41,\ 24=42\ ,$$

where we write i instead of s_i.

For $J \subseteq S$, let $R_J=\{w \in W \mid \mathscr{R}(w)=J\}$. For $X \subseteq W$, define $X^*=Xw_0$ where w_0 is the maximal length element in W ($w_0=432312343231234323123121$).

It is clear that $R_J^* = R_{S-J}$ and we write $X+Y+\cdots$, to indicate the union of disjoint subsets $X, Y, \cdots$, of W.

We shall now construct certain subsets of W from the subsets R_J, $J \subseteq S$.

First, define

$$A_1 = R_{12} \cdot 12132 \cap R_{34},\ A_2 = A_1 \cdot 2,\ A_3 = A_2 \cdot 3,\ A_4 = A_2 \cdot 1\ ,$$
$$A_5 = A_3 \cdot 1,\ A_6 = A_5 \cdot 2,\ A_7 = A_6 \cdot 3,\ A_8 = A_7 \cdot 4\ .$$

Put $A = \cup A_i$.

Next define

$$B_1 = R_{34} \cdot 34323 \cap R_{12},\ B_2 = B_1 \cdot 3,\ B_3 = B_2 \cdot 2,\ B_4 = B_2 \cdot 4\ ,$$
$$B_5 = B_4 \cdot 2,\ B_6 = B_5 \cdot 3,\ B_7 = B_6 \cdot 2,\ B_8 = B_7 \cdot 1\ .$$

Set $B = \cup B_i$.

Define

$$C_1 = R_{12} \cdot 213234 \cap R_{123} = A_8^*,\ C_2 = C_1 \cdot 4 = A_7^*,\ C_3 = C_2 \cdot 3 = A_6^*\ ,$$
$$C_4 = C_3 \cdot 2 = A_5^*,\ C_5 = C_4 \cdot 3 = A_4^*,\ C_6 = C_4 \cdot 1 = A_3^*\ ,$$
$$C_7 = C_5 \cdot 1 = A_2^*,\ C_8 = C_7 \cdot 2 = A_1^*\ .$$

Set $C = \cup C_i = \cup A_i^* = A^*$.

Define

$$D_1 = R_{34} \cdot 324321 \cap R_{234} = B_8^*,\ D_2 = D_1 \cdot 1 = B_7^*,\ D_3 = D_2 \cdot 2 = B_6^*\ ,$$
$$D_4 = D_3 \cdot 3 = B_5^*,\ D_5 = D_4 \cdot 2 = B_4^*,\ D_6 = D_4 \cdot 4 = B_3^*\ ,$$
$$D_7 = D_6 \cdot 2 = B_2^*,\ D_8 = D_7 \cdot 3 = B_1^*\ .$$

Set $D = \cup D_i = \cup B_i^* = B^*$.

Define

$$E_1 = (R_4 \cdot 432 - A_8 \cdot 432) \cap R_{13},\ E_2 = (R_1 \cdot 123 - B_8 \cdot 123) \cap R_{24}\ ,$$
$$E_3 = E_2 \cdot 3,\ E_4 = E_1 \cdot 2,\ E_5 = E_4 \cdot 3,\ E_6 = E_3 \cdot 2,\ E_7 = E_5 \cdot 4\ ,$$
$$E_8 = E_6 \cdot 1,\ E_9 = (E_2 \cdot 1 \cup R_1 \cdot 1232) \cap (R_{14} - B_4)\ .$$

Set $E = \cup E_i$.

Define

$$F_1 = R_{13} \cdot 321 - D_1 = E_8^*,\ F_2 = E_7^*,\ F_3 = (R_{24} \cdot 23 - D_7 \cdot 23) \cap R_{124} = E_5^*\ ,$$
$$F_4 = F_1 \cdot 1 = E_6^*,\ F_5 = F_3 \cdot 3 = E_4^*,\ F_6 = F_5 \cdot 2 = E_1^*,\ F_7 = F_4 \cdot 2 = E_3^*\ ,$$
$$F_8 = E_9^*,\ F_9 = F_7 \cdot 3 = E_2^*\ .$$

Set $F = \cup F_i = \cup E_i^* = E^*$.

Define

$$G_1=R_3\cdot 32\cap R_{134},\ G_2=G_1\cdot 2,\ G_3=G_2\cdot 3\ .$$

Set $G=\cup G_i$.

Define

$$H_1=R_2\cdot 23\cap R_{124}=G_3^*,\ H_2=H_1\cdot 3=G_2^*,\ H_3=H_2\cdot 2=G_1^*\ .$$

Set $H=\cup H_i=\cup G_i^*=G^*$.

The subset I, J are defined by the following equations;

$$R_{12}=B_1+C_3+I\ ,$$
$$R_{34}=A_1+D_3+J\ .$$

We have $J=I^*$.

The subset K_i ($i=1, 2, 3, 4$) are defined by the following equations;

$$R_{23}=C_5+D_5+F_8+K_1\ ,$$
$$R_{24}=A_2+B_5+C_4+D_7+E_3+F_6+G_2+K_2\ ,$$
$$R_{13}=A_5+B_2+C_7+D_4+E_2+F_9+H_2+K_3\ ,$$
$$R_{14}=A_4+B_4+E_1+K_4\ .$$

We have $K_3=K_2^*$ and $K_4=K_1^*$. Set $K=\cup K_i$.

The subset L_i, M_i, N_i and O_i are given by

$$R_1=B_8+E_9+L_1\ ,$$
$$R_2=A_6+B_3+B_7+E_5+E_7+H_3+L_2\ ,$$
$$R_4=A_8+E_8+M_1\ ,$$
$$R_3=A_3+A_7+B_6+E_4+E_6+G_3+M_2\ ,$$
$$R_{123}=C_1+F_2+N_1\ ,$$
$$R_{124}=C_2+C_6+D_3+F_3+F_7+H_1+N_2\ ,$$
$$R_{234}=D_1+F_1+O_1\ ,$$
$$R_{134}=C_8+D_2+D_6+F_4+F_5+G_1+O_2\ .$$

We have $N_i=M_i^*$ and $O_i=L_i^*$ ($i=1, 2$). We set $L=\cup L_i$, $M=\cup M_i$, $N=\cup N_i=M^*$ and $O=\cup O_i=L^*$.

Finally, define

$$P=P_1=R_\varnothing=\{e\}\ ,\quad \text{and}\quad Q=Q_1=R_S=\{w_0\}=P^*\ .$$

After the polynomials $P_{y,w}$ were calculated, a computer was used to determine the relation $\leqq_{L}$ and $\leqq_{LR}$ on W, and we obtain the following result.

Theorem 3.1. (1) *The left cells of W are the subsets A_i, B_i, C_i, D_i, E_i, F_i, G_i, H_i, I, J, K_i, L_i, M_i, N_i, O_i, P and Q. There are 72 left cells.*

(2) *The two-sided cells of W are the subsets $A\cup B$, $C\cup D$, E, F, $G\cup H\cup I\cup J\cup K$, $L\cup M$, $N\cup O$, P and Q. There are 9 two-sided cells.*

Now we recall the definition of a W-graph. A W-graph is a combinatorial object which defines a representation of $\mathscr{H}$. It consists of a graph with a set of vertices X and a subset Y of $X\times X$ consisting of edges, together with two additional data: for each vertex $x\in X$, there is associated a subset $I_x\subseteq S$, and for each edge $(y, x)\in Y$, there is assigned an integer $\mu(y, x)\neq 0$. Let $E(X)$ be the free $\mathscr{A}$-module with basis X. For each $s\in S$, let τ_s be the endomorphism of $E(X)$ defined by

$$\tau_s(x)=\begin{cases}-x & \text{if } s\in I_x\,,\\ qx+q^{1/2}\displaystyle\sum_{\substack{y\in X\\ x-y\\ s\in I_y}}\mu(y, x)y & \text{if } s\notin I_x\,.\end{cases}$$

The preceding data defines a W-graph provided that for any $s\neq t$ such that st has finite order m,

$$\underbrace{\tau_s\tau_t\tau_s\cdots\cdots}_{m\text{ factors}}=\underbrace{\tau_t\tau_s\tau_t\cdots\cdots}_{m\text{ factors}}\,.$$

We define a representation $\varphi\colon\mathscr{H}\to\mathrm{End}_A(E)$ setting by $\varphi(T_s)=\tau_s$, and it is a representation of W by setting $q=1$. We say that the W-graph is irreducible if the representation corresponding to the W-graph is irreducible.

Let Γ be a left cell in W and W_Γ be the W-graph defined as follows: the set of vertices of W_Γ be the set Γ, the set of edges are the set $\{(y, x)\in\Gamma\times\Gamma \mid \mu(y, x)\neq 0\}$, where $\mu(y, x)$ is the integer defined in § 2, and $I_x=\mathscr{L}(x)$, for each $x\in\Gamma$. We shall call the W-graph obtained in this way the natural W-graph.

Let $\varphi_{\Gamma,q}$ be the matrix representation of $\mathscr{H}$, which have entries in $\mathscr{A}$, afforded by the natural W-graph W_Γ. So $\varphi_{\Gamma,1}$ is a matrix representation of W.

Since we determined the left cells, we can describe a natural W-graph W_Γ corresponding to each left cell Γ (up to equivalence) and construct the matrix representation of W afforded by each natural W-graph. The character table of W was determined by Kondo [6]. There are 25 irreducible characters, which we denote by $\chi_{i,j}$ the j-th character of degree i as encountered in his character table, except for the "isolated" character of degree 4, 12 and 16, which we denote by χ_4, χ_{12} and χ_{16}, respectively, as Shoji did in [11]. After constructing these matrix representations, we decompose the characters of each matrix representation of W into the irreducible characters. So, we obtain the following theorem. On the other hand, these representations and their decompositions were already obtained by Lusztig by making use of another method, and he called them cells (see [7]).

THEOREM 3.2. *The natural W-graph W_Γ corresponding to the left cell Γ is given in Figures 1~6. Each character afforded by the natural W-graph W_Γ is decomposed as follows.*

$|P|=1:\ \chi_{1,1}$
$|Q|=1:\ \chi_{1,4}$
$|L_i|=6\ (1\leqq i\leqq 2):\ \chi_{2,3}+\chi_{4,1}$
$|M_i|=6\ (1\leqq i\leqq 2):\ \chi_{2,1}+\chi_{4,1}$
$|N_i|=6\ (1\leqq i\leqq 2):\ \chi_{2,2}+\chi_{4,4}$
$|O_i|=6\ (1\leqq i\leqq 2):\ \chi_{2,4}+\chi_{4,4}$
$|A_i|=8\ (1\leqq i\leqq 8):\ \chi_{8,1}$
$|B_i|=8\ (1\leqq i\leqq 8):\ \chi_{8,3}$
$|C_i|=8\ (1\leqq i\leqq 8):\ \chi_{8,2}$
$|D_i|=8\ (1\leqq i\leqq 8):\ \chi_{8,4}$
$|E_i|=9\ (1\leqq i\leqq 9):\ \chi_{9,1}$
$|F_i|=9\ (1\leqq i\leqq 9):\ \chi_{9,4}$
$|G_i|=47\ (1\leqq i\leqq 3):\ \chi_{4,2}+\chi_{6,1}+\chi_{9,2}+\chi_{12}+\chi_{16}$
$|H_i|=47\ (1\leqq i\leqq 3):\ \chi_{4,3}+\chi_{6,1}+\chi_{9,3}+\chi_{12}+\chi_{16}$
$|I|=57:\ \chi_{1,3}+\chi_{4,3}+\chi_{6,2}+2\chi_{9,3}+\chi_{12}+\chi_{16}$
$|J|=57:\ \chi_{1,2}+\chi_{4,2}+\chi_{6,2}+2\chi_{9,2}+\chi_{12}+\chi_{16}$
$|K_i|=72\ (1\leqq i\leqq 4):\ \chi_4+\chi_{6,2}+\chi_{9,2}+\chi_{9,3}+\chi_{12}+2\chi_{16}$

Let $a: W\to \boldsymbol{N}\cup\{\infty\}$ be a special function which is called "a-function" (see [9]), and let $\delta(w)$ be the degree of $P_{1,w}$. Let $\mathscr{D}=\{w\in W \mid w^2=e,\ a(w)=l(w)-2\delta(w)\}$. The element of $\mathscr{D}$ is called the Duflo involution (see [3], [10]). We know that there is a unique Duflo involution in each left cell (see [10, 1.10]).

Let $w\in W$. We say w is a quasi-involution if $w\in\Gamma\cap\Gamma^{-1}$ for some left cell Γ in W. Clearly involutions are quasi-involutions.

In the present case, we obtain the following corollaries by examining each case. We also notice that the following corollary 3.3 has been proved for classical Weyl groups by Lusztig (see [8, 12.17]).

COROLLARY 3.3. *If Γ is a left cell in W, then the number of irreducible components of the character afforded by the natural W-graph W_Γ is equal to the number of involutions in Γ.*

COROLLARY 3.4. *For irreducible natural W-graph, let Γ and Γ' be different left cells which are in the same equivalence classes. Let x be the Duflo involution in Γ and let x' be the Duflo involution in Γ'. Then x and x' appear at the different position of the vertices of the natural W-graph* (see Figure 1, type A, B, C, D, E and F).

COROLLARY 3.5. *When Γ is of type I, J or K, $\Gamma \cap \Gamma^{-1}$ contains exactly two quasi-involutions which are not involutions. These elements have order* 12 *and length* 12, *and they are conjugate in W.*

We consider the relation between Duflo involutions $\mathscr{D}$ and conjugate classes in W. We obtain the following results by examining each case.

PROPOSITION 3.6. *Let Γ and Γ' be two left cells which have the same natural W-graph. Let x be the Duflo involution in Γ and let x' be the Duflo involution in Γ'. Then x and x' are conjugate in W.*

The figures of the natural W-graph W_Γ corresponding to each left cell Γ are as follows (see Theorem 3.2). In Figures $1\sim 6$, we use the following notation.

For a left cell Γ, a position of vertex $x \in \Gamma$ is indicated by $\boxed{I_x}$ and we denote $\boxed{I_x}-\boxed{I_y}$, if $x \sim_L y$, for x, $y \in \Gamma$. If Γ is of type X_i by the notation in Theorem 3.1 and $d \in \Gamma$ is the Duflo involution, then we write x_i at the nearest position to $\boxed{I_d}$ (see Corollary 3.4).

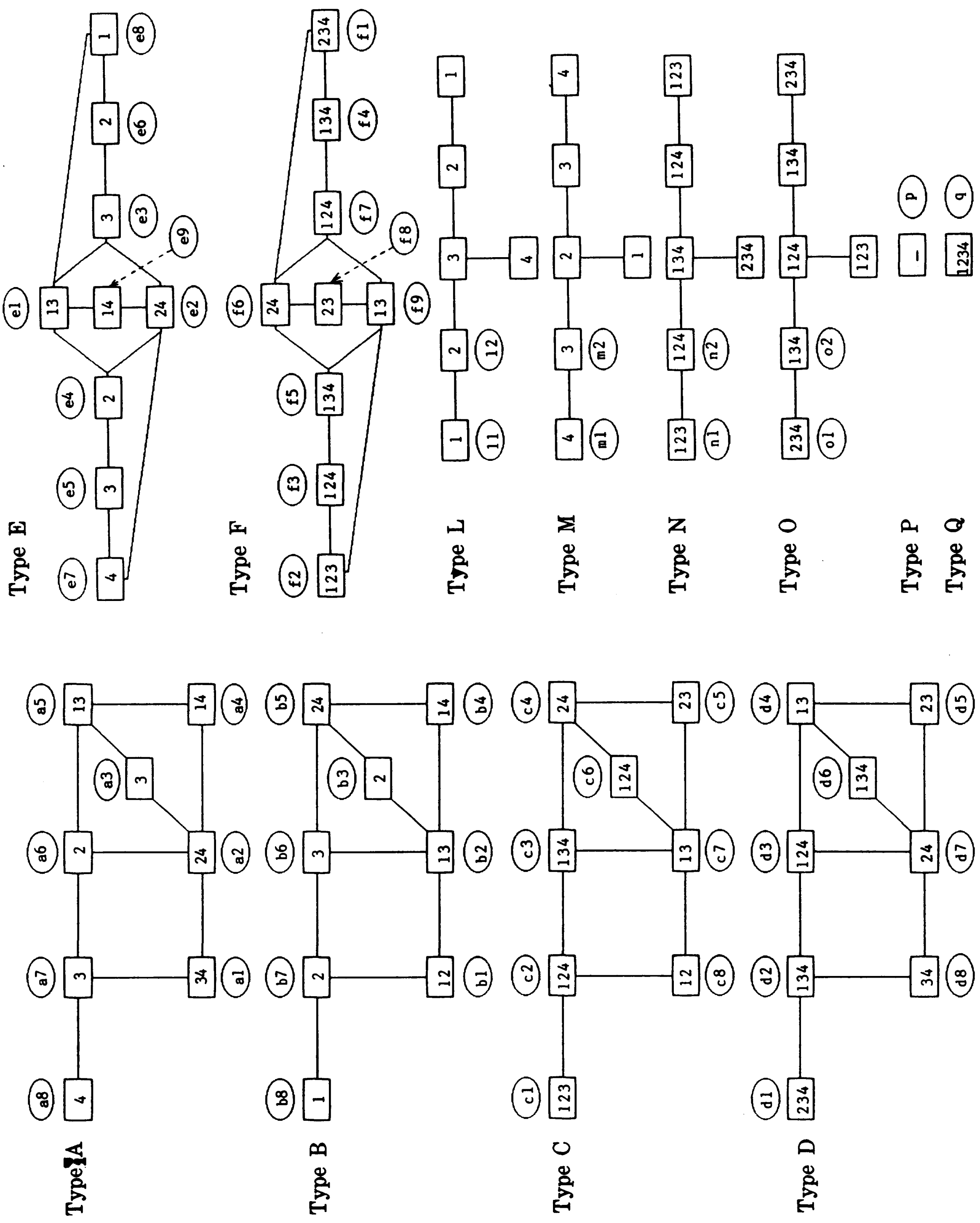

FIGURE 1. The *W*-graphs for type A, B, C, D, E, F, L, M, N, O, P and Q

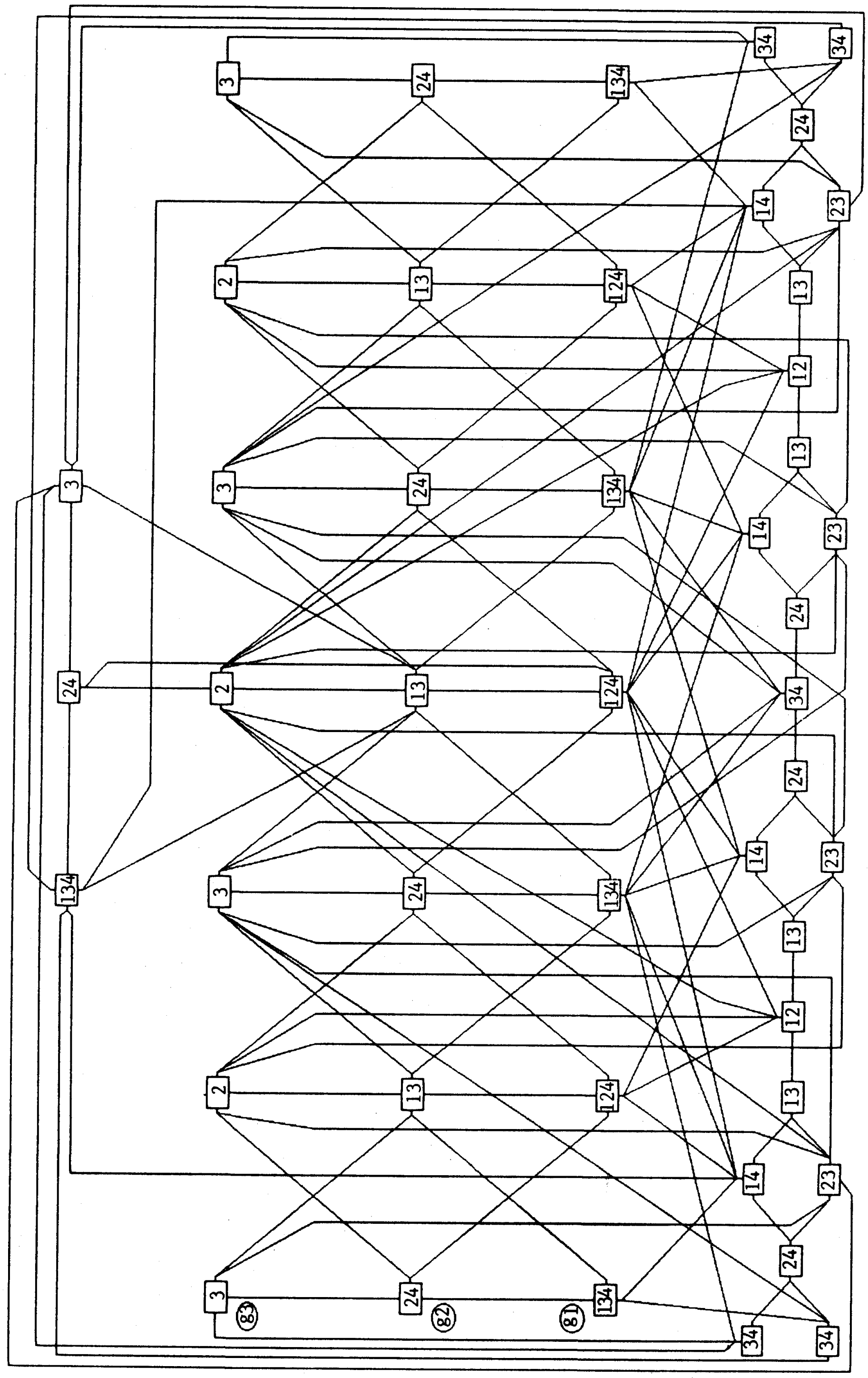

FIGURE 2. The *W*-graph for type G

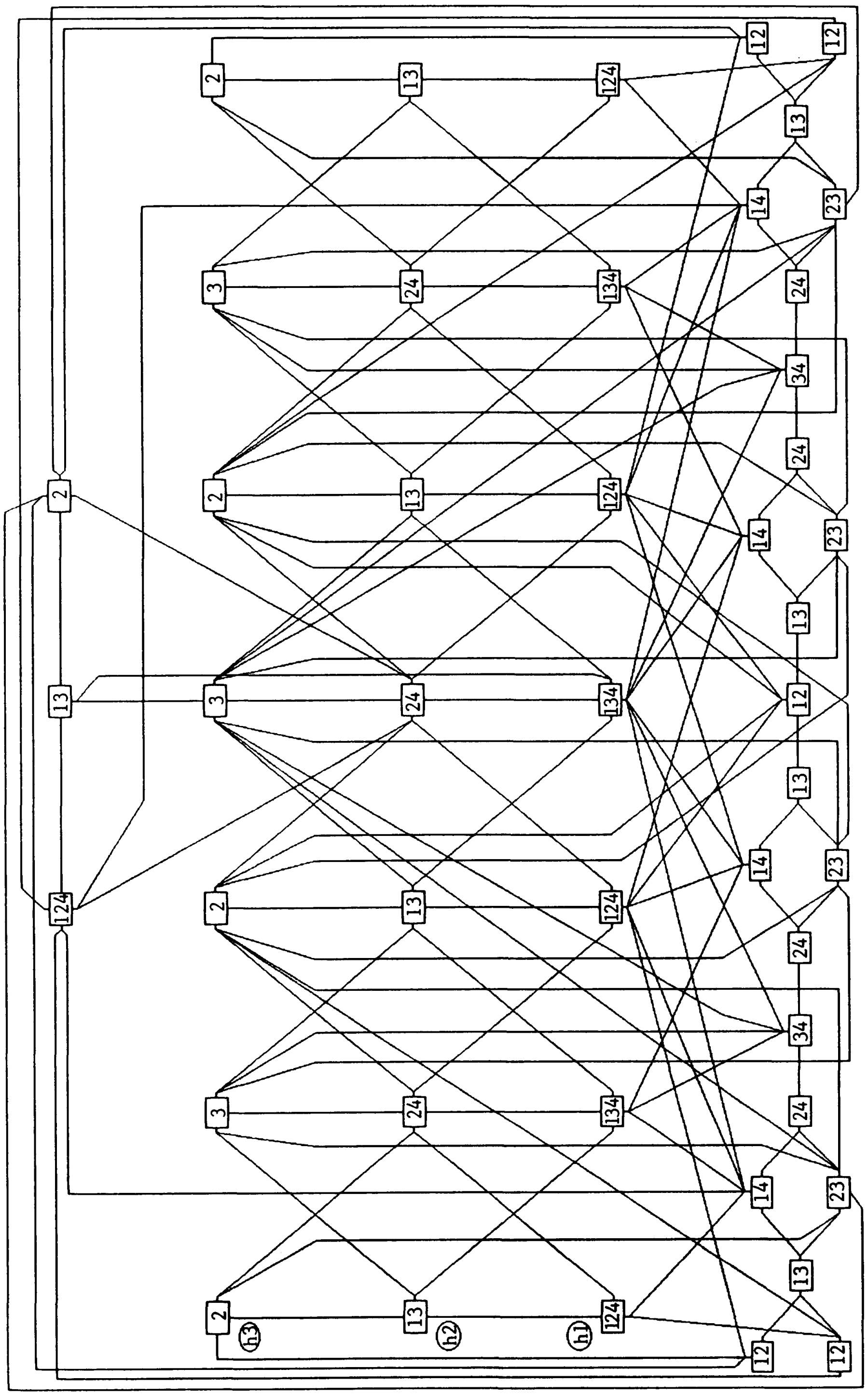

FIGURE 3. The W-graph for type H

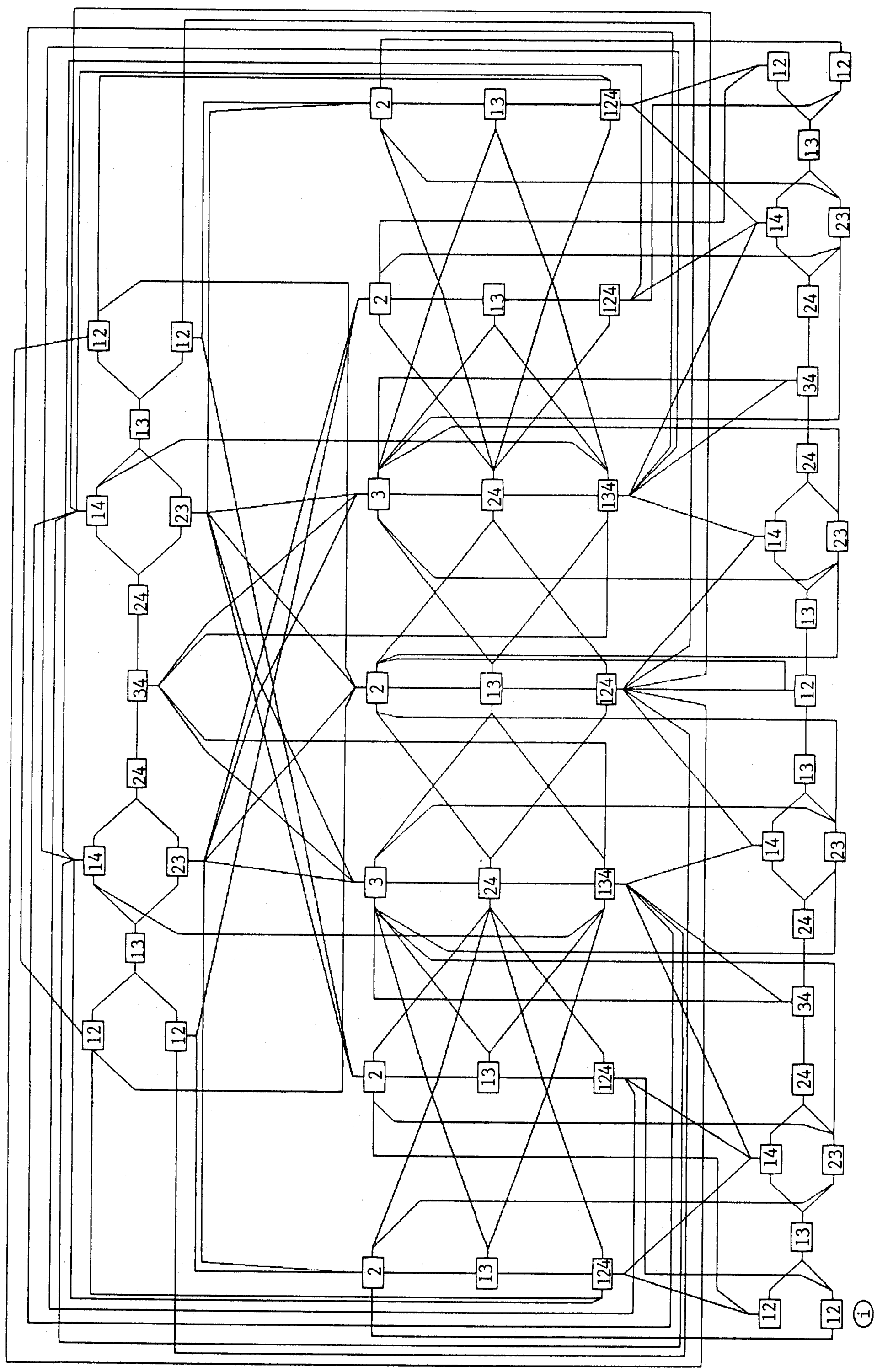

FIGURE 4. The W-graph for type I

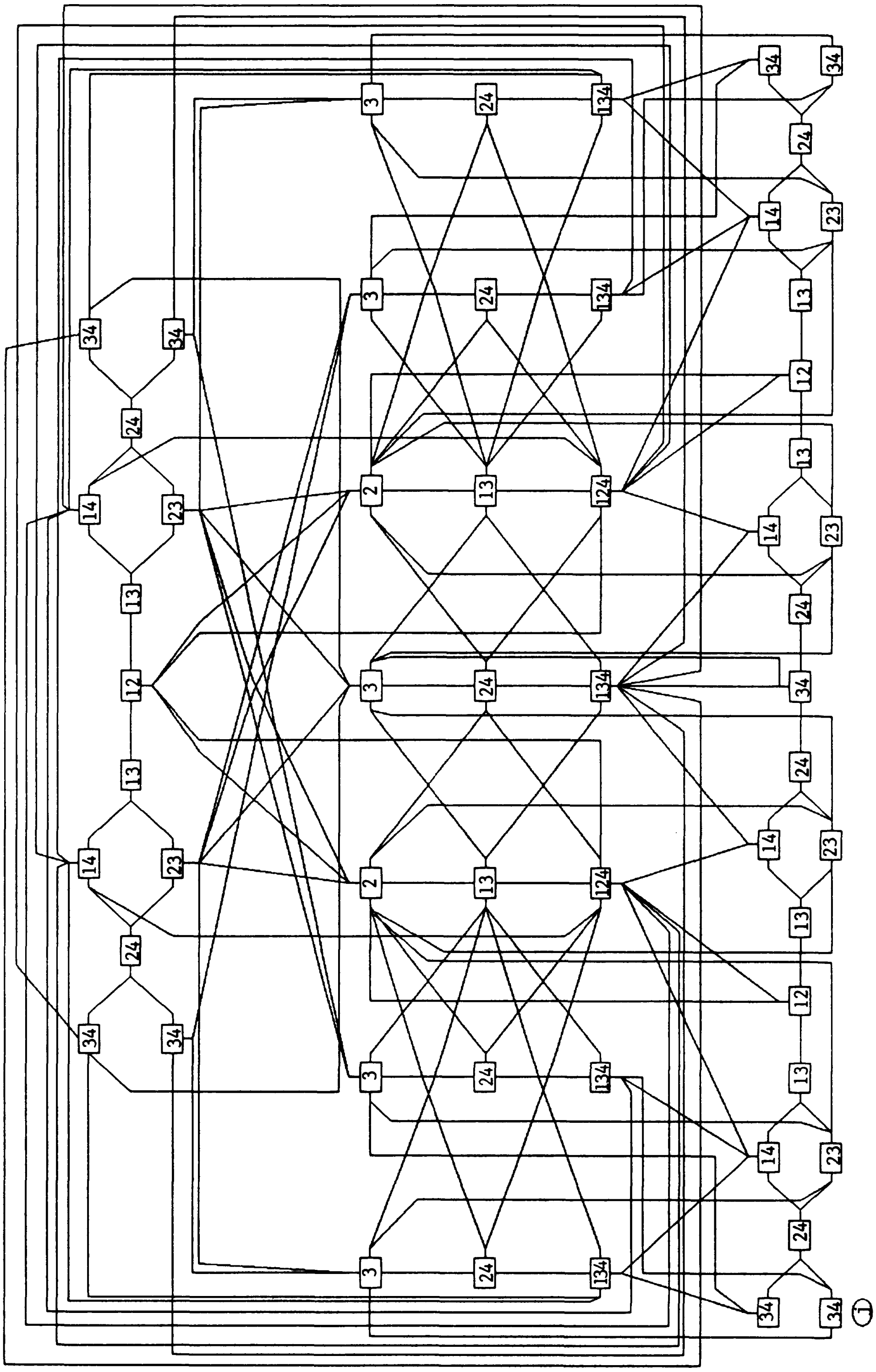

FIGURE 5. The W-graph for type J

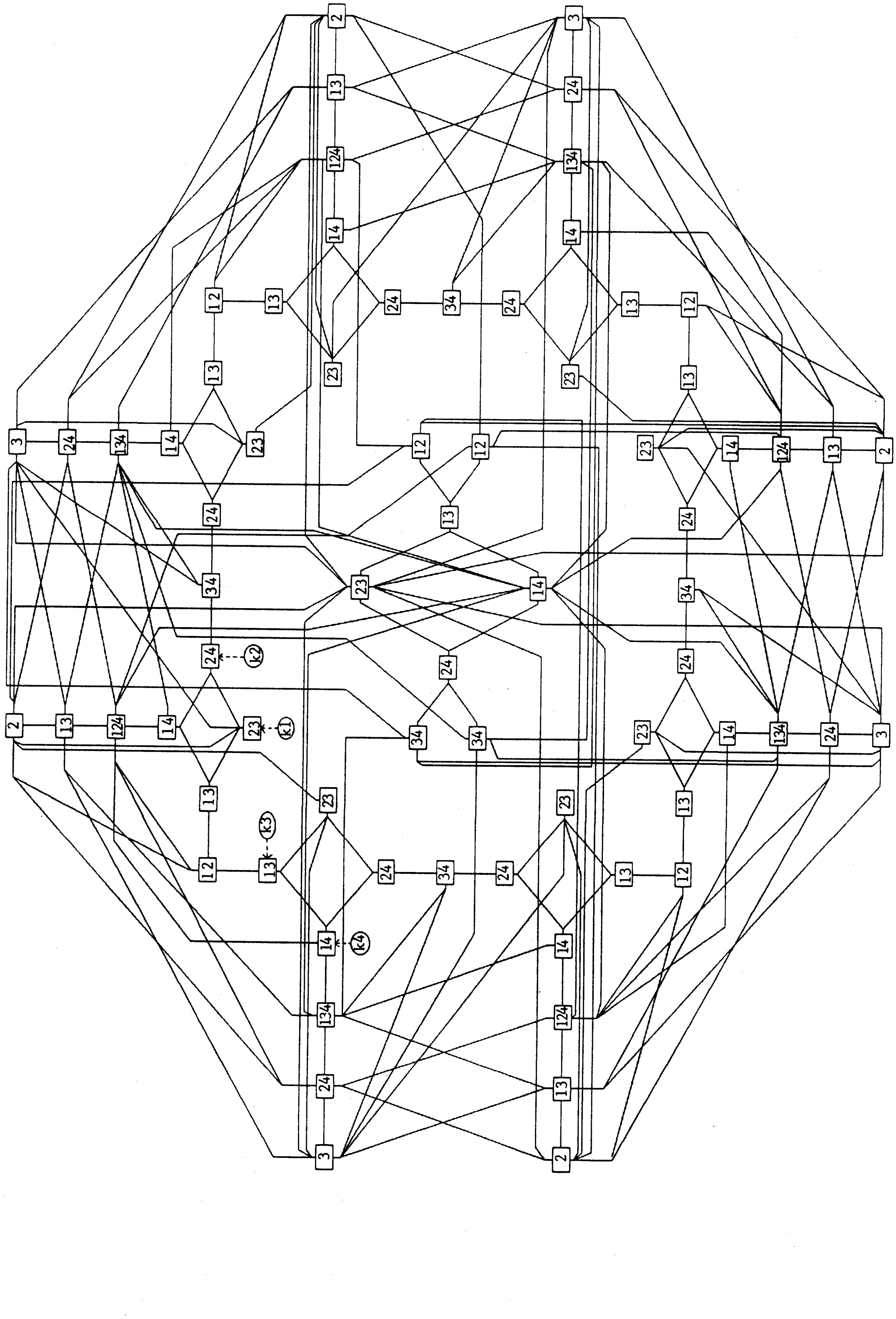

FIGURE 6. The W-graph for type K

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