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On the Existence and Uniqueness of the Stationary Solution to the Equations of Natural Convection

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1. Notations and results.

In this paper, we discuss the existence of weak solutions of a system of equations which describes the motion of fluid with natural convection (Boussinesq approximation) in a bounded domain Ω in $\boldsymbol{R}^n$, $2\leq n$. We consider the following system of differential equations:

$$\begin{cases} (u\cdot\nabla)u=-\dfrac{1}{\rho}\nabla p+\nu\Delta u+\beta g\theta\,, & \\ \operatorname{div} u=0\,, & \text{in} \quad \Omega \\ (u\cdot\nabla)\theta=\chi\Delta\theta\,, & \end{cases} \tag{1}$$

where $u\cdot\nabla=\sum_j u_j\partial/\partial x_j$. Here u is the fluid velocity, p is the pressure, θ is the temperature, g is the gravitational vector function, and ρ (density), ν (kinematic viscosity), β (coefficient of volume expansion), χ (thermal diffusivity) are positive constants. We study this system of equations with mixed boundary condition for θ. Let $\partial\Omega$ (the boundary of Ω) be divided into two parts Γ_1, Γ_2 such that

$$\partial\Omega=\Gamma_1\cup\Gamma_2\,,\qquad \Gamma_1\cap\Gamma_2=\varnothing\,.$$

The boundary conditions are as follows.

$$\begin{cases} u=0\,, \quad \theta=\xi\,, & \text{on} \quad \Gamma_1\,, \\ u=0\,, \quad \dfrac{\partial\theta}{\partial n}=\eta\,, & \text{on} \quad \Gamma_2\,, \end{cases} \tag{2}$$

where ξ (resp. η) is a given function on Γ_1 (resp. Γ_2), n is the outward normal vector to $\partial\Omega$.

In this paper, we show the existence of weak solution of this problem for bounded domain Ω in $\boldsymbol{R}^n$, $2\leq n$, using the Galerkin method (Theorem 1). Some uniqueness result

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is also obtained (Theorem 2). In the previous paper [7], we treated this problem only for the case $n=3$.

In order to state the definition of weak solution and our results, we introduce some function spaces. The functions considered in this paper are all real valued. L^p and the Sobolev space W_p^m are defined as usual. We also denote $H^m = W_2^m$. Whether the elements of the space are scalar or vector functions is understood from the context unless stated explicitly. For the inner product and the norm of $L^2(\Omega)$, we use the notation $(u, v)_\Omega$ and $\|u\|_\Omega$, or simply, (u, v) and $\|u\|$.

$D_\sigma = \{\text{vector function } \varphi \in C^\infty(\Omega) \mid \text{supp } \varphi \subset \Omega, \text{ div } \varphi = 0 \text{ in } \Omega\}$,
H = completion of D_σ under the $L^2(\Omega)$-norm,
V = completion of D_σ under the $H^1(\Omega)$-norm,
$\tilde{V}$ = completion of D_σ under the norm $\|u\|_{H^1(\Omega)} + \|u\|_{L^n(\Omega)}$,
$D_0 = \{\text{scalar function } \varphi \in C^\infty(\bar{\Omega}) \mid \varphi \equiv 0 \text{ in a neighborhood of } \Gamma_1\}$,
W = completion of D_0 under the $H^1(\Omega)$-norm,
$\tilde{W}$ = completion of D_0 under the norm $\|u\|_{H^1(\Omega)} + \|u\|_{L^n(\Omega)}$.

Consider L^2 inner product of the first equation of (1) with v in $\tilde{V}$, and the third equation of (1) with τ in $\tilde{W}$. Using the integration by parts, we obtain:

$$\begin{cases} \nu(\nabla u, \nabla v) + B(u, u, v) - (\beta g\theta, v) = 0\,, & \text{for all } v \in \tilde{V}\,, \\ \chi(\nabla\theta, \nabla\tau) + b(u, \theta, \tau) = \chi(\eta, \tau)_{\Gamma_2}\,, & \text{for all } \tau \in \tilde{W}\,, \end{cases} \tag{3}$$

where

$$B(u, v, w) = ((u\cdot\nabla)v, w) = \int_\Omega \sum_{i,j=1}^n u_j(x)\frac{\partial v_i(x)}{\partial x_j} w_i(x)dx\,,$$

and

$$b(u, \theta, \tau) = ((u\cdot\nabla)\theta, \tau) = \int_\Omega \sum_{j=1}^n u_j(x)\frac{\partial\theta(x)}{\partial x_j}\tau(x)dx\,.$$

Now, we define the weak solution of (1), (2).

DEFINITION 1. A pair of functions $\{u, \theta\}$ is called a weak solution of (1), (2), if there exists a function θ_0 in $H^1(\Omega)$ such that $u \in V$, $\theta - \theta_0 \in W$, $\theta_0 = \xi$ on Γ_1, and, $\{u, \theta\}$ satisfies (3).

For the domain Ω, we assume that Ω is a bounded domain in $\mathbf{R}^n$ with C^1 boundary. Concerning the partition of $\partial\Omega$ into Γ_1, Γ_2 appearing in (2), we further assume that the following condition is satisfied;

CONDITION (H).

$$\partial\Omega = \Gamma_1 \cup \Gamma_2\,, \quad \Gamma_1 \cap \Gamma_2 = \emptyset\,, \quad \text{measure of } \Gamma_1 \neq 0\,,$$

and the intersection

$$\bar{\Gamma}_1 \cap \bar{\Gamma}_2$$

is an $n-2$ dimensional C^1 manifold.

Now, we state our results.

THEOREM 1. *Let Ω be a bounded domain in $\boldsymbol{R}^n$ with C^1 boundary satisfying Condition* (H). *If $g(x)$ is in $L^\infty(\Omega)$, ξ is in $C^1(\bar{\Gamma}_1)$, and η is in $L^2(\Gamma_2)$, then there exists a weak solution of* (1), (2).

REMARK 1. Generally, $\tilde{V} \subset V \cap L^n(\Omega)$ and $\tilde{W} \subset W \cap L^n(\Omega)$. For $2 \leq n \leq 4$, $\tilde{V} = V$ and $\tilde{W} = W$ (cf. Masuda [6], Giga [3]). Therefore our theorem contains the result of [7].

Let $g_\infty = \|g\|_{L^\infty(\Omega)}$, and c, c_1, c_2 be constants in Lemma 3 (Section 2). As for the uniqueness, we have:

THEOREM 2. *The weak solution $\{u, \theta\}$ of* (1), (2) *satisfying*

(i) $u \in L^n(\Omega)$, $\theta \in L^n(\Omega)$,

(ii) $c\|u\|_n + \dfrac{\beta g_\infty c c_1 c_2}{\chi} \|\theta\|_n < \nu$, *when* $n \geq 3$,

((ii)′ $c\|u\|_p + \dfrac{\beta g_\infty c c_1 c_2}{\chi} \|\theta\|_p < \nu$, *for some $p > 2$, when $n = 2$*),

is, if it exists, unique.

REMARK 2. The condition (i) is automatically satisfied when $2 \leq n \leq 4$.

REMARK 3. If we set

$$Re = \frac{c}{\nu} \|u\|_n \qquad \text{(Reynolds number)},$$

$$Ra = \frac{\beta g_\infty c c_1 c_2}{\nu \chi} \|\theta\|_n \qquad \text{(Rayleigh number)},$$

then the condition (ii) reads as

$$Re + Ra < 1 .$$

See also Joseph [5].

2. Some lemmas.

Here, we state some lemmas for the convenience of reference. For instance, the constants in Lemma 3 appear in the statement of Theorem 2.

LEMMA 1. *$\tilde{V}$ and $\tilde{W}$ are separable Banach spaces.*

PROOF. A subset of separable metric space is separable (e.g. Brezis [2]). If we show $V \cap L^n(\Omega)$ is separable, Lemma 1 is proved. We can identify $V \cap L^n(\Omega)$ as a subset

$$F = \left\{ \left(v, \frac{\partial v}{\partial x_1}, \cdots, \frac{\partial v}{\partial x_n} \right) ; \; v \in V \cap L^n(\Omega) \right\}$$

of $L^n(\Omega) \times L^2(\Omega) \times \cdots \times L^2(\Omega)$. Since the latter space is separable, the set F is also separable and Lemma 1 is proved.

Lemma 2 (Sobolev imbedding theorem), Lemma 3 (Inequalities of Poincaré type) and Lemma 4 (Continuity of the trace operator) are essentially well-known. On the other hand, Lemma 5 and Lemma 6 are concerned with the trilinear form familiar in the study of the Navier-Stokes equation (see e.g. Temam [8]).

LEMMA 2. *Sobolev space $H^1(\Omega)$ is continuously imbedded in $L^q(\Omega)$, where $q = 2n/(n-2)$ for $n \geq 3$, and $+\infty > q \geq 1$ for $n = 2$.*

For the proof, see Adams [1].

LEMMA 3. *There exist constants c_1, c_2, c depending on Ω and n such that*

(i) $\|u\| \leq c_1 \|\nabla u\| \quad$ *for* $\forall u \in V$,

(ii) $\|u\|_q \leq c \|\nabla u\| \quad$ *for* $\forall u \in V \quad \begin{cases} q = \dfrac{2n}{n-2} & (n \geq 3), \\ q = 4 & (n = 2), \end{cases}$

(iii) $\|\theta\| \leq c_2 \|\nabla \theta\| \quad$ *for* $\forall \theta \in W$.

The inequalities (i), (iii) are well known, and (ii) follows from (i) and Lemma 2.

For the boundary value of H^1 functions, we have:

LEMMA 4. *There exists a positive constant C such that*

$$\|v\|_{L^2(\partial\Omega)} \leq C \|v\|_{H^1(\Omega)} \qquad \text{for all } v \text{ in } H^1(\Omega).$$

In particular there is a positive constant c_3 such that

$$\|\theta\|_{L^2(\Gamma_2)} \leq c_3 \|\nabla \theta\|_{L^2(\Omega)} \qquad \text{for all } \theta \text{ in } W.$$

By Hölder's inequality and Lemmas 2, 3, we have:

LEMMA 5. *Let $n \geq 3$. There exists a constant c_B dependeing on Ω and n such that*

$$\begin{aligned} |B(u, v, w)| &\leq c_B \|\nabla u\| \, \|\nabla v\| \, \|w\|_n \qquad \text{for } \forall u \in V, \; \forall v \in H^1(\Omega), \; \forall w \in L^n(\Omega), \\ |b(u, \theta, \tau)| &\leq c_B \|\nabla u\| \, \|\nabla \theta\| \, \|\tau\|_n \qquad \text{for } \forall u \in V, \; \forall \theta \in H^1(\Omega), \; \forall \tau \in L^n(\Omega), \end{aligned} \tag{4}$$

hold.

Using the integration by parts, we obtain:

LEMMA 6.

(i) $B(u, v, w) = -B(u, w, v)$ *for* $\forall u \in V$, $\forall v, w \in H^1 \cap L^n$

holds. In particular,

$$B(u, v, v) = 0 \quad for \quad \forall u \in V, \quad \forall v \in H^1 \cap L^n .$$

(ii) $b(u, \theta, \tau) = -b(u, \tau, \theta)$ *for* $\forall u \in V$, $\forall \theta, \tau \in H^1 \cap L^n$

holds. In particular,

$$b(u, \theta, \theta) = 0 \quad for \quad \forall u \in V, \quad \forall \theta \in H^1 \cap L^n .$$

LEMMA 7. *Let Ω be a bounded domain in $\boldsymbol{R}^n$ satisfying the condition* (H). *If ξ is a C^1 function defined on $\bar{\Gamma}_1$, then for any positive number ε and any $p \geq 1$, there exists an extension θ_0 of ξ such that*

$$\theta_0 \in C_0^1(\boldsymbol{R}^n), \qquad \theta_0 = \xi \quad on \ \bar{\Gamma}, \qquad \|\theta_0\|_p < \varepsilon .$$

For the proof, see e.g. [4] Lemma 6.38.

3. Proof of Theorem 1.

Under our assumptions, we can extend ξ to a $C_0^1(\boldsymbol{R}^n)$ function which we denote by θ_0. Using the Galerkin method, we construct approximate solutions of (3). Let $\{\varphi_j\}$ be a sequence of functions in D_σ, linearly independent and total in $\tilde{V}$. We can assume

$$(\nabla \varphi_j, \nabla \varphi_k) = \delta_{jk}$$

without loss of generality. Let $\{\psi_j\}$ be a sequence of functions in D_0, linearly independent and total in $\tilde{W}$. We can assume

$$(\nabla \psi_j, \nabla \psi_k) = \delta_{jk} .$$

Since $\tilde{V}$ (resp. $\tilde{W}$) is separable and D_σ (resp. D_0) is dense there, we can find these functions. We put

$$u^{(m)} = \sum_{j=1}^{m} \xi_j \varphi_j ,$$

$$\theta^{(m)} = \sum_{j=1}^{m} \xi_{m+j} \psi_j ,$$

and we consider the following system of equations:

$$\nu(\nabla u^{(m)}, \nabla \varphi_j) + ((u^{(m)} \cdot \nabla) u^{(m)}, \varphi_j) - (\beta g \theta^{(m)}, \varphi_j) - (\beta g \theta_0, \varphi_j) = 0 , \qquad 1 \leq j \leq m . \tag{5}$$

$$\chi(\nabla\theta^{(m)}, \nabla\psi_j)+((u^{(m)}\cdot\nabla)\theta^{(m)}, \psi_j)+((u^{(m)}\cdot\nabla)\theta_0, \psi_j)$$
$$+\chi(\nabla\theta_0, \nabla\psi_j)-\chi(\eta, \psi_j)_{\Gamma_2}=0\,, \qquad 1\leq j\leq m\,. \tag{6}$$

Substituting $u^{(m)}$, $\theta^{(m)}$ into these equations, we obtain:

$$\xi_j+\frac{1}{\nu}\sum_{k,l}\xi_k\xi_l((\varphi_k\cdot\nabla)\varphi_l, \varphi_j)-\frac{1}{\nu}\sum_k\xi_{m+k}(\beta g\psi_k, \varphi_j)$$
$$-\frac{1}{\nu}(\beta g\theta_0, \varphi_j)=0\,, \qquad 1\leq j\leq m\,, \tag{7}$$

$$\xi_{m+j}+\frac{1}{\chi}\sum_{k,l}\xi_k\xi_{m+k}((\varphi_k\cdot\nabla)\psi_l, \psi_j)+\frac{1}{\chi}\sum_k\xi_k((\varphi_k\cdot\nabla)\theta_0, \psi_j)$$
$$+(\nabla\theta_0, \nabla\psi_j)-(\eta, \psi_j)_{\Gamma_2}=0\,, \qquad 1\leq j\leq m\,. \tag{8}$$

The left hand sides of (7) and (8) determine polynomials which we denote by

$$\xi_j-P_j(\xi_1, \xi_2, \cdots, \xi_{2m})\,, \qquad 1\leq j\leq 2m\,.$$

P_j is a polynomial in $\xi=(\xi_1, \cdots, \xi_{2m})$ of degree 2. Let P be a mapping from $\boldsymbol{R}^{2m}$ to $\boldsymbol{R}^{2m}$ defined by $P(\xi)=(P_1(\xi), \cdots, P_{2m}(\xi))$. Then the fixed point of P, if it exists, is a solution of (7), (8). We show the existence of a fixed point of P. Let $\xi=\xi(\lambda)$ be any solution of $\xi=\lambda P(\xi)$, $0\leq\lambda\leq 1$. First we treat the case $n\geq 3$. Multiplying (7) by ξ_j and summing with respect to j, we have:

$$\sum_{j=1}^{m}|\xi_j|^2=\|\nabla u^{(m)}\|^2=\lambda\sum_{j=1}^{m}P_j(\xi)\xi_j$$
$$=-\frac{\lambda}{\nu}\sum_{j,k,l}\xi_j\xi_k\xi_l((\varphi_k\cdot\nabla)\varphi_l, \varphi_j)+\frac{\lambda\beta}{\nu}\sum_{j,k}\xi_{m+k}\xi_j(g\psi_k, \varphi_j)+\frac{\lambda\beta}{\nu}\sum_j\xi_j(g\theta_0, \varphi_j)$$
$$=-\frac{\lambda}{\nu}((u^{(m)}\cdot\nabla)u^{(m)}, u^{(m)})+\frac{\lambda\beta}{\nu}\{(g\theta^{(m)}, u^{(m)})+(g\theta_0, u^{(m)})\}$$
$$\leq\frac{\lambda\beta g_\infty}{\nu}\{\|\theta^{(m)}\|+\|\theta_0\|\}\|u^{(m)}\|$$
$$\leq\frac{\lambda\beta g_\infty c_1}{\nu}\{c_2\|\nabla\theta^{(m)}\|+\|\theta_0\|\}\|\nabla u^{(m)}\|$$

where we have used Lemmas 3, 6. Therefore,

$$\|\nabla u^{(m)}\|\leq\frac{\lambda\beta g_\infty c_1}{\nu}\{c_2\|\nabla\theta^{(m)}\|+\|\theta_0\|\}\,. \tag{9}$$

Similarly,

$$\sum_{j=1}^{m} |\xi_{m+j}|^2 = \|\nabla\theta^{(m)}\|^2 = \lambda \sum_{j=1}^{m} P_{m+j}(\xi)\xi_{m+j}$$

$$= -\frac{\lambda}{\chi}\{((u^{(m)}\cdot\nabla)\theta^{(m)}, \theta^{(m)}) - ((u^{(m)}\cdot\nabla)\theta^{(m)}, \theta_0)\} - \lambda(\nabla\theta_0, \nabla\theta^{(m)}) + \lambda(\eta, \theta^{(m)})_{\Gamma_2}$$

$$\leq \frac{\lambda}{\chi}\|u^{(m)}\|_{2n/(n-2)}\|\nabla\theta^{(m)}\|\,\|\theta_0\|_n + \lambda\|\nabla\theta^{(m)}\|\,\|\nabla\theta_0\| + \lambda\|\eta\|_{\Gamma_2}\|\theta^{(m)}\|_{\Gamma_2}$$

(by Hölder's inequality)

$$\leq \frac{\lambda c}{\chi}\|\nabla u^{(m)}\|\,\|\nabla\theta^{(m)}\|\,\|\theta_0\|_n + \lambda\{\|\nabla\theta_0\| + c_3\|\eta\|_{\Gamma_2}\}\|\nabla\theta^{(m)}\| \quad \text{(by Lemma 3)}.$$

For $n=2$, we have

$$\|\nabla\theta^{(m)}\|^2 \leq \frac{\lambda c}{\chi}\|\nabla u^{(m)}\|\,\|\nabla\theta^{(m)}\|\,\|\theta_0\|_4 + \lambda\{\|\nabla\theta_0\| + c_3\|\eta\|_{\Gamma_2}\}\|\nabla\theta^{(m)}\|.$$

Therefore,

$$\|\nabla\theta^{(m)}\| \leq \frac{\lambda c}{\chi}\|\theta_0\|_p\|\nabla u^{(m)}\| + \lambda\{\|\nabla\theta_0\| + c_3\|\eta\|_{\Gamma_2}\} \tag{10}$$

where $p=n$ when $n\geq 3$, and $p=4$ when $n=2$. Substituting (10) into (9), we obtain:

$$\left(1 - \frac{cc_1c_2\beta g_\infty\lambda^2}{\chi\nu}\|\theta_0\|_p\right)\|\nabla u^{(m)}\| \leq \frac{\lambda c_1\beta g_\infty}{\nu}(c_2\lambda\|\nabla\theta_0\| + \lambda c_2c_3\|\eta\|_{\Gamma_2} + \|\theta_0\|).$$

According to Lemma 7, we can choose θ_0 satisfying the estimate

$$1 - \frac{cc_1c_2\beta g_\infty}{\chi\nu}\|\theta_0\|_p > \frac{1}{2}. \tag{11}$$

Then, we have

$$\begin{aligned}\|\nabla u^{(m)}\| &\leq \frac{2\lambda c_1\beta g_\infty}{\nu}(c_2\lambda\|\nabla\theta_0\| + \|\theta_0\| + \lambda c_2c_3\|\eta\|_{\Gamma_2}) \\ &\leq \frac{2c_1\beta g_\infty}{\nu}(c_2\|\nabla\theta_0\| + \|\theta_0\| + c_2c_3\|\eta\|_{\Gamma_2}) \equiv \rho_1.\end{aligned} \tag{12}$$

From (11), θ_0 satisfies the inequality:

$$\|\theta_0\|_p < \frac{\chi\nu}{2cc_1c_2\beta g_\infty}.$$

Therefore the estimate

$$\|\nabla\theta^{(m)}\| \leq 2\|\nabla\theta_0\| + \frac{1}{c_2}\|\theta_0\| + 2c_3\|\eta\|_{\Gamma_2} \equiv \rho_2 \tag{13}$$

follows from (10), (12). Note that ρ_1 and ρ_2 are constants independent of λ and m. Therefore the solution ξ of $\xi = \lambda P(\xi)$ satisfies:

$$\sum_{j=1}^{2m} |\xi_j|^2 \leq \rho_1^2 + \rho_2^2 \equiv \rho^2, \qquad \text{for} \quad 0 \leq \forall\lambda \leq 1.$$

Brouwer's theorem [4] tells us the existence of a fixed point of the mapping $P: \xi = P(\xi)$, such that $|\xi| \leq \rho$.

Thus we have obtained the solutions $u^{(m)}$, $\theta^{(m)}$ of (5), (6). Moreover, they satisfy the estimates:

$$\|\nabla u^{(m)}\| \leq \rho_1, \qquad \|\nabla\theta^{(m)}\| \leq \rho_2.$$

Since V (resp. W) is compactly imbedded in H (resp. L^2), we can choose subsequences of $\{u^{(m)}, \theta^{(m)}\}$ which we denote by the same symbols, and elements $u \in V$, $\tilde{\theta} \in W$ such that the following convergences hold:

$$u^{(m)} \to u \qquad \text{weakly in } V, \text{ strongly in } H \tag{14}$$

$$\theta^{(m)} \to \tilde{\theta} \qquad \text{weakly in } W, \text{ strongly in } L^2(\Omega). \tag{15}$$

For these convergent sequences, the following lemma holds:

Lemma 8.

$$B(u^{(m)}, u^{(m)}, v) \to B(u, u, v), \qquad \text{for} \quad \forall v \in D_\sigma,$$

$$b(u^{(m)}, \theta^{(m)}, \tau) \to b(u, \tilde{\theta}, \tau), \qquad \text{for} \quad \forall \tau \in D_0.$$

The proof is found in [8] and omitted. Using this lemma for (5), (6), we find

$$\nu(\nabla u, \nabla v) + B(u, u, v) - (\beta g\tilde{\theta}, v) - (\beta g\theta_0, v) = 0, \tag{16}$$

$$\chi(\nabla\tilde{\theta}, \nabla\tau) + b(u, \tilde{\theta}, \tau) + b(u, \theta_0, \tau) + \chi(\nabla\theta_0, \nabla\tau) - \chi(\eta, \tau)_{\Gamma_2} = 0, \tag{17}$$

hold for $v = \varphi_j$, $\tau = \psi_j$, $\forall j$. By Lemma 5, we see the linear functional

$$v \to B(u, u, v) \qquad (\text{resp. } \tau \to b(u, \tilde{\theta}, \tau))$$

is continuous in L^n. Therefore the linear functional

$$v \to \text{the left hand side of (16)}$$
$$(\text{resp. } \tau \to \text{the left hand side of (17)})$$

is continuous in $V \cap L^n$ (resp. $W \cap L^n$). Since $\{\varphi_j\}$ (resp. $\{\psi_j\}$) is total in $\tilde{V}$ (resp. $\tilde{W}$), (16) (resp. (17)) holds for any v in $\tilde{V}$ (resp. for any τ in $\tilde{W}$). Therefore $\{u, \theta\}$ $(\theta = \tilde{\theta} + \theta_0)$

is a required weak solution.

4. Proof of Theorem 2.

Let $\{u_i, \theta_i\}$, $i=1,2$, be weak solutions of (1), (2) satisfying (i), (ii). For $i=1,2$, there is a function $\theta_0^{(i)}$ satisfying the condition in Definition 1, and u_i and θ_i satisfy (3). Since the trace of $\theta_0^{(1)}-\theta_0^{(2)}$ is 0 on Γ_1, $\theta_0^{(1)}-\theta_0^{(2)}$ belongs to W. Therefore, $\theta_1-\theta_2$ is also in W. Put $u=u_1-u_2$, $\theta=\theta_1-\theta_2$. Then, they satisfy the following relations:

$$\begin{aligned} &\nu(\nabla u, \nabla v)+B(u, u_1, v)+B(u_2, u, v)-(g\beta\theta, v)=0\,, && \forall v\in\tilde{V}\,,\\ &\chi(\nabla\theta, \nabla\tau)+b(u, \theta_1, \tau)+b(u_2, \theta, \tau)=0\,, && \forall\tau\in\tilde{W}\,. \end{aligned} \tag{18}$$

From the condition (i), we see

$$u\in\tilde{V}\,,\qquad \theta\in\tilde{W}\,.$$

Therefore, we can take $v=u$, $\tau=\theta$ and we have

$$\begin{aligned} &\nu\|\nabla u\|^2=B(u, u, u_1)+\beta(g\theta, u)\,,\\ &\chi\|\nabla\theta\|^2=b(u, \theta, \theta_1)\,. \end{aligned} \tag{19}$$

Here we have used Lemma 6.

Let $n\geq 3$. Making use of the Hölder's inequality to estimate (19), we have

$$\begin{aligned} &\nu\|\nabla u\|^2\leq\|u\|_{2n/(n-2)}\|\nabla u\|\,\|u_1\|_n+g_\infty\beta\|\theta\|\,\|u\|\,,\\ &\chi\|\nabla\theta\|^2\leq\|u\|_{2n/(n-2)}\|\nabla\theta\|\,\|\theta_1\|_n\,. \end{aligned}$$

By Lemma 3, we estimate the right hand side of the above equations, and we obtain:

$$\begin{aligned} &\nu\|\nabla u\|\leq c\|u_1\|_n\|\nabla u\|+\beta g_\infty c_1c_2\|\nabla\theta\|\,,\\ &\chi\|\nabla\theta\|\leq c\|\theta_1\|_n\|\nabla u\|\,. \end{aligned}$$

Therefore,

$$\nu\|\nabla u\|\leq\left\{c\|u_1\|_n+\frac{\beta g_\infty cc_1c_2}{\chi}\|\theta_1\|_n\right\}\|\nabla u\|$$

holds. Since u_1, θ_1 satisfy the condition (ii):

$$c\|u_1\|_n+\frac{\beta g_\infty cc_1c_2}{\chi}\|\theta_1\|_n<\nu\,,$$

therefore $\|\nabla u\|=\|\nabla\theta\|=0$. Since $u=0$ on $\partial\Omega$ and $\theta=0$ on Γ_1, we see $u=0$, $\theta=0$ in Ω. Therefore $u_1=u_2$, $\theta_1=\theta_2$ in Ω.

When $n=2$, we have

$$\nu\|\nabla u\|^2 \leq \|u\|_{p'}\|\nabla u\|\,\|u_1\|_p + \beta g_\infty\|\theta\|\,\|u\| ,$$

$$\chi\|\nabla\theta\|^2 \leq \|u\|_{p'}\|\nabla\theta\|\,\|\theta_1\|_p ,$$

where $1/p+1/p'=1/2$. We discuss in a similar way to the case $n\geq 3$, and we have $u=0$, $\theta=0$. Theorem 2 is proved.

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