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Stable Sheaves of Rank 2 on a 3-Dimensional Rational Scroll

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§0. Introduction.

In a joint work ([HI]), we showed that a family of stable vector bundles of rank 2 on a 3-dimensional rational scroll forms a complement of a dual 3-dimensional rational scroll. It is natural to ask what is its closure in the moduli of stable sheaves of same rank and same Chern classes (cf [M2]). Is the moduli of stable sheaves connected? Does it have other irreducible components? We will answer such questions in this paper.

Let (X, H) be a 3-dimensional rational scroll and M be the moduli of H-stable sheaves of rank 2 on X with $C_1=C_1(\mathscr{L})$, $C_2=D\cdot F$ and $C_3=0$ (see (2.1) for notations).

In section 1, we list up several formulas and describe the normal bundle of an n-dimensional rational scroll. In section 2, we show that there are 6 types of stable sheaves. The hierarchy of such types will be settled. In section 3, we construct limits of extensions. This is the main tool of this paper. The idea comes from monad ([BH]) and elementary transformation ([M1]). The main theorem (Theorem 3.13) says that the moduli M is connected and has two irreducible components M_0 and M_1. M_0 contains all vector bundles and M_1 contains no vector bundles. The dimension of M_1 is greater than that of M_0. In section 4, we construct a family for the difference $M_0 \setminus M_1$. The description of the normal bundle of a 3-dimensional rational scroll will be used to construct the family. In section 5, we construct a family for M_1. The construction will be sketched without proofs.

§1. Preliminaries.

In this section, we list up some basic formulas and describe the normal bundle of an n-dimensional rational scroll. The former will be used mainly

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in section 2 and the latter will be used in section 4.

Let k be an algebraically closed field of arbitrary characteristic. In this paper, the ground field k will be fixed.

Formulas. Let X be a non-singular projective 3-fold.

(1.1) RIEMANN-ROCH FORMULA. *For a coherent sheaf $\mathscr{E}$ of rank r on X,*

$$\chi(\mathscr{E})=\frac{C_1(\mathscr{E})(C_1(\mathscr{E})-K_X)(2C_1(\mathscr{E})-K_X)-6C_2(\mathscr{E})(C_1(\mathscr{E})-K_X)+6C_3(\mathscr{E})}{12}+\frac{1}{12}C_1(\mathscr{E})\cdot C_2(X)+r\chi(\mathscr{O}_X)\,.$$

(1.2) *Let $\mathscr{E}$ be a coherent sheaf of rank r on X and D be a divisor on X. Then we have*

$$C_1(\mathscr{E}(D))=C_1(\mathscr{E})+rD\,,$$

$$C_2(\mathscr{E}(D))=C_2(\mathscr{E})+(r-1)C_1(\mathscr{E})\cdot D+\frac{r(r-1)}{2}D^2\,,$$

$$C_3(\mathscr{E}(D))=C_3(\mathscr{E})+(r-2)C_2(\mathscr{E})\cdot D+\frac{(r-1)(r-2)}{2}C_1(\mathscr{E})\cdot D^2+\frac{r(r-1)(r-2)}{6}D^3\,.$$

LEMMA 1.3. *Let $\mathscr{F}$ be a coherent sheaf on X.*

(1) *If* $\dim \operatorname{Supp}\mathscr{F}\leqq 0$ *then* $C_1(\mathscr{F})=C_2(\mathscr{F})=0$, $C_3(\mathscr{F})=2h^0(\mathscr{F})$.

(2) *If* $\dim \operatorname{Supp}\mathscr{F}\leqq 1$ *then* $C_1(\mathscr{F})=0$, $C_2(\mathscr{F})\cdot D\leqq 0$ *for any base point free divisor D.*

PROOF. (1) It is obvious that $C_1=C_2=0$. By the Riemann-Roch formula for $\mathscr{F}$, $h^0(\mathscr{F})=\frac{1}{2}C_3(\mathscr{F})$. (2) It is also obvious that $C_1=0$. For a general member D' in $|D|$, $\dim \operatorname{Supp}(\mathscr{F}\otimes\mathscr{O}_{D'})\leqq 0$, and there is a short exact sequence

$$0\longrightarrow\mathscr{F}(-D')\longrightarrow\mathscr{F}\longrightarrow\mathscr{F}\otimes\mathscr{O}_{D'}\longrightarrow 0\,.$$

Since $C_2(\mathscr{F}(-D'))=C_2(\mathscr{F})$ and $C_3(\mathscr{F}(-D'))=C_3(\mathscr{F})+2C_2(\mathscr{F})\cdot D'$,

$$\begin{aligned}0&\leqq\chi(\mathscr{F}\otimes\mathscr{O}_{D'})\\&=\chi(\mathscr{F})-\chi(\mathscr{F}(-D'))\\&=-C_2(\mathscr{F})\cdot D'\,.\end{aligned}$$

LEMMA 1.4. *For a torsion free coherent sheaf $\mathscr{E}$ on X, let $\mathscr{E}^{\vee\vee}$ be the double dual of $\mathscr{E}$ and $\mathscr{F}$ be the cokernel of $\mathscr{E}\hookrightarrow\mathscr{E}^{\vee\vee}$. Then*

$$C_1(\mathscr{E}^{\vee\vee})=C_1(\mathscr{E}) ,$$
$$C_2(\mathscr{E}^{\vee\vee})=C_2(\mathscr{E})+C_2(\mathscr{F}) ,$$
$$C_3(\mathscr{E}^{\vee\vee})=C_3(\mathscr{E})+C_1(\mathscr{E})\cdot C_2(\mathscr{F})+C_3(\mathscr{F}) ,$$

and $C_2(\mathscr{E}^{\vee\vee})\cdot D\leqq C_2(\mathscr{E})\cdot D$, *for any base point free divisor* D *on* X. *Moreover if* $\dim \operatorname{Supp}\mathscr{F}\leqq 0$ *then*

$$C_2(\mathscr{E}^{\vee\vee})=C_2(\mathscr{E}) ,$$
$$C_3(\mathscr{E}^{\vee\vee})=C_3(\mathscr{E})+2h^0(\mathscr{F}) .$$

PROOF. Since $\dim \operatorname{Supp}\mathscr{F}\leqq 1$, it is clear by Lemma 1.3 and the calculation of the Chern polynomial

$$\begin{aligned} C_t(\mathscr{E}^{\vee\vee}) &= (1+C_1(\mathscr{E})t+C_2(\mathscr{E})t^2+C_3(\mathscr{E})t^3)(1+C_2(\mathscr{F})t^2+C_3(\mathscr{F})t^3) \\ &= 1+C_1(\mathscr{E})t+(C_2(\mathscr{E})+C_2(\mathscr{F}))t^2+(C_3(\mathscr{E})+C_1(\mathscr{E})\cdot C_2(\mathscr{F})+C_3(\mathscr{F}))t^3 . \end{aligned}$$

(1.5) ([H]) *For a reflexive coherent sheaf* $\mathscr{E}$ *of rank* 2 *on* X,

$$C_1(\mathscr{E})=C_1(\det\mathscr{E}) , \qquad \mathscr{E}^{\vee}\cong\mathscr{E}\otimes(\det\mathscr{E})^{-1} ,$$
$$C_3(\mathscr{E})=h^0(\mathscr{E}xt^1(\mathscr{E}, \mathscr{O}_X)) .$$

If $\mathscr{E}$ *has a section whose scheme* Y *of zeros has codimension* $\geqq 2$, *then* $C_2(\mathscr{E})=Y$ *and there is an exact sequence*

$$0\longrightarrow\mathscr{O}_X\longrightarrow\mathscr{E}\longrightarrow\mathscr{I}_Y\otimes\det\mathscr{E}\longrightarrow 0 .$$

The normal bundle of an n-dimensional rational scroll. Let $\mathscr{V}$ be a vector bundle of rank n on the projective line $\boldsymbol{P}^1$. If each component of $\mathscr{V}$ has positive degree, the tautological line bundle $\mathscr{O}_X(1)$ of the projective bundle $\pi: X=\boldsymbol{P}(\mathscr{V})\to\boldsymbol{P}^1$ is very ample and $\mathscr{O}_X(1)\otimes\pi^*\mathscr{O}_{P^1}(-1)$ is generated by global sections. The complete linear system $|\mathscr{O}_X(1)|$ (resp. $|\mathscr{O}_X(1)\otimes\pi^*\mathscr{O}_{P^1}(-1)|$) defines an embedding $X\to\boldsymbol{P}^N$ (resp. a morphism $\phi: X\to\boldsymbol{P}^{N-n}$), where $N+1=h^0(X, \mathscr{O}_X(1))=\deg\mathscr{V}+n$. The pair $(X, \mathscr{O}_X(1))$ is called an n-dimensional rational scroll. The aim of the rest of this section is to prove the following theorem.

THEOREM 1.6. *Let* $\mathscr{N}$ *be the normal bundle of an* n*-dimensional rational scroll* $(X, \mathscr{O}_X(1))$ *in* $\boldsymbol{P}^N$. *Then* $\mathscr{N}\cong\pi^*\mathscr{T}_{P^1}\otimes\phi^*\mathscr{T}_{P^{N-n}}$, *where* $\mathscr{T}_Y$ *stands for the tangent bundle of* Y. *More precisely, the pull back of the Euler sequence*

$$0\longrightarrow\mathscr{O}_{P^{N-n}}\longrightarrow\mathscr{O}_{P^{N-n}}(1)^{\oplus(N+1-n)}\longrightarrow\mathscr{T}_{P^{N-n}}\longrightarrow 0$$

by the morphism $\phi: X\to\boldsymbol{P}^{N-n}$ *is*

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\phi} \mathcal{O}_X(1)\otimes\pi^*\mathcal{O}_{P^1}(-1)^{\oplus(N+1-n)} \longrightarrow \mathcal{N}\otimes\pi^*\Omega_{P^1} \longrightarrow 0 .$$

PROOF. Let $\mathcal{F}$ be the cokernel of the composite homomorphism $\mathcal{T}_{X/P^1}\hookrightarrow\mathcal{T}_X\hookrightarrow\mathcal{T}_{P^N}\otimes\mathcal{O}_X$. Then there is the following exact sequence

$$0 \longrightarrow \pi^*\mathcal{T}_{P^1} \longrightarrow \mathcal{F} \longrightarrow \mathcal{N} \longrightarrow 0 . \tag{1.6.1}$$

On the other hand, if $\mathcal{Q}$ is the kernel of the surjective homomorphism $\mathcal{O}_{P^1}^{\oplus N+1}\to\mathcal{V}$, then $\mathcal{F}\cong(\pi^*\mathcal{Q}^\vee)\otimes\mathcal{O}_X(1)$. Since the homomorphism $\mathcal{O}_{P^1}^{\oplus N+1}\to\mathcal{V}$ implies an isomorphism of global sections and $H^1(\boldsymbol{P}^1, \mathcal{O}_{P^1})=(0)$, we have $H^0(\boldsymbol{P}^1, \mathcal{Q})=H^1(\boldsymbol{P}^1, \mathcal{Q})=(0)$. Hence we have $\mathcal{Q}\cong\mathcal{O}_{P^1}(-1)^{\oplus N+1-n}$. Tensoring (1.6.1) with $\pi^*\Omega_{P^1}$, we get

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\phi'} \mathcal{O}_X(1)\otimes\pi^*\mathcal{O}_{P^1}(-1)^{\oplus(N+1-n)} \longrightarrow \mathcal{N}\otimes\pi^*\Omega_{P^1} \longrightarrow 0 .$$

We only have to show the homomorphism ϕ' is complete. If $\mathcal{V}=\bigoplus\mathcal{O}_{P^1}(a_i)$ $(a_i>0)$, for homogeneous coordinates $(x:y)$ of $\boldsymbol{P}^1$, let

$$\psi_k=x^{(N-(k-1)-\Sigma_{i=1}^{k}a_i)}\cdot y^{(k-1+\Sigma_{i=1}^{k-1}a_i)} \qquad (k=1, \cdots, n) .$$

Then $\psi=(\psi_1, \cdots, \psi_n)$ defines a surjective homomorphism $\mathcal{V}\to\mathcal{O}_{P^1}(N)$ which implies an isomorphism of global sections. The morphism ψ corresponds to a section C of π and C is a rational normal curve of degree N in $\boldsymbol{P}^N$. The normal bundle sequence of $C\hookrightarrow X\hookrightarrow\boldsymbol{P}^N$ is

$$0 \longrightarrow \mathcal{N}_{C/X} \longrightarrow \mathcal{N}_{C/P^N} \longrightarrow \mathcal{N}\otimes\mathcal{O}_C \longrightarrow 0 .$$

The normal bundle of a rational normal curve is balanced ([K] (3.5)), that is,

$$\mathcal{N}_{C/P^N}\cong\mathcal{O}_{P^1}(N+2)^{\oplus N-1} .$$

If the morphism ϕ' were not complete, the normal bundle $\mathcal{N}$ would have $\mathcal{O}_X(1)\otimes\pi^*\mathcal{O}_{P^1}(1)$ as a direct summand and $\mathcal{N}_{C/P^N}$ would have $\mathcal{O}_X(1)\otimes\pi^*\mathcal{O}_{P^1}(1)\otimes\mathcal{O}_C\cong\mathcal{O}_{P^1}(N+1)$ as a quotient line bundle. This is a contradiction.

COROLLARY 1.7. *For $n>1$, the normal bundle of an n-dimensional rational scroll is simple.*

PROOF. Put $\mathcal{O}_X(1)\otimes\pi^*\mathcal{O}_{P^1}(-1)=\mathcal{L}$, $\mathcal{N}\otimes\pi^*\Omega_{P^1}=\mathcal{E}$ and $N+1-n=t$. Taking the dual of the sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\phi} \mathcal{L}^{\oplus t} \longrightarrow \mathcal{E} \longrightarrow 0 , \tag{1.7.1}$$

we get

(1.7.2) $$0 \longrightarrow \mathscr{E}^{\vee} \longrightarrow \mathscr{L}^{\vee \oplus t} \longrightarrow \mathscr{O}_X \longrightarrow 0 .$$

Since $n>1$, $h^i(X, \mathscr{L}^{\vee})=0$ $(i=0, 1)$. Thus $h^1(X, \mathscr{E}^{\vee})=1$. Tensoring the sequence (1.7.2) with $\mathscr{L}$, we get

$$0 \longrightarrow \mathscr{E}^{\vee} \otimes \mathscr{L} \longrightarrow \mathscr{O}_X^{\oplus t} \longrightarrow \mathscr{L} \longrightarrow 0 .$$

By the completeness of the morphism ϕ, $h^i(X, \mathscr{E}^{\vee} \otimes \mathscr{L})=0$ $(i=0, 1)$. Tensor the sequence (1.7.1) with $\mathscr{E}^{\vee}$ and take the long exact sequence of cohomology groups. Then

$$\cdots \longrightarrow H^0(X, \mathscr{E}^{\vee} \otimes \mathscr{L})^{\oplus t} \longrightarrow H^0(X, \mathscr{E}nd\, \mathscr{E}) \longrightarrow H^1(X, \mathscr{E}^{\vee}) \longrightarrow H^1(X, \mathscr{E}^{\vee} \otimes \mathscr{L})^{\oplus t} \longrightarrow \cdots .$$

Thus $h^0(X, \mathscr{E}nd\, \mathscr{E})=1$.

§ 2. Hierarchy of stable sheaves.

The notation below will be fixed.

(2.1) NOTATION. For integers a, b such that $a \leqq b \leqq 0$,
$\mathscr{V} := \mathscr{O}_{P^1}(a) \oplus \mathscr{O}_{P^1}(b) \oplus \mathscr{O}_{P^1}$.
$\pi : X = \boldsymbol{P}(\mathscr{V}) \to \boldsymbol{P}^1$.
D : a divisor on X such that $\pi_* \mathscr{O}_X(D) \cong \mathscr{V}$.
For a closed point x in $\boldsymbol{P}^1$,
$F := \pi^{-1}(x)$.
For an integer q such that $q \geqq 1-a$,
$H := D+qF$.
$p := (D \cdot H^2) = 2q+a+b$.
$\mathscr{L} := \mathscr{O}_X(-D+(p+1)F)$.

The couple (X, H) is a 3-dimensional rational scroll. We will work on X and consider H-stable sheaves of rank 2 on X. We will use the words "stable sheaf" instead of "H-stable sheaf" on X. The following theorem was settled for stable vector bundles of rank 2 ([HI] Theorem 1.5). The same proof is valid for stable reflexive sheaves of rank 2.

THEOREM 2.2. *For integers α, x and y, let $\mathscr{E}$ be a stable reflexive sheaf of rank 2 with $C_1(\mathscr{E}) = -\alpha D + (\alpha p + 1)F$ and $C_2(\mathscr{E}) = xD^2 + yD \cdot F$. If $\alpha > 0$ and $x \leqq 0$, then there are integers $l \geqq 0$ and m such that $\mathscr{E}(-lD - mF)$ has a section whose scheme Y of zeros has codimension $\geqq 2$ and there is a short exact sequence*

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}(-lD-mF) \longrightarrow \mathcal{I}_Y(-(\alpha+2l)D+(\alpha p+1-2m)F) \longrightarrow 0 .$$

And the following inequalities hold

$$\begin{aligned} & l(l+\alpha)+x \geqq 0 , \\ & y \geqq l(\alpha p+1)-(\alpha+2l)m-b\{l(l+\alpha)+x\} \\ & \quad \geqq l(\alpha p+1)-(\alpha+2l)m \\ & \quad \geqq 2l(l+\alpha)p+l . \end{aligned}$$

COROLLARY 2.3. *Let $\mathcal{E}$ be as in Theorem 2.2. If the additional assumption $y \leqq 0$ holds, then $x=y=0$ and there is a short exact sequence*

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X(-\alpha D+(\alpha p+1)F) \longrightarrow 0 .$$

In particular $\mathcal{E}$ is locally free and $C_3(\mathcal{E})=0$.

PROOF. Let l, m and Y be as in Theorem 2.2. Then one sees that $x=y=l=m=0$ and $Y=\varnothing$.

COROLLARY 2.4. *For an integer x such that $x \leqq 0$, let $\mathcal{E}$ be a stable reflexive sheaf of rank 2 with $C_1(\mathcal{E})=C_1(\mathcal{L})$ and $C_2(\mathcal{E})=xD^2+D\cdot F$. Then $x=0$ and either*

(1) *there is an exact sequence*

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}(F) \longrightarrow \mathcal{L}(2F) \longrightarrow 0 ,$$

in particular $\mathcal{E}$ is locally free and $C_3(\mathcal{E})=0$, or

(2) *there is an exact sequence*

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_Y \otimes \mathcal{L} \longrightarrow 0 ,$$

where Y is a line in a fibre of π, and $C_3(\mathcal{E})=2$, in particular $\mathcal{E}$ is not locally free.

PROOF. If l, m and Y are as in Theorem 2.2, then one sees that $x=l=0$ and $m=-1$ or 0. Now assume $m=-1$. Since $C_2(\mathcal{E}(F))=0$, one obtains $Y=\varnothing$ and an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}(F) \longrightarrow \mathcal{L}(2F) \longrightarrow 0 .$$

If $m=0$, then $C_2(\mathcal{E})=D\cdot F$, and hence Y is a line in a fibre of π. Since $\mathcal{E}xt^1(\mathcal{I}_Y \otimes \mathcal{L}, \mathcal{O}_X) \cong \mathcal{E}xt^2(\mathcal{O}_Y(-1), \mathcal{O}_X) \cong \det \mathcal{N}_{Y/X} \otimes \mathcal{O}_Y(1) \cong \mathcal{O}_Y(2)$, taking the dual of the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_Y \otimes \mathcal{L} \longrightarrow 0 ,$$

one obtains the following exact sequence

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_Y(2) \longrightarrow \mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_X) \longrightarrow 0 ,$$

and $C_3(\mathcal{E}) = h^0(\mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_X)) = 2$.

(2.5) For a stable sheaf $\mathcal{E}$ of rank 2 with $C_1(\mathcal{E}) = C_1(\mathcal{L})$, $C_2(\mathcal{E}) = D \cdot F$ and $C_3(\mathcal{E}) = 0$, denote the double dual of $\mathcal{E}$ by $\mathcal{E}^{\vee\vee}$ and the cokernel of the inclusion $\mathcal{E} \hookrightarrow \mathcal{E}^{\vee\vee}$ by $\mathcal{F}$. $\mathcal{E}$ falls into one of the three cases:

Case (I) $\mathcal{E}$ is locally free,

Case (II) $\mathcal{E}$ is not locally free but $\mathcal{E}^{\vee\vee}$ is locally free,

Case (III) neither $\mathcal{E}$ nor $\mathcal{E}^{\vee\vee}$ is locally free.

We investigate each case separately and set up the hierarchy of such stable sheaves.

LEMMA 2.6. *Under the above situation,*
Case (I): *There is an exact sequence*

$$0 \longrightarrow \mathcal{O}_X(-F) \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}(F) \longrightarrow 0 .$$

Case (II): *There is an exact sequence*

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}^{\vee\vee} \longrightarrow \mathcal{L} \longrightarrow 0 .$$

In this case, we see $C_2(\mathcal{E}^{\vee\vee}) = C_3(\mathcal{E}^{\vee\vee}) = 0$.
Case (III): *There is an exact sequence*

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}^{\vee\vee} \longrightarrow \mathcal{I}_Y \otimes \mathcal{L} \longrightarrow 0 ,$$

where Y is a line in a fibre of π. In this case, we see $C_2(\mathcal{E}^{\vee\vee}) = D \cdot F$ *and* $C_3(\mathcal{E}^{\vee\vee}) = 2$.

PROOF. Let $C_2(\mathcal{E}^{\vee\vee}) = xD^2 + yD \cdot F$. Since both divisors F and $D - aF$ are base point free, by Lemma 1.4, one obtains the inequalities $x \leqq 0$ and $bx + y \leqq 1$, and therefore $y \leqq 1$. If $y = 1$, by Corollary 2.4, one gets Case (I) or Case (III). If $y \leqq 0$, by Corollary 2.3, one gets Case (II).

LEMMA 2.7. *If Case* (II) *occurs, then either* $\mathcal{F} \cong \mathcal{O}_Y$ *or there is an exact sequence*

$$0 \longrightarrow k(x) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_Y(-1) \longrightarrow 0 ,$$

where Y is a line in a fibre of π and x is a closed point of X.

PROOF. Since $C_2(\mathcal{E}^{\vee\vee}) = C_3(\mathcal{E}^{\vee\vee}) = 0$, one sees that $C_2(\mathcal{F}) = -D \cdot F$ and $C_3(\mathcal{F}) = -1$ by Lemma 1.4. Since $C_2(\mathcal{F}) = -D \cdot F$, the codimension

2 part of Supp $\mathscr{F}$ is a line in a fibre of π. Let Y be the codimension 2 component of $(\mathrm{Supp}\,\mathscr{F})_{\mathrm{red}}$. Let $\mathscr{F}'$ be the quotient of $\mathscr{F}\otimes\mathcal{O}_Y$ modulo the torsion part of $\mathscr{F}\otimes\mathcal{O}_Y$ as $\mathcal{O}_Y$-modules, and let $\mathscr{K}$ be the kernel of the surjective homomorphism $\mathscr{F}\to\mathscr{F}'$. Since $C_2(\mathscr{F})=-D\cdot F$, the rank of $\mathscr{F}'$ as $\mathcal{O}_Y$-modules is 1, and $\dim\mathrm{Supp}\,\mathscr{K}\leqq 0$. Let $\mathscr{F}'\cong\mathcal{O}_Y(l)$ and $h^0(\mathscr{K})=n$. Since $C_3(\mathcal{O}_Y(l))=2l-1$ and $C_3(\mathscr{K})=2n$, $C_3(\mathscr{F})=2(l+n)-1$. Thus $n+l=0$. Since there is a surjective homomorphism $\mathscr{E}^{\vee\vee}\to\mathscr{F}\to\mathcal{O}_Y(l)$ and since $\mathscr{E}^{\vee\vee}\otimes\mathcal{O}_Y\cong\mathcal{O}_Y\oplus\mathcal{O}_Y(-1)$, $l\geqq -1$. If $l=0$ then $\mathscr{F}\cong\mathcal{O}_Y$. If $l=-1$ then there is an exact sequence

$$0\longrightarrow k(x)\longrightarrow\mathscr{F}\longrightarrow\mathcal{O}_Y(-1)\longrightarrow 0\,.$$

PROPOSITION 2.8. *If Case* (II) *occurs, then $\mathscr{E}$ is obtained by one of the following non-trivial extensions*

(2.8.1) $$0\longrightarrow\mathscr{I}_Y\longrightarrow\mathscr{E}\longrightarrow\mathscr{L}\longrightarrow 0\,,$$

(2.8.2) $$0\longrightarrow\mathscr{I}_x\longrightarrow\mathscr{E}\longrightarrow\mathscr{I}_Y\otimes\mathscr{L}\longrightarrow 0\,,$$

(2.8.3) $$0\longrightarrow\mathcal{O}_X\longrightarrow\mathscr{E}\longrightarrow\mathscr{I}_{Y'}\otimes\mathscr{L}\longrightarrow 0\,,$$

where Y is a line in a fibre of π, x is a closed point of X and where $\mathscr{I}_{Y'}$ is defined by the following exact sequence

$$0\longrightarrow\mathscr{I}_{Y'}\longrightarrow\mathscr{I}_Y\longrightarrow k(x)\longrightarrow 0\,.$$

Moreover, all the above extensions are locally trivial at every closed point.

PROOF. We have to consider two possibilities of $\mathscr{F}$. One is that $\mathscr{F}\cong\mathcal{O}_Y$ where Y is a line in a fibre of π. The other is that $\mathscr{F}$ is obtained by an exact sequence

$$0\longrightarrow k(x)\longrightarrow\mathscr{F}\longrightarrow\mathcal{O}_Y(-1)\longrightarrow 0\,,$$

where Y is a line in a fibre of π and x is a closed point of X. If $\mathscr{F}\cong\mathcal{O}_Y$ then there are following two exact sequences

$$0\longrightarrow\mathcal{O}_X\longrightarrow\mathscr{E}^{\vee\vee}\longrightarrow\mathscr{L}\longrightarrow 0\,,$$
$$0\longrightarrow\mathscr{E}\longrightarrow\mathscr{E}^{\vee\vee}\longrightarrow\mathcal{O}_Y\longrightarrow 0\,.$$

Since there are no surjective homomorphisms from $\mathscr{L}$ to $\mathcal{O}_Y$, one gets the extension (2.8.1). If the latter possibility happens, then there are following three exact sequences

$$0\longrightarrow\mathcal{O}_X\stackrel{f}{\longrightarrow}\mathscr{E}^{\vee\vee}\longrightarrow\mathscr{L}\longrightarrow 0\,,$$

$$0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{E}^{\vee\vee} \xrightarrow{g} \mathscr{F} \longrightarrow 0\,,$$
$$0 \longrightarrow k(x) \longrightarrow \mathscr{F} \xrightarrow{h} \mathscr{O}_Y(-1) \longrightarrow 0\,.$$

Since there are no non-zero homomorphisms from $\mathscr{O}_X$ to $\mathscr{O}_Y(-1)$, the composite homomorphism $h \circ g \circ f$ is a zero homomorphism. If $g \circ f \neq 0$ one gets the extension (2.8.2). If $g \circ f = 0$ one gets the extension (2.8.3). The last assertion is clear, since the extension

$$0 \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{E}^{\vee\vee} \longrightarrow \mathscr{L} \longrightarrow 0$$

is locally trivial.

PROPOSITION 2.9. *If Case* (III) *occurs, then* $\mathscr{E}$ *is obtained by one of the following extensions*

(2.9.1) $$0 \longrightarrow \mathscr{I}_x \longrightarrow \mathscr{E} \longrightarrow \mathscr{I}_Y \otimes \mathscr{L} \longrightarrow 0\,,$$

(2.9.2) $$0 \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{E} \longrightarrow \mathscr{I}_{Y'} \otimes \mathscr{L} \longrightarrow 0\,,$$

where the notation is the same as in Proposition 2.8. *Moreover, the above extensions are not locally trivial at a general closed point of* Y.

PROOF. Since $C_2(\mathscr{E}^{\vee\vee}) = C_2(\mathscr{E})$ and $C_3(\mathscr{E}^{\vee\vee}) = 2$, $\dim \operatorname{Supp} \mathscr{F} \leqq 0$ and $h^0(X, \mathscr{F}) = 1$ by Lemma 1.4. Therefore, there exists a closed point x of X such that $\mathscr{F} \cong k(x)$. There are following two exact sequences

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{f} \mathscr{E}^{\vee\vee} \longrightarrow \mathscr{I}_Y \otimes \mathscr{L} \longrightarrow 0\,,$$
$$0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{E}^{\vee\vee} \xrightarrow{g} k(x) \longrightarrow 0\,.$$

If $g \circ f \neq 0$ one gets the extension (2.9.1). If $g \circ f = 0$ one gets the extension (2.9.2). For the last assertion, since $\mathscr{E}$ and $\mathscr{E}^{\vee\vee}$ differ only at one closed point, what we need to prove is the same assertion for the extension

$$0 \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{E}^{\vee\vee} \longrightarrow \mathscr{I}_Y \otimes \mathscr{L} \longrightarrow 0\,.$$

It is clear because of reflexivity of $\mathscr{E}^{\vee\vee}$.

(2.10) By the proofs and the last assertions of propositions (2.8) and (2.9), we can draw the following diagram

$$\begin{array}{ccc} (2.8.2) & \longleftarrow & (2.9.1) \\ \downarrow & & \downarrow \\ (2.8.3) & \longleftarrow & (2.9.2)\,, \end{array}$$

where arrows indicate specializations. The vertical arrows are given by the composite homomorphism $g \circ f$ being a zero homomorphism. The horizontal arrows are given by the local triviality of the extensions. In the next section, we prove that each stable sheaf of type (2.8.1) is a limit of stable vector bundles. We also prove that a stable sheaf of type (2.8.2) is a limit of stable sheaves of type (2.8.1) (Proposition 3.11).

§ 3. Limits of extensions.

In this section, we establish the remaining hierarchy of stable sheaves in the last section and prove the main theorem. First we construct limits of extensions in a general situation.

(3.1) Let X be a scheme over k and $\mathscr{E}$, $\mathscr{F}_i$, $\mathscr{G}_i$ $(i=1, 2)$ and $\mathscr{T}$ be coherent $\mathscr{O}_X$-modules which fit in the following exact commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{F}_2 & \xrightarrow{f} & \mathscr{G}_2 & \xrightarrow{g} & \mathscr{T} & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle i} & & \\
0 & \longrightarrow & \mathscr{F}_2 & \longrightarrow & \mathscr{E} & \longrightarrow & \mathscr{F}_1 & \longrightarrow & 0\,. \\
 & & & & \uparrow & & \uparrow{\scriptstyle h} & & \\
 & & & & \mathscr{G}_1 & = & \mathscr{G}_1 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The commutative diagram

$$\begin{array}{ccc}
\mathrm{Ext}^1(\mathscr{F}_1, \mathscr{F}_2) & \longrightarrow & \mathrm{Ext}^1(\mathscr{F}_1, \mathscr{G}_2) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^1(\mathscr{G}_1, \mathscr{F}_2) & \longrightarrow & \mathrm{Ext}^1(\mathscr{G}_1, \mathscr{G}_2)
\end{array}$$

defines a homomorphism

$$c : \mathrm{Ext}^1(\mathscr{F}_1, \mathscr{F}_2) \longrightarrow \mathrm{Ext}^1(\mathscr{G}_1, \mathscr{G}_2)\,.$$

For elements $\xi \in \mathrm{Ext}^1(\mathscr{F}_1, \mathscr{F}_2)$ and $c(\xi) \in \mathrm{Ext}^1(\mathscr{G}_1, \mathscr{G}_2)$, we consider the corresponding extensions

$$\xi : 0 \longrightarrow \mathscr{F}_2 \xrightarrow{\ j\ } \mathscr{E}_\xi \xrightarrow{\ k\ } \mathscr{F}_1 \longrightarrow 0\,,$$
$$c(\xi) : 0 \longrightarrow \mathscr{G}_2 \longrightarrow \mathscr{E}_{c(\xi)} \longrightarrow \mathscr{G}_1 \longrightarrow 0\,.$$

(3.2) In the above situation, we can form a complex of coherent sheaves, which is a so called monad ([BH]), over $X\times \boldsymbol{P}^1$ as follows

$$\mathscr{F}_2 \boxtimes \mathscr{O}_{\boldsymbol{P}^1} \xrightarrow{(f,\,-j)} (\mathscr{G}_2 \oplus \mathscr{E}_\xi)\boxtimes \mathscr{O}_{\boldsymbol{P}^1} \xrightarrow{\begin{pmatrix} y\cdot g \\ x\cdot i\circ k\end{pmatrix}} \mathscr{T}\boxtimes \mathscr{O}_{\boldsymbol{P}^1}(1),$$

where x, y are homogeneous coordinates of $\boldsymbol{P}^1$. Let E denote its cohomology sheaf. E is flat over $\boldsymbol{P}^1$ since the cokernel of the first arrow is the pull back of a coherent sheaf on X and the second arrow is surjective. Let E_0 and E_∞ be the restrictions of E to the fibres over $x=0$ and $y=0$, respectively. It is easy to see that $E_0 \cong \mathscr{E}_\xi$ and $E_\infty \cong \mathscr{E}_{c(\xi)}$ by the construction. Denote the kernel of the composite homomorphism $E\otimes p^*\mathscr{O}_{\boldsymbol{P}^1}(1)\to E_\infty \to \mathscr{G}_1$ by E', where p is the projection $X\times\boldsymbol{P}^1\to\boldsymbol{P}^1$. By the exact commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathscr{G}_1 & = & \mathscr{G}_1 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & E & \longrightarrow & E\otimes p^*\mathscr{O}_{\boldsymbol{P}^1}(1) & \longrightarrow & E_\infty & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & E & \longrightarrow & E' & \longrightarrow & \mathscr{G}_2 & \longrightarrow & 0\,, \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

E is regained from E'. This process is a so called elementary transformation ([M1]).

Proposition 3.3. *There is an exact sequence*

$$0 \longrightarrow \mathscr{F}_2\boxtimes\mathscr{O}_{\boldsymbol{P}^1}(1) \longrightarrow E' \longrightarrow \mathscr{F}_1\boxtimes\mathscr{O}_{\boldsymbol{P}^1} \longrightarrow 0$$

and $E'_\infty \cong \mathscr{E}$ *on the fibre at* $y=0$.

Proof. Let $\mathscr{E}_1$ be the cokernel of the homomorphism $(f,-j)\colon \mathscr{F}_2 \to \mathscr{G}_2\oplus\mathscr{E}_\xi$. Then one gets an exact sequence

$$0 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{E}_1 \longrightarrow \mathscr{T} \oplus \mathscr{F}_1 \longrightarrow 0 .$$

Let L be the kernel of the homomorphism

$$\begin{pmatrix} y\cdot \mathrm{id} \\ x\cdot i \end{pmatrix} : (\mathscr{T} \oplus \mathscr{F}_1) \boxtimes \mathscr{O}_{P^1} \longrightarrow \mathscr{T} \boxtimes \mathscr{O}_{P^1}(1) .$$

Then one gets the following exact commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & L & \longrightarrow & (\mathscr{T} \oplus \mathscr{F}_1) \boxtimes \mathscr{O}_{P^1} & \longrightarrow & \mathscr{T} \boxtimes \mathscr{O}_{P^1}(1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & E & \longrightarrow & \mathscr{E}_1 \boxtimes \mathscr{O}_{P^1} & \longrightarrow & \mathscr{T} \boxtimes \mathscr{O}_{P^1}(1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathscr{F}_2 \boxtimes \mathscr{O}_{P^1} & = & \mathscr{F}_2 \boxtimes \mathscr{O}_{P^1} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

and $L_\infty \cong \mathscr{T} \oplus \mathscr{G}_1$ on the fibre at $y=0$. Let L' be the kernel of the composite homomorphism $L \otimes p^* \mathscr{O}_{P^1}(1) \to L_\infty \to \mathscr{G}_1$. Then one gets the following exact sequence

$$0 \longrightarrow \mathscr{F}_2 \boxtimes \mathscr{O}_{P^1}(1) \longrightarrow E' \longrightarrow L' \longrightarrow 0 .$$

Now consider the following exact commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{G}_1 & \longrightarrow & \mathscr{F}_1 & \longrightarrow & \mathscr{T} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & L \otimes p^* \mathscr{O}_{P^1}(1) & \longrightarrow & (\mathscr{T} \oplus \mathscr{F}_1) \boxtimes \mathscr{O}_{P^1}(1) & \xrightarrow{\begin{pmatrix} y\cdot \mathrm{id} \\ x\cdot i \end{pmatrix}} & \mathscr{T} \boxtimes \mathscr{O}_{P^1}(2) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle (\mathrm{id},\, y\cdot \mathrm{id})} & & \uparrow{\scriptstyle y} & & \\
0 & \longrightarrow & L' & \longrightarrow & \mathscr{T} \boxtimes \mathscr{O}_{P^1}(1) \oplus \mathscr{F}_1 \boxtimes \mathscr{O}_{P^1} & \xrightarrow{\begin{pmatrix} \mathrm{id} \\ x\cdot i \end{pmatrix}} & \mathscr{T} \boxtimes \mathscr{O}_{P^1}(1) & \longrightarrow & 0 . \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since the bottom row splits, $L' \cong \mathscr{F}_1 \boxtimes \mathcal{O}_{P^1}$. The last assertion is obtained by restricting the upper half of the following exact commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{F}_2 & \longrightarrow & \mathscr{G}_2 & \longrightarrow & \mathscr{T} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{F}_2 \boxtimes \mathcal{O}_{P^1}(1) & \longrightarrow & E' & \longrightarrow & \mathscr{F}_1 \boxtimes \mathcal{O}_{P^1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{F}_2 \boxtimes \mathcal{O}_{P^1} & \longrightarrow & E & \longrightarrow & L & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

to the fibre at $y=0$.

(3.4) Let (X, H) be a non-singular polarized variety over k. For torsion free coherent sheaves $\mathscr{F}_1$ and $\mathscr{F}_2$ of rank 1 with $\deg_H \mathscr{F}_1 > \deg_H \mathscr{F}_2$, we consider a non-trivial extension

(3.4.1) $$0 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{E} \longrightarrow \mathscr{F}_1 \longrightarrow 0 .$$

If $\mathscr{E}$ is unstable, then $\mathscr{E}$ has a unique rank 1 subsheaf $\mathscr{G}_1$, which we call the destabilizing subsheaf, such that the quotient $\mathscr{G}_2 = \mathscr{E}/\mathscr{G}_1$ is torsion free and $\deg_H \mathscr{G}_1 > \deg_H \mathscr{G}_2$. Then we get the following exact commutative diagram

(3.4.2) $$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{F}_2 & \longrightarrow & \mathscr{G}_2 & \longrightarrow & \mathscr{T} & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{F}_2 & \longrightarrow & \mathscr{E} & \longrightarrow & \mathscr{F}_1 & \longrightarrow & 0 , \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathscr{G}_1 & = & \mathscr{G}_1 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $\mathscr{T} \cong \mathscr{F}_1/\mathscr{G}_1 \cong \mathscr{G}_2/\mathscr{F}_2$ is a non-zero torsion sheaf. We call the above diagram *the destabilizing diagram* of the unstable extension (3.4.1). Applying the functor Hom(, $\mathscr{F}_2$) to the sequence

$$0 \longrightarrow \mathscr{G}_1 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{T} \longrightarrow 0\ ,$$

we have the long exact sequence

$$\cdots \longrightarrow \mathrm{Hom}(\mathscr{G}_1, \mathscr{F}_2) \longrightarrow \mathrm{Ext}^1(\mathscr{T}, \mathscr{F}_2) \longrightarrow \mathrm{Ext}^1(\mathscr{F}_1, \mathscr{F}_2) \longrightarrow \mathrm{Ext}^1(\mathscr{G}_1, \mathscr{F}_2) \longrightarrow \cdots .$$

For a non-trivial extension (3.4.1), $\mathscr{G}_1$ lifts as a subsheaf of $\mathscr{E}$ if and only if the image of the extension (3.4.1) in $\mathrm{Ext}^1(\mathscr{G}_1, \mathscr{F}_2)$ is trivial. Since $\deg_H \mathscr{G}_1 > \deg_H \mathscr{F}_2$, $\mathrm{Hom}(\mathscr{G}_1, \mathscr{F}_2) = (0)$. Thus each non-trivial extension

$$0 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{G}_2 \longrightarrow \mathscr{T} \longrightarrow 0\ ,$$

where $\mathscr{G}_2$ has no torsion, corresponds to precisely a destabilizing diagram (3.4.2). If we assume $\mathrm{Hom}(\mathscr{F}_2, \mathscr{F}_1) = (0)$, then for a non-trivial extension (3.4.1), $\mathscr{E}$ is simple. The projective space $M = \boldsymbol{P}(\mathrm{Ext}^1(\mathscr{F}_1, \mathscr{F}_2)^\vee)$ parametrizes mutually distinct sheaves. If the unstable locus of M is a non-empty proper closed subscheme A, to compactify the open set $U = M \backslash A$ by attaching stable sheaves, you may replace each closed point x of A by the projective space $\boldsymbol{P}(c(\mathrm{Ext}^1(\mathscr{G}_1, \mathscr{G}_2))^\vee)$ associated to the destabilizing diagram of the unstable extension corresponding to x. It may also happen that $\boldsymbol{P}(c(\mathrm{Ext}^1(\mathscr{G}_1, \mathscr{G}_2))^\vee)$ has unstable points. In such cases, the same steps should be proceeded.

Now we return to the situation in section 2. Let (X, H) be a 3-dimensional rational scroll and $\mathscr{L} = \mathscr{O}_X(-D + (p+1)F)$ as in (2.1).

LEMMA 3.5. *For an unstable extension*

$$0 \longrightarrow \mathscr{O}_X(-F) \longrightarrow \mathscr{E} \longrightarrow \mathscr{L}(F) \longrightarrow 0\ ,$$

the destabilizing diagram is as follows

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{O}_X(-F) & \longrightarrow & \mathscr{J}_Y & \longrightarrow & \mathscr{O}_F(-1) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{O}_X(-F) & \longrightarrow & \mathscr{E} & \longrightarrow & \mathscr{L}(F) & \longrightarrow & 0\ , \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathscr{L} & = & \mathscr{L} & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where Y is a line in a fibre F of π.

PROOF. Let $\mathscr{G}_1$ be the destabilizing subsheaf of $\mathscr{E}$. Since $\mathscr{E}$ is locally free, $\mathscr{G}_1$ is also locally free. Since $\deg_H \mathscr{G}_1 = 1$, one sees that $\mathscr{G}_1 \cong \mathscr{L}$. $\mathscr{E} \otimes \mathscr{L}^{-1}$ has a section whose scheme Y of zeros has codimension $\geqq 2$. Since $C_2(\mathscr{E} \otimes \mathscr{L}^{-1}) = D \cdot F$, Y is a line in a fibre F of π and the quotient $\mathscr{G}_2 = \mathscr{E}/\mathscr{L} \cong \mathscr{I}_Y$.

LEMMA 3.6. *Let Y be a line in a fibre of π. For an unstable extension*

$$0 \longrightarrow \mathscr{I}_Y \longrightarrow \mathscr{E} \longrightarrow \mathscr{L} \longrightarrow 0,$$

the destabilizing diagram is as follows

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{I}_Y & \longrightarrow & \mathscr{I}_x & \longrightarrow & \mathcal{O}_Y(-1) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{I}_Y & \longrightarrow & \mathscr{E} & \longrightarrow & \mathscr{L} & \longrightarrow & 0, \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathscr{L} \otimes \mathscr{I}_Y & = & \mathscr{L} \otimes \mathscr{I}_Y & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where x is a closed point of Y.

PROOF. Let $\mathscr{G}_1$ be the destabilizing subsheaf of $\mathscr{E}$. Since $\deg_H \mathscr{G}_1 = 1$, one sees that $\mathscr{G}_1 \cong \mathscr{L} \otimes \mathscr{I}_Z$ where Z is a closed subscheme X of codimension $\geqq 2$ and the quotient $\mathscr{G}_2 = \mathscr{E}/\mathscr{G}_1$ is the ideal sheaf of a closed subscheme Z' of Y. The quotient $\mathscr{T} = \mathscr{I}_{Z'}/\mathscr{I}_Y$ is the ideal sheaf of Z' in Y, therefore $\mathscr{I}_{Z'}/\mathscr{I}_Y \cong \mathcal{O}_Y(-\text{length } Z')$, and this is also isomorphic to the quotient $\mathscr{L}/\mathscr{L} \otimes \mathscr{I}_Z$. Thus one sees that $Z = Y$ and length $Z' = 1$.

LEMMA 3.7. *Let Y be a line in a fibre of π and x be a closed point of X. For an unstable extension*

$$0 \longrightarrow \mathscr{I}_x \longrightarrow \mathscr{E} \longrightarrow \mathscr{L} \otimes \mathscr{I}_Y \longrightarrow 0,$$

the destabilizing diagram is as follows

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{I}_x & \longrightarrow & \mathscr{O}_X & \longrightarrow & k(x) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{I}_x & \longrightarrow & \mathscr{E} & \longrightarrow & \mathscr{L}\otimes\mathscr{I}_Y & \longrightarrow & 0\,, \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathscr{L}\otimes\mathscr{I}_{Y'} & = & \mathscr{L}\otimes\mathscr{I}_{Y'} & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $\mathscr{I}_{Y'}$ *is an ideal sheaf defined by an exact sequence*

$$0 \longrightarrow \mathscr{I}_{Y'} \longrightarrow \mathscr{I}_Y \longrightarrow k(x) \longrightarrow 0 .$$

PROOF. The proof is almost the same as the above and easy.

LEMMA 3.8. *Let* $\mathscr{I}_{Y'}$ *be an ideal sheaf defined by an exact sequence*

$$0 \longrightarrow \mathscr{I}_{Y'} \longrightarrow \mathscr{I}_Y \longrightarrow k(x) \longrightarrow 0 ,$$

where Y *is a line in a fibre of* π *and* x *is a closed point of* X. *If a coherent sheaf* $\mathscr{E}$ *is obtained by a non-trivial extension*

$$0 \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{E} \longrightarrow \mathscr{L}\otimes\mathscr{I}_{Y'} \longrightarrow 0 ,$$

then $\mathscr{E}$ *is stable.*

PROOF. If $\mathscr{E}$ were unstable, considering the destabilizing diagram, the quotient $\mathscr{G}_2=\mathscr{E}/\mathscr{G}_1$ would be an ideal sheaf which contains $\mathscr{O}_X$ as a subsheaf. This contradicts to the non-triviality of the extension.

LEMMA 3.9. *For the destabilizing diagram in Lemma* 3.5, *the homomorphism*

$$c : \operatorname{Ext}^1(\mathscr{L}(F), \mathscr{O}_X(-F)) \longrightarrow \operatorname{Ext}^1(\mathscr{L}, \mathscr{I}_Y)$$

is surjective and

$$\dim \operatorname{Ext}^1(\mathscr{L}(F), \mathscr{O}_X(-F)) = N+1 ,$$
$$\dim \operatorname{Ext}^1(\mathscr{L}, \mathscr{I}_Y) = N-3 ,$$

where $N=3p-a-b+5$.

PROOF. Since $\mathscr{L}(F)$ is locally free and $\pi_*\mathscr{O}_X(D)=\mathscr{O}_{P^1}(a)\oplus\mathscr{O}_{P^1}(b)\oplus\mathscr{O}_{P^1}$, $\dim \mathrm{Ext}^1(\mathscr{L}(F), \mathscr{O}_X(-F))=\dim H^1(X, \mathscr{L}^{-1}(-2F))=\dim H^1(\boldsymbol{P}^1, \pi_*\mathscr{L}^{-1}(-2F))=-\deg \pi_*\mathscr{L}^{-1}(-2F)-3=3p-a-b+6$. The others are obtained from the exact sequences

$$0\longrightarrow \mathscr{L}^{-1}(-2F)\longrightarrow \mathscr{L}^{-1}(-F)\longrightarrow \mathscr{O}_F(1)\longrightarrow 0\ ,$$
$$0\longrightarrow \mathscr{L}^{-1}(-F)\longrightarrow \mathscr{L}^{-1}\otimes\mathscr{I}_Y\longrightarrow \mathscr{O}_F\longrightarrow 0\ .$$

LEMMA 3.10. *For the destabilizing diagram in Lemma* 3.6, *the homomorphism*

$$c:\ \mathrm{Ext}^1(\mathscr{L}, \mathscr{I}_Y)\longrightarrow \mathrm{Ext}^1(\mathscr{L}\otimes\mathscr{I}_Y, \mathscr{I}_x)$$

is not surjective. More precisely

$$\dim \mathrm{Ext}^1(\mathscr{L}\otimes\mathscr{I}_Y, \mathscr{I}_x)=N-1\ ,$$
$$\dim \mathrm{Im}(c)=N-5\ .$$

And $\mathrm{Im}(c)$ *consists of extensions of type* (2.8.2).

PROOF. By virtue of the spectral sequence of local and global Ext, there is an exact sequence

(3.10.1) $$0\longrightarrow H^1(X, \mathscr{H}om(\mathscr{L}\otimes\mathscr{I}_Y, \mathscr{I}_x))\longrightarrow \mathrm{Ext}^1(\mathscr{L}\otimes\mathscr{I}_Y, \mathscr{I}_x)$$
$$\longrightarrow H^0(X, \mathscr{E}xt^1(\mathscr{L}\otimes\mathscr{I}_Y, \mathscr{I}_x))\longrightarrow H^2(X, \mathscr{H}om(\mathscr{L}\otimes\mathscr{I}_Y, \mathscr{I}_x))\ .$$

Now apply the functor $\mathscr{H}om(\mathscr{L}\otimes\mathscr{I}_Y,\quad)$ to the sequence

$$0\longrightarrow \mathscr{I}_x\longrightarrow \mathscr{O}_X\longrightarrow k(x)\longrightarrow 0\ .$$

Since $x\in Y$, it is easy to see that $\mathscr{H}om(\mathscr{L}\otimes\mathscr{I}_Y, \mathscr{I}_x)\cong\mathscr{L}^{-1}$, $\mathscr{H}om(\mathscr{L}\otimes\mathscr{I}_Y, k(x))\cong k(x)^{\oplus 2}$ and $\mathscr{E}xt^1(\mathscr{L}\otimes\mathscr{I}_Y, k(x))\cong k(x)$. Thus there is an exact sequence

$$0\longrightarrow k(x)^{\oplus 2}\longrightarrow \mathscr{E}xt^1(\mathscr{L}\otimes\mathscr{I}_Y, \mathscr{I}_x)\longrightarrow \mathscr{O}_Y(2)\longrightarrow k(x)\longrightarrow 0\ .$$

Therefore $\dim H^0(\mathscr{E}xt^1(\mathscr{L}\otimes\mathscr{I}_Y, \mathscr{I}_x))=4$. Since $H^2(\mathscr{L}^{-1})=0$, $\dim \mathrm{Ext}^1(\mathscr{L}\otimes\mathscr{I}_Y, \mathscr{I}_x)=\dim H^1(\mathscr{L}^{-1})+4=N-1$. Since $\mathrm{Ext}^1(\mathscr{L}, \mathscr{O}_Y(-1))\cong H^1(X, \mathscr{L}^{-1}\otimes\mathscr{O}_Y(-1))=(0)$, the homomorphism $\mathrm{Ext}^1(\mathscr{L}, \mathscr{I}_Y)\to\mathrm{Ext}^1(\mathscr{L}, \mathscr{I}_x)$ is surjective. Therefore the image of c coincides with that of the homomorphism

(3.10.2) $$\mathrm{Ext}^1(\mathscr{L}, \mathscr{I}_x)\longrightarrow \mathrm{Ext}^1(\mathscr{L}\otimes\mathscr{I}_Y, \mathscr{I}_x)\ .$$

Applying the functor $\mathscr{H}om(\quad, \mathscr{I}_x)$ to the sequence

$$0\longrightarrow \mathscr{L}\otimes\mathscr{I}_Y\longrightarrow \mathscr{L}\longrightarrow \mathscr{O}_Y(-1)\longrightarrow 0\ ,$$

we see that $\mathscr{E}xt^1(\mathcal{O}_Y(-1), \mathscr{I}_x) \cong k(x)$. Since $\mathrm{Hom}(\mathscr{L} \otimes \mathscr{I}_Y, \mathscr{I}_x) = (0)$ and $\mathrm{Ext}^1(\mathcal{O}_Y(-1), \mathscr{I}_x) \cong H^0(X, k(x)) = k$, the dimension of the image of the homomorphism (3.10.2) is $\dim \mathrm{Ext}^1(\mathscr{L}, \mathscr{I}_x) - 1 = N-5$. Since all extensions in the image of (3.10.2) are locally trivial, by the dimension estimate, we can see the last assertion.

PROPOSITION 3.11. *Let Y be a line in a fibre of π and x be a closed point of X.*

(1) *Every stable sheaf of type* (2.8.1) *is a limit of stable vector bundles.*

(2) *If $x \in Y$ every stable sheaf of type* (2.8.2) *is a limit of stable sheaves of type* (2.8.1).

PROOF. (1) Combining the sequences

$$0 \longrightarrow \mathcal{O}_X(-F) \longrightarrow \mathscr{I}_Y \longrightarrow \mathcal{O}_F(-1) \longrightarrow 0 \,,$$
$$0 \longrightarrow \mathscr{L} \longrightarrow \mathscr{L}(F) \longrightarrow \mathcal{O}_F(-1) \longrightarrow 0 \,,$$

one gets a destabilizing diagram as in Lemma 3.5. The assertion follows from Lemma 3.9.

(2) Use the sequences

$$0 \longrightarrow \mathscr{I}_Y \longrightarrow \mathscr{I}_x \longrightarrow \mathcal{O}_Y(-1) \longrightarrow 0 \,,$$
$$0 \longrightarrow \mathscr{L} \otimes \mathscr{I}_Y \longrightarrow \mathscr{L} \longrightarrow \mathcal{O}_Y(-1) \longrightarrow 0 \,,$$

and Lemma 3.10.

LEMMA 3.12. *Let Y be a line in a fibre of π and x be a closed point of X, then*

$$\dim \mathrm{Ext}^1(\mathscr{L} \otimes \mathscr{I}_Y, \mathscr{I}_x) = N-1 \,.$$

PROOF. If $x \in Y$ then it has been already shown in Lemma 3.10. Now assume $x \notin Y$, then $\mathscr{E}xt^1(\mathscr{L} \otimes \mathscr{I}_Y, \mathscr{I}_x) \cong \mathcal{O}_Y(2)$. By the sequence (3.10.1), $\dim \mathrm{Ext}^1(\mathscr{L} \otimes \mathscr{I}_Y, \mathscr{I}_x) = \dim H^1(X, \mathscr{L}^{-1} \otimes \mathscr{I}_x) + \dim H^0(X, \mathcal{O}_Y(2)) = N-4+3 = N-1$.

THEOREM 3.13. *Let M be the moduli of stable sheaves of rank 2 with $C_1 = C_1(\mathscr{L})$, $C_2 = D \cdot F$ and $C_3 = 0$. M has two irreducible components M_0 and M_1. The difference $M_0 \setminus M_1$ consists of all vector bundles and all stable sheaves of type* (2.8.1). *The intersection $M_0 \cap M_1$ contains at least all stable sheaves described in Proposition* 3.11, (2). *The dimension of M_0 is N and that of M_1 is $N+4$.*

PROOF. In ([HI] Theorem 3.19), we showed that the moduli of stable vector bundles of rank 2 and the given Chern classes is an open subscheme of the projective space $\boldsymbol{P}(\mathrm{Ext}^1(\mathscr{L}(F), \mathscr{O}_X(-F))^\vee)$. Let M_0 be its closure in M. The dimension of M_0 is N by Lemma 3.9. Since $\dim \mathrm{Ext}^1(\mathscr{L}\otimes\mathscr{I}_Y, \mathscr{I}_x)=N-1$ for every line Y in a fibre of π and every closed point x of X by Lemma 3.12, all the stable sheaves of type (2.9.1) form an irreducible family of dimension $(N-2)+3+3=N+4$. Let M_1 be its closure in M. By the hierarchy of stable sheaves (2.10), M_1 consists of all stable sheaves of types (2.9.1), (2.9.2), (2.8.2) and (2.8.3). The statements for $M_0\setminus M_1$ and $M_0\cap M_1$ are easily shown by using Proposition 3.11. By Lemma 2.6 and propositions (2.8) and (2.9), we see that $M=M_0\cup M_1$.

In the next section, we will construct a family of stable sheaves for $M_0\setminus M_1$ by the method (3.4). We don't proceed the same step to compactify the family. In section 5, we will construct a family of stable sheaves for M_1.

§ 4. Construction of a family for $M_0\setminus M_1$.

In addition to the notation (2.1), we introduce the following notation.

(4.1) NOTATION.

$\mathscr{V}^\vee$: the dual of the vector bundle $\mathscr{V}=\mathscr{O}_{P^1}(a)\oplus\mathscr{O}_{P^1}(b)\oplus\mathscr{O}_{P^1}$.

$\tilde{\pi}: \tilde{X}=\boldsymbol{P}(\mathscr{V}^\vee)\to \boldsymbol{P}^1$.

$\tilde{D}$: a divisor on $\tilde{X}$ such that $\tilde{\pi}_*\mathscr{O}_{\tilde{X}}(\tilde{D})\cong\mathscr{V}^\vee$.

For a closed point x in $\boldsymbol{P}^1$,

$\tilde{F}:=\tilde{\pi}^{-1}(x)$.

$\tilde{H}:=\tilde{D}+(p+1)\tilde{F}$.

$\tilde{Y}\subset\tilde{X}\times X$: the universal family of all lines in fibres of π defined in ([HI] (3.4)).

$\mathscr{I}_{\tilde{Y}}$: the ideal sheaf of $\tilde{Y}$ in $\tilde{X}\times X$.

$q:\tilde{X}\times X\to X$, $\tilde{q}:\tilde{X}\times X\to\tilde{X}$: the projections.

$\boldsymbol{P}:=\boldsymbol{P}(\mathrm{Ext}^1(\mathscr{L}(F), \mathscr{O}_X(-F))^\vee)$.

(4.2) For the convenience of the reader, we summarize the results in ([HI] § 3).

On $\boldsymbol{P}\times X$, there is an exact sequence

$$0\longrightarrow\mathscr{O}_P(1)\boxtimes\mathscr{O}_X(-F)\longrightarrow\mathscr{E}_P\longrightarrow\mathscr{O}_P\boxtimes\mathscr{L}(F)\longrightarrow 0\,. \tag{4.2.1}$$

There is a morphism

$$\Psi : \tilde{X} \longrightarrow \boldsymbol{P}$$

defined by the complete linear system $|\tilde{H}|$. *Let* $\mathscr{E}_{\tilde{X}}$ *be the pull back of* $\mathscr{E}_{\boldsymbol{P}}$ *by the morphism* $\Psi \times \mathrm{id} : \tilde{X} \times X \to \boldsymbol{P} \times X$. *Then we have an exact sequence*

$$(4.2.2) \qquad 0 \longrightarrow \mathscr{O}_X(-\tilde{F}) \boxtimes \mathscr{L} \longrightarrow \mathscr{E}_{\tilde{X}} \longrightarrow \mathscr{I}_{\tilde{Y}} \otimes \tilde{q}^* \mathscr{O}_{\tilde{X}}(\tilde{H}+\tilde{F}) \longrightarrow 0 .$$

Put $\boldsymbol{P} \setminus \Psi(\tilde{X}) = U$. *Let* $\mathscr{E}_U$ *be the restriction of* $\mathscr{E}_{\boldsymbol{P}}$ *to* $U \times X$. $(U, \mathscr{E}_U)$ *is a universal family of stable vector bundles of rank* 2 *with* $C_1 = C_1(\mathscr{L})$, $C_2 = D \cdot F$ *and* $C_3 = 0$.

(4.3) Let $\bar{\boldsymbol{P}} \to \boldsymbol{P}$ be the blow up of $\boldsymbol{P}$ along $\Psi(\tilde{X})$ and $\bar{X}$ be the exceptional divisor. Now pull back the sequences (4.2.1) and (4.2.2) to $\bar{\boldsymbol{P}} \times X$ and $\bar{X} \times X$, respectively, to obtain

$$(4.3.1) \qquad 0 \longrightarrow \mathscr{O}_{\bar{\boldsymbol{P}}}(1) \boxtimes \mathscr{O}_X(-F) \longrightarrow \mathscr{E}_{\bar{\boldsymbol{P}}} \longrightarrow \mathscr{O}_{\bar{\boldsymbol{P}}} \boxtimes \mathscr{L}(F) \longrightarrow 0 ,$$

$$(4.3.2) \qquad 0 \longrightarrow \mathscr{O}_{\bar{X}}(-\bar{F}) \boxtimes \mathscr{L} \longrightarrow \mathscr{E}_{\bar{X}} \longrightarrow \mathscr{I}_{\bar{Y}} \otimes \bar{q}^* \mathscr{O}_{\bar{X}}(\bar{H}+\bar{F}) \longrightarrow 0,$$

where $\bar{q} : \bar{X} \times X \to \bar{X}$ is the projection. Let $\mathscr{F}$ be the kernel of the surjective homomorphism

$$\mathscr{E}_{\bar{\boldsymbol{P}}} \longrightarrow \mathscr{E}_{\bar{X}} \longrightarrow \mathscr{I}_{\bar{Y}} \otimes \bar{q}^* \mathscr{O}_{\bar{X}}(\bar{H}+\bar{F}) .$$

We are interested in what happened on $\bar{X} \times X$. Let's consider the following exact commutative diagrams

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathscr{J} & = & \mathscr{J} & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{E}_{\bar{\boldsymbol{P}}}(-\bar{X} \times X) & \longrightarrow & \mathscr{E}_{\bar{\boldsymbol{P}}} & \longrightarrow & \mathscr{E}_{\bar{X}} & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{E}_{\bar{\boldsymbol{P}}}(-\bar{X} \times X) & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{O}_{\bar{X}}(-F) \boxtimes \mathscr{L} & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

and

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathscr{O}_{\bar X}(-F)\boxtimes\mathscr{L} & = & \mathscr{O}_{\bar X}(-F)\boxtimes\mathscr{L} & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{F}(-\bar X\times X) & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{F}_{\bar X} & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{F}(-\bar X\times X) & \longrightarrow & \mathscr{E}_{\bar F}(-\bar X\times X) & \longrightarrow & \mathscr{J}\otimes\mathscr{O}_{\bar X\times X}(-\bar X\times X) & \longrightarrow & 0\,, \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where $\mathscr{F}_{\bar X}$ is the restriction of $\mathscr{F}$ to $\bar X\times X$ and $\mathscr{J}=\mathscr{I}_{\bar Y}\otimes\bar q^*\mathscr{O}_{\bar X}(\bar H\times\bar F)$. Let $\mathscr{O}_{\bar X}(1)$ be the tautological line bundle of $\bar X=\boldsymbol{P}(\mathscr{N}^\vee)$, where $\mathscr{N}$ is the normal bundle of $\Psi(\tilde X)$ in $\boldsymbol{P}$. Since $\mathscr{O}_{\bar X\times X}(-\bar X\times X)\cong\bar q^*\mathscr{O}_{\bar X}(1)$, the right column of the above diagram

$$
(4.3.3)\qquad 0\longrightarrow\mathscr{J}\otimes\mathscr{O}_{\bar X\times X}(-\bar X\times X)\longrightarrow\mathscr{F}_{\bar X}\longrightarrow\mathscr{O}_{\bar X}(-\bar F)\boxtimes\mathscr{L}\longrightarrow 0
$$

defines an element of

$$
\begin{aligned}
&\mathrm{Ext}^1(\mathscr{O}_{\bar X}(-\bar F)\boxtimes\mathscr{L},\ \mathscr{I}_{\bar Y}\otimes\bar q^*\mathscr{O}_{\bar X}(\bar H+\bar F)\otimes\bar q^*\mathscr{O}_{\bar X}(1))\\
&\quad\cong H^1(\bar X\times X,\ \mathscr{I}_{\bar Y}\otimes((\mathscr{O}_{\bar X}(\bar H+2\bar F)\otimes\mathscr{O}_{\bar X}(1))\boxtimes\mathscr{L}^{-1}))\\
&\quad\cong H^0(\bar X,\ \mathscr{R}^1\bar q_*(\mathscr{I}_{\bar Y}\otimes(\mathscr{O}_{\bar X}\boxtimes\mathscr{L}^{-1}))\otimes\mathscr{O}_{\bar X}(\bar H+2\bar F)\otimes\mathscr{O}_{\bar X}(1))\ .
\end{aligned}
$$

The last isomorphism follows from the vanishing of $\bar q_*(\mathscr{I}_{\bar Y}\otimes(\mathscr{O}_{\bar X}\boxtimes\mathscr{L}^{-1}))$. By the commutative diagram

$$
\begin{array}{ccc}
\bar X & \xrightarrow{\ \bar q\ } & \bar X\times X \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
\tilde X & \xrightarrow[\ \tilde q\ ]{} & \tilde X\times X\,,
\end{array}
$$

$\mathscr{R}^1\bar q_*(\mathscr{I}_{\bar Y}\otimes(\mathscr{O}_{\bar X}\boxtimes\mathscr{L}^{-1}))\cong f^*\mathscr{R}^1\tilde q_*(\mathscr{I}_{\tilde Y}\otimes(q^*\mathscr{L}^{-1}))$ since f is flat. Therefore

$$
\begin{aligned}
&H^0(\bar X,\ \mathscr{R}^1\bar q_*(\mathscr{I}_{\bar Y}\otimes(\mathscr{O}_{\bar X}\boxtimes\mathscr{L}^{-1}))\otimes\mathscr{O}_{\bar X}(\bar H+2\bar F)\otimes\mathscr{O}_{\bar X}(1))\\
&\quad\cong H^0(\tilde X,\ \mathscr{R}^1\tilde q_*(\mathscr{I}_{\tilde Y}\otimes(q^*\mathscr{L}^{-1})\otimes\mathscr{O}_{\tilde X}(\tilde H+2\tilde F)\otimes\mathscr{N}^\vee)\ .
\end{aligned}
$$

LEMMA 4.4.

$$
\mathscr{R}^1\tilde q_*(\mathscr{I}_{\tilde Y}\otimes(q^*\mathscr{L}^{-1}))\otimes\mathscr{O}_{\tilde X}(\tilde H+2\tilde F)\cong\mathscr{N}\qquad\textit{and}
$$
$$
\mathrm{Ext}^1(\mathscr{O}_{\bar X}(-\bar F)\boxtimes\mathscr{L},\ \mathscr{I}_{\bar Y}\otimes\bar q^*\mathscr{O}_{\bar X}(\bar H+\bar F)\otimes\bar q^*\mathscr{O}_{\bar X}(1))\cong\mathrm{End}(\mathscr{N})\cong k\ .
$$

PROOF. Tensoring the sequence

$$0 \longrightarrow \mathcal{O}_{\tilde{X}}(-\tilde{F}) \boxtimes \mathcal{O}_X(-F) \longrightarrow \mathcal{I}_{\tilde{Y}} \longrightarrow \mathcal{O}_{\tilde{X}\times_{P^1}X}(-\tilde{Y}) \longrightarrow 0$$

with $\mathcal{O}_{\tilde{X}}(\tilde{H}+2\tilde{F})\boxtimes\mathcal{L}^{-1}$, we get

$$0 \longrightarrow \mathcal{O}_{\tilde{X}}(\tilde{H}+\tilde{F}) \boxtimes \mathcal{L}^{-1}(-F) \longrightarrow \mathcal{I}_{\tilde{Y}} \otimes (\mathcal{O}_{\tilde{X}}(\tilde{H}+2\tilde{F}) \boxtimes \mathcal{L}^{-1}) \longrightarrow \tilde{q}^*\mathcal{O}_{\tilde{X}}(2\tilde{F}) \otimes \mathcal{O}_{\tilde{X}\times_{P^1}X} \longrightarrow 0 .$$

Taking the direct images of the above sequence, we get

$$0 \longrightarrow \mathcal{O}_{\tilde{X}}(2\tilde{F}) \xrightarrow{\ \phi\ } \mathcal{O}_{\tilde{X}}(\tilde{H}+\tilde{F}) \otimes H^1(\mathcal{L}^{-1}(-F)) \longrightarrow \mathcal{R}^1\tilde{q}_*(\mathcal{I}_{\tilde{Y}} \otimes (q^*\mathcal{L}^{-1})) \otimes \mathcal{O}_{\tilde{X}}(\tilde{H}+2\tilde{F}) \longrightarrow 0 .$$

By Theorem 1.6, to prove the first assertion, it suffices to show that the homomorphism

$$\phi \otimes \mathcal{O}_{\tilde{X}}(-2\tilde{F}) : \mathcal{O}_{\tilde{X}} \longrightarrow \mathcal{O}_{\tilde{X}}(\tilde{H}-\tilde{F}) \otimes H^1(\mathcal{L}^{-1}(-F))$$

is complete. If it were not, then there would be a surjective homomorphism

$$\mathcal{R}^1\tilde{q}_*(\mathcal{I}_{\tilde{Y}} \otimes (q^*\mathcal{L}^{-1})) \longrightarrow \mathcal{O}_{\tilde{X}}(-\tilde{F}) .$$

Tensoring the sequence

$$0 \longrightarrow \mathcal{I}_{\tilde{Y}} \longrightarrow \mathcal{O}_{\tilde{X}\times X} \longrightarrow \mathcal{O}_{\tilde{Y}} \longrightarrow 0$$

with $q^*\mathcal{L}^{-1}$ and taking the direct images, we get

$$0 \longrightarrow \tilde{q}^*(\mathcal{O}_{\tilde{Y}} \otimes (q^*\mathcal{L}^{-1})) \longrightarrow \mathcal{R}^1\tilde{q}_*(\mathcal{I}_{\tilde{Y}} \otimes (q^*\mathcal{L}^{-1})) \longrightarrow \mathcal{O}_{\tilde{X}} \otimes H^1(X, \mathcal{L}^{-1}) \longrightarrow 0 .$$

Since $\tilde{q}_*(\mathcal{O}_{\tilde{Y}} \otimes (q^*\mathcal{L}^{-1})) \cong \mathcal{T}_{\tilde{X}/P^1} \otimes \mathcal{O}_{\tilde{X}}(-\tilde{D}-(p+1)\tilde{F})$, there are no non-zero homomorphisms from $\mathcal{R}^1\tilde{q}_*(\mathcal{I}_{\tilde{Y}} \otimes (q^*\mathcal{L}^{-1}))$ to $\mathcal{O}_{\tilde{X}}(-\tilde{F})$. Therefore the homomorphism $\phi \otimes \mathcal{O}_{\tilde{X}}(-2\tilde{F})$ is complete. The second assertion follows from Corollary 1.7.

LEMMA 4.5. *The extension* (4.3.3) *is non-trivial and induces an isomorphism*

$$\mathcal{R}^1\tilde{q}_*(\mathcal{I}_{\tilde{Y}} \otimes (q^*\mathcal{L}^{-1})) \otimes \mathcal{O}_{\tilde{X}}(\tilde{H}+2\tilde{F}) \cong \mathcal{N} .$$

PROOF. Let Y be a line in a fibre of π and η be a non-zero element of $\mathrm{Ext}^1(\mathcal{L}, \mathcal{I}_Y)$. By Lemma 3.9, there exists an element ξ_0 in $\mathrm{Ext}^1(\mathcal{L}(F), \mathcal{O}_X(-F))$ such that $c(\xi_0)=\eta$. Let ξ_∞ be an element of

$\mathrm{Ext}^1(\mathscr{L}(F), \mathscr{O}_X(-F))$ corresponding to a destabilizing diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{O}_X(-F) & \longrightarrow & \mathscr{I}_Y & \longrightarrow & \mathscr{O}_F(-1) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{O}_X(-F) & \longrightarrow & \mathscr{E}_{\xi_\infty} & \longrightarrow & \mathscr{L}(F) & \longrightarrow & 0\,. \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathscr{L} & = & \mathscr{L} & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

ξ_∞ is unique up to the multiplication by a non-zero constant. Let l be the line joining $[\xi_0]$ and $[\xi_\infty]$ in $\boldsymbol{P}$. For general ξ_0, l intersects $\Psi(\tilde{X})$ at $[\xi_\infty]$ transversely. By Proposition 3.3, there is a rational map $\phi\colon \boldsymbol{P}^1\to\boldsymbol{P}$ such that $\phi(0)=[\xi_0]$ and $\phi(\infty)=[\xi_\infty]$. Since ϕ is not constant and defined by a homomorphism $\mathrm{Ext}^1(\mathscr{L}(F), \mathscr{O}_X(-F))^\vee\otimes\mathscr{O}_{\boldsymbol{P}^1}\to\mathscr{O}_{\boldsymbol{P}^1}(1)$, ϕ is an isomorphism onto its image and $\phi(\boldsymbol{P}^1)=l$. The sequence given by Proposition 3.3 is the pull back of the sequence (4.2.1) by $\phi\times\mathrm{id}: \boldsymbol{P}^1\times X\to\boldsymbol{P}\times X$. Let $\bar{l}$ be the strict transform of l by the blow up $\bar{\boldsymbol{P}}\to\boldsymbol{P}$. Let $\bar{\infty}$ be the closed point of $\bar{l}$ lying over $[\xi_\infty]$. By the construction, the restriction of the extension (4.3.3) to $\bar{\infty}\times X$ is η up to constant. Therefore the extension (4.3.3) is non-trivial and induces an isomorphism

$$\mathscr{R}^1\tilde{q}_*(\mathscr{I}_{\tilde{Y}}\otimes(q^*\mathscr{L}^{-1}))\otimes_{\tilde{X}}(\tilde{H}+2\tilde{F})\cong\mathscr{N}$$

by Lemma 4.4.

PROPOSITION 4.6. *There is a closed subscheme A of $\bar{\boldsymbol{P}}$ such that for $V=\bar{\boldsymbol{P}}\setminus A$, $(V, \mathscr{F}_V)$ is a family for $M_0\setminus M_1$*

PROOF. Let Y be a line in a fibre of π and y be the closed point of $\tilde{X}$ corresponding to Y. Since $\mathscr{R}^1\tilde{q}_*(\mathscr{I}_{\tilde{Y}}\otimes(q^*\mathscr{L}^{-1}))\otimes k(y)\cong H^1(X, \mathscr{L}^{-1}\otimes\mathscr{I}_Y)$, $\tilde{X}=\boldsymbol{P}(\mathscr{N}^\vee)$ parametrizes all coherent sheaves of type (2.8.1), which contains also unstable sheaves. Since the extension (4.3.3) induces an isomorphism

$$\mathscr{R}^1\tilde{q}_*(\mathscr{I}_{\tilde{Y}}\otimes(q^*\mathscr{L}^{-1}))\otimes\mathscr{O}_{\tilde{X}}(\tilde{H}+2\tilde{F})\cong\mathscr{N}\ ,$$

the extension (4.3.3) is a universal family of all extensions of type (2.8.1) up to constant. Let A be the unstable locus of $\bar{X}$. Since $(\bar{\boldsymbol{P}}\setminus\bar{X}, \mathscr{F}_{\bar{P}\setminus\bar{X}})=(U, \mathscr{E}_U)$ is a universal family of stable vector bundles, for $V=\bar{\boldsymbol{P}}\setminus A$, $(V, \mathscr{F}_V)$ is a required family.

§ 5. Construction of a family for M_1.

In this section, we will sketch the construction of a family for M_1. All proofs are omitted because they are tedious but easy.

Lemma 5.1. *Let $\mathscr{E}$ be a torsion free coherent sheaf of rank 2 on X with $C_1(\mathscr{E})=C_1(\mathscr{L})$, $C_2(\mathscr{E})=D\cdot F$ and $C_3(\mathscr{E})=2$. $\mathscr{E}$ is stable if and only if $\mathscr{E}$ is a non-trivial extension of $\mathscr{L}\otimes\mathscr{I}_Y$ by $\mathscr{O}_X$;*

$$0\longrightarrow\mathscr{O}_X\longrightarrow\mathscr{E}\longrightarrow\mathscr{L}\otimes\mathscr{I}_Y\longrightarrow 0\,,$$

where Y is a line in a fibre of π.

(5.2) For a line Y in a fibre of π, there is a unique unstable vector bundle $\mathscr{E}_Y$ which is a non-trivial extension of $\mathscr{I}_Y$ by $\mathscr{L}$ ([HI] (3.3));

$$0\longrightarrow\mathscr{L}\longrightarrow\mathscr{E}_Y\longrightarrow\mathscr{I}_Y\longrightarrow 0\,.$$

The family $(\tilde{X}, \mathscr{E}_{\tilde{X}})$ is a universal family of $\mathscr{E}_Y$'s ([HI] (3.6)).

Lemma 5.3. *There is an exact sequence*

$$0\longrightarrow \mathrm{Ext}^1(\mathscr{L}\otimes\mathscr{I}_Y, \mathscr{O}_X)\longrightarrow H^1(X, \mathscr{E}_Y^\vee\otimes\mathscr{L}^{-1})\longrightarrow H^1(X, \mathscr{L}^{-1})\longrightarrow 0\,.$$

(5.4) Taking dual of the second arrow of the exact sequence (4.2.2)

$$0\longrightarrow\mathscr{O}_X(-\tilde{F})\boxtimes\mathscr{L}\longrightarrow\mathscr{E}_{\tilde{X}}\longrightarrow\mathscr{I}_{\tilde{Y}}\otimes\tilde{q}^*\mathscr{O}_{\tilde{X}}(\tilde{H}+\tilde{F})\longrightarrow 0$$

and applying the functor $\mathscr{R}^1\tilde{q}_*(\quad\otimes q^*\mathscr{L}^{-1})$, by Lemma 5.3, we get an exact sequence of vector bundles on $\tilde{X}$

$$(5.4.1)\quad 0\longrightarrow\mathscr{R}\longrightarrow\mathscr{R}^1\tilde{q}_*(\mathscr{E}_{\tilde{X}}^\vee\otimes q^*\mathscr{L}^{-1})\longrightarrow\mathscr{R}^1\tilde{q}_*(\mathscr{O}_{\tilde{X}}(\tilde{F})\otimes q^*\mathscr{L}^{-2})\longrightarrow 0$$

such that for a line Y in a fibre of π, $\mathscr{R}\otimes k(y)\cong\mathrm{Ext}^1(\mathscr{L}\otimes\mathscr{I}_Y, \mathscr{O}_X)$ where y is the closed point of $\tilde{X}$ corresponding to Y. The rank of $\mathscr{R}$ is $\dim\mathrm{Ext}^1(\mathscr{L}\otimes\mathscr{I}_Y, \mathscr{O}_X)=N-2$. Put $\boldsymbol{P}(\mathscr{R}^1\tilde{q}_*(\mathscr{E}_{\tilde{X}}^\vee\otimes q^*\mathscr{L}^{-1})^\vee)=W$ and $\boldsymbol{P}(\mathscr{R}^\vee)=Z$. Let $g: W\to\tilde{X}$ and $f: Z\to\tilde{X}$ be projections, and $h=g\times\mathrm{id}: W\times X\to\tilde{X}\times X$. There is a family of extensions on $W\times X$

$$0\longrightarrow\mathscr{O}_W(1)\boxtimes\mathscr{L}^{-1}\longrightarrow\mathscr{F}\longrightarrow h^*\mathscr{E}_{\tilde{X}}\longrightarrow 0\,.$$

Let $\mathscr{G}$ be the kernel of the composite homomorphism

$$\mathscr{F} \longrightarrow h^*\mathscr{E}_{\tilde X} \longrightarrow h^*(\mathscr{I}_{\tilde Y}\otimes \tilde q^*\mathcal{O}_{\tilde X}(\tilde H+\tilde F)) .$$

Put $h^*(\mathscr{I}_{\tilde Y}\otimes \tilde q^*\mathcal{O}_{\tilde X}(\tilde H+\tilde F))=\mathscr{J}$. There is an exact commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathscr{J} & = & \mathscr{J} & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_W(1)\boxtimes\mathscr{L}^{-1} & \longrightarrow & \mathscr{F} & \longrightarrow & h^*\mathscr{E}_{\tilde X} & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_W(1)\boxtimes\mathscr{L}^{-1} & \longrightarrow & \mathscr{G} & \longrightarrow & h^*(\mathcal{O}_{\tilde X}(-\tilde F)\boxtimes\mathscr{L}) & \longrightarrow & 0 . \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

By the exact sequence (5.4.1), the restriction of the bottom row of the above diagram to $Z\times X$ splits. Now consider the restriction of the middle column of the above diagram to $Z\times X$. Define a coherent sheaf $\mathscr{H}$ on $Z\times X$ by the following exact commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \mathcal{O}_Z(1)\boxtimes\mathscr{L}^{-1} & \longrightarrow & \mathscr{H} & \longrightarrow & \mathscr{J}_Z & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & \mathscr{G}_Z & \longrightarrow & \mathscr{F}_Z & \longrightarrow & \mathscr{J}_Z & \longrightarrow & 0 . \\
 & & \uparrow & & \uparrow & & & & \\
 & & h^*(\mathcal{O}_{\tilde X}(-\tilde F)\boxtimes\mathscr{L})_Z & = & h^*(\mathcal{O}_{\tilde X}(-\tilde F)\boxtimes\mathscr{L})_Z & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

PROPOSITION 5.5. *The family* $(Z, \mathscr{H}\otimes p^*\mathscr{L})$ *is a family of all the stable sheaves of rank* 2 *with* $C_1=C_1(\mathscr{L})$, $C_2=D\cdot F$ *and* $C_3=2$ *where* $p: Z\times X\to X$ *is the projection.*

(5.6) Put $\boldsymbol{P}(\mathscr{H}\otimes p^*\mathscr{L})=\bar Z$. Let $\mathscr{K}$ be the pull back of $\mathscr{H}\otimes p^*\mathscr{L}$ to $\bar Z$. Let $i:\bar Z=\bar Z\times_X X\hookrightarrow \bar Z\times X$ be the closed immersion. Let $\mathscr{M}$ be the kernel of the composite homomorphism

$$\mathscr{K}\boxtimes\mathcal{O}_X \longrightarrow \mathcal{O}_{\bar{Z}}(1)\boxtimes\mathcal{O}_X \longrightarrow i^*(\mathcal{O}_{\bar{Z}}(1)\boxtimes\mathcal{O}_X)\cong\mathcal{O}_{\bar{Z}}(1)\ .$$

PROPOSITION 5.7. *$(\bar{Z}, \mathscr{M})$ is a family for M_1.*

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