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Replacements in the Conway Third Identity

Yasutaka NAKANISHI

Kobe University

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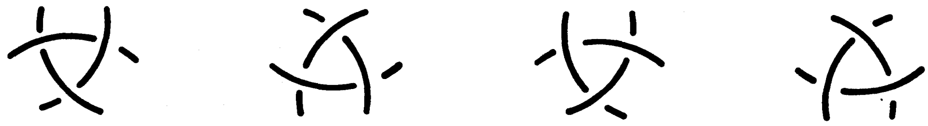
Dedicated to Professor Shôrô Araki on his sixtieth birthday

In this note we study knots and links in the 3-sphere S^3. J. H. Conway introduced the potential function for a link with labels and stated three Identities in [1]. It is well-known that each replacement appearing in the Conway First Identity is a kind of unknotting operation. In the Conway Second Identity, two replacements are an (ordinary) unknotting operation and the other one is unknown even if we ignore labels (cf. [3], [4]). Here, we will consider replacements appearing in the Conway Third Identity. Let L_1, L_2, L_3, and L_4 be four links which differ only in one place as shown in Fig. 1.



L_1 L_2 L_3 L_4

FIGURE 1.

A Δ_{ij}-move is defined to be a local move on a link diagram between L_i and L_j. If a diagram of a link L' is a result of a Δ_{ij}-move on a diagram of a link L, then we say that L is deformed into L' by a Δ_{ij}-move. Δ_{14}- and Δ_{23}- moves are Δ-unknotting operations defined by H. Murakami and the author in [2]. Our purpose in this note is to show that each Δ_{ij}-move ($i \neq j$) is a kind of unknotting operation and which kind of equivalence relation for links is generated by each Δ_{ij}-move.

1. Definitions and theorems.

It is clear that Δ_{ij}-moves never change the number of components of links. And

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we can see the following relationship among Δ_{ij}-moves.

Proposition 1. (1) *Δ_{12}- and Δ_{34}- moves are the same move.*

(2) *Δ_{13}- and Δ_{24}- moves are the same move.*

(3) *Δ_{14}- and Δ_{23}- moves are equivalent moves, i.e., each one can be realized by a finite sequence of the other.*

(4) *Δ_{12}- and Δ_{13}- moves are equivalent moves.*

(5) *A Δ_{14}-move can be realized by a finite sequence of Δ_{12}-moves (and so, by a finite sequence of Δ_{13}-moves).*

The proof will be given in the section 2.

Abstractly, six replacements are considered in the Conway Third Identity. By the above Proposition 1, we will consider two replacements Δ_{12}- and Δ_{14}- moves as representatives from now.

Observations. (1) *For knots, a Δ_{14}-move must change the knot types. For links, a Δ_{14}-move may not do so.*

(2) *A Δ_{12}- (or Δ_{13}-) move may not change the knot (link) types.*

Proof. (1) For a knot, a Δ_{14}-move must change the Arf invariant and the coefficient of z^2 in the Conway polynomial by one as in [2] and M. Okada [5]. So, we have the former part. (Similarly for a proper link, a Δ_{14}-move must change the Arf invariant by one. So, a Δ_{14}-move must change the *oriented link types* for proper links.) For the later part, we have examples as in Fig. 2. (We remark that for the second example the oriented link type is changed, however. There is an orientation reversing homeomorphism which maps the original link onto the result link.)

(2) It is easily checked by Fig. 3.

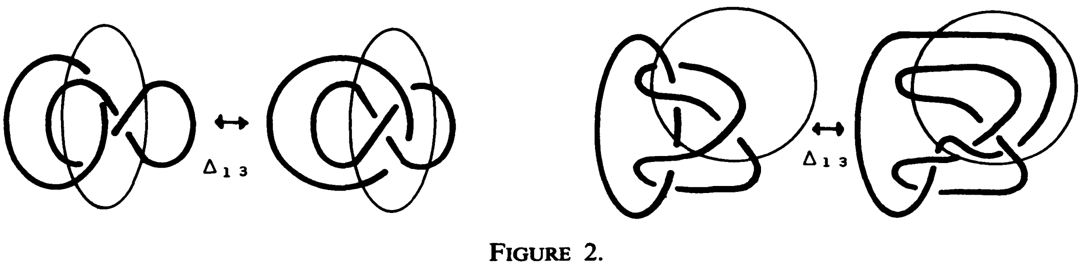

Figure 2.

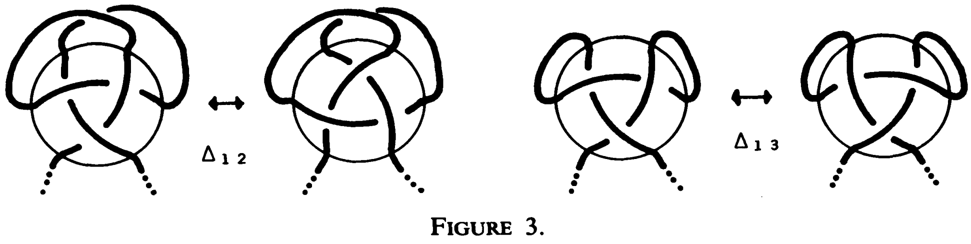

Figure 3.

NOTE. The example of the later part of the Observation (1) was firstly observed by M. Okada, whose example is a 2-component link with linking number 3.

For two n-component links $L=K_1 \cup K_2 \cup \cdots \cup K_n$ and $L'=K'_1 \cup K'_2 \cup \cdots \cup K'_n$, we characterize Δ_{ij}-moves as follows.

THEOREM 2 ([2]). *L can be deformed into L' by a finite sequence of Δ_{14}-moves if and only if, after suitably oriented and/or ordered if necessary, the linking numbers of L and L' are identical: $lk(K_i, K_j)=lk(K'_i, K'_j)$ for each pair i and j $(1 \leq i<j \leq n)$.*

THEOREM 3. *L can be deformed into L' by a finite sequence of Δ_{12}-moves if and only if, after suitably oriented and/or ordered if necessary, $\sum_{j \neq i} lk(K_i, K_j) \equiv \sum_{j \neq i} lk(K'_i, K'_j)$* mod 2 *for each i $(i=1, 2, \cdots, n)$.*

The proof will be given in the section 2.

COROLLARY 4. *For the equivalence relation generated by Δ_{12}-moves, the number of equivalence classes is* 2^{n-1}.

The proof will be given in the section 2.
We will consider the oriented version of Δ_{ij}-moves in the section 3.

2. Proofs.

PROOF OF PROPOSITION 1. (1) and (2): It is easily checked by rotating the figures in Fig. 1. (3): The proof is indicated in Fig. 1.1 (c) in [2]. (4): Fig. 4 indicates that a Δ_{12}-move can be realized by two Δ_{13}-moves. And Fig. 5 indicates that a Δ_{13}-move can be realized by two Δ_{12}-moves. (5): Fig. 6 indicates that a Δ_{14}-move can be realized by three Δ_{12}-moves. The proof is completed.

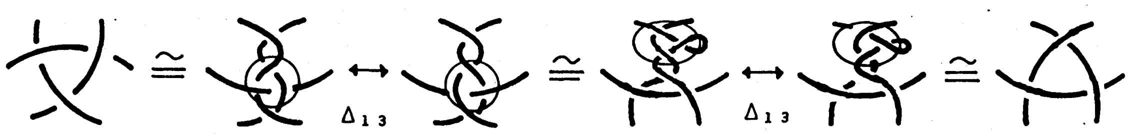

FIGURE 4.

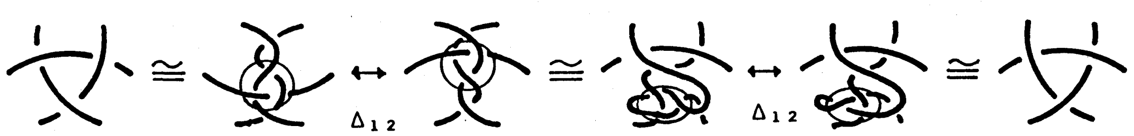

FIGURE 5.

The proof of Theorem 3 follows the following two Lemmas A and B.

LEMMA A. *L can be deformed into L' by a finite sequence of Δ_{12}-moves if and only if, after suitably oriented and/or ordered if necessary, the linking numbers of L and L'*

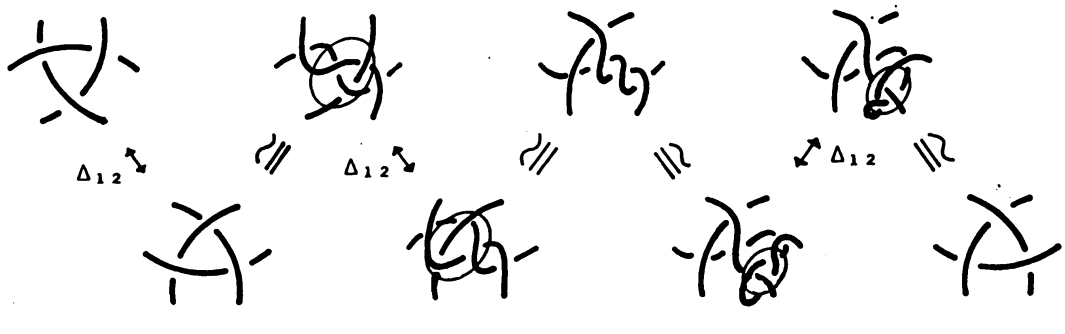

FIGURE 6.

are connected by a finite sequence of the following relations:

(1) $lk(K_i, K_j) \equiv lk(K'_i, K'_j)$ mod 2 *for each pair i and j* $(1 \leq i < j \leq n)$.

(2) *There exists a triple* $i<j<k$ *such that* $lk(K_i, K_j)+1 = lk(K'_i, K'_j)$, $lk(K_j, K_k)+1 = lk(K'_j, K'_k)$, $lk(K_k, K_i)+1 = lk(K'_k, K'_i)$, *and* $lk(K_r, K_s) = lk(K'_r, K'_s)$ *for each pair* $\{r, s\} \notin \{i, j, k\}$.

PROOF. First, we show the "only if" part. For (1): If there is a pair i and j such that $lk(K_i, K_j) \neq lk(K'_i, K'_j)$, then we take subarcs of K_i and K_j. By applying the move as in Fig. 7, we can change $lk(K_i, K_j)$ by two. Performing such operations succesively, we

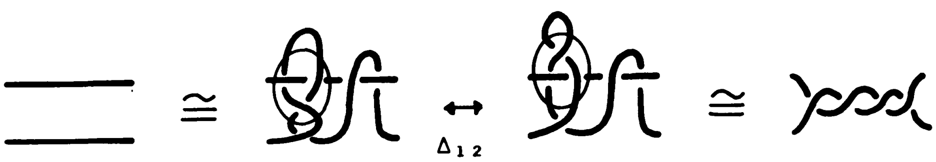

FIGURE 7.

can regard $lk(K_i, K_j) = lk(K'_i, K'_j)$ for each pair i and j. For (2): We take subarcs of K_i, K_j, and K_k. By applying the move as in Fig. 8, we can change the linking numbers suitably.

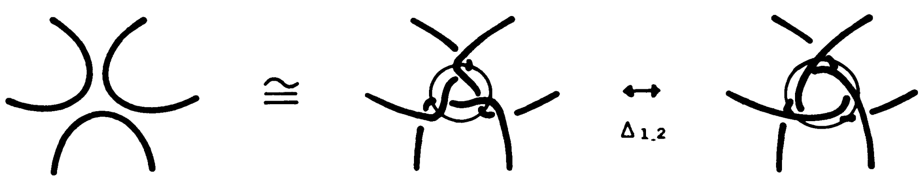

FIGURE 8.

Therefore, L can be deformed into L'' whose linking numbers are identical to those of L' by a finite sequence of Δ_{12}-moves. By Theorem 2, L'' can be deformed into L' by a finite sequence of Δ_{14}-moves, and so by a finite sequence of Δ_{12}-moves from Proposition 1. Hence, L can be deformed into L' by a finite sequence of Δ_{12}-moves. The "if" part is trivial, remarking the difference of linking numbers due to a Δ_{12}-move. The proof is completed.

LEMMA B. *Let* l_{ij}, l'_{ij} $(1 \leq i, j \leq n, i \neq j)$ *be integers such that* $l_{ij} = l_{ji}$ *and* $l'_{ij} = l'_{ji}$. *The following two equivalence relations* (I) *and* (II) *are equivalent.*

(I) $\sum_{j \neq i} l_{ij} \equiv \sum_{j \neq i} l'_{ij} \bmod 2$ *for each* i $(i = 1, 2, \cdots, n)$.

(II) *The sets of integers* $\{l_{ij}\}$ *and* $\{l'_{ij}\}$ *are connected by a finite sequence of the following relations*:

(1) $l_{ij} \equiv l'_{ij} \bmod 2$ *for each pair* i *and* j $(1 \leq i < j \leq n)$.

(2) *There exists a triple* $i < j < k$ *such that* $l_{ij} + 1 = l'_{ij}$, $l_{jk} + 1 = l'_{jk}$, $l_{ki} + 1 = l'_{ki}$, *and* $l_{rs} = l'_{rs}$ *for each pair* $\{r, s\} \notin \{i, j, k\}$.

PROOF. It is obvious that $\{\sum_{j \neq i} l_{ij} \pmod 2\}$ is an invariant under the relation (1). Since the relation (2) changes exactly two or none of $l_{i1}, l_{i2}, \cdots, l_{in}$ by one, $\{\sum_{j \neq i} l_{ij} \pmod 2\}$ is an invariant under the relation (2), too. Therefore, $\{\sum_{j \neq i} l_{ij} \pmod 2\}$ is an invariant for the equivalence relation (II). Conversely, suppose that $\sum_{j \neq i} l_{ij} \equiv \sum_{j \neq i} l'_{ij} \bmod 2$ for each i $(i = 1, 2, \cdots, n)$. If there is a pair i and j $(1 < i < j \leq n)$ such that $l_{ij} \not\equiv l'_{ij} \bmod 2$, then we apply the operation in (2) to get a set of integers $\{l''_{ij}$ $(1 \leq i, j \leq n;$ $i \neq j);$ $l''_{ij} = l''_{ji}\}$ so that $l_{ij} + 1 = l''_{ij}$, $l_{1i} + 1 = l''_{1i}$, $l_{1j} + 1 = l''_{1j}$, and $l_{rs} = l''_{rs}$ for each pair $\{r, s\} \notin \{1, i, j\}$. Performing such operations succesively, we can get $\{l''_{ij}\}$ such that $\{l''_{ij}\}$ and $\{l_{ij}\}$ are connected by a finite sequence of the relation (2) and $l''_{ij} \equiv l'_{ij} \bmod 2$ for each pair i and j $(1 < i < j \leq n)$. Since $\sum_{j \neq i} l''_{ij} \equiv \sum_{j \neq i} l_{ij} \equiv \sum_{j \neq i} l'_{ij} \bmod 2$ and $l''_{ij} \equiv l'_{ij} \bmod 2$ for each pair i and j $(1 < i < j \leq n)$, we have $l''_{i1} \equiv l'_{i1} \bmod 2$ for each i $(1 < i \leq n)$. Therefore, $\{l''_{ij}\}$ and $\{l'_{ij}\}$ are connected by the relation (1). Hence $\{l_{ij}\}$ and $\{l'_{ij}\}$ are connected by a finite sequence of the relations (1) and (2). The proof is completed.

PROOF OF COROLLARY 4. It can be seen that there are at least 2^{n-1} types of $\{\sum_{j \neq i} lk(K_i, K_j) \pmod 2\}$; which are represented by $lk(K_1, K_i) = 0$ or 1 $(i = 2, 3, \cdots, n)$ and $lk(K_i, K_j) = 0$ for each pair i and j $(1 < i < j \leq n)$. On the other hand, $\sum_i \sum_{j \neq i} lk(K_i, K_j) = 2 \sum_{i < j} lk(K_i, K_j) \equiv 0 \bmod 2$. So, $\sum_{j \neq i} lk(K_i, K_j)$ $(i = 1, 2, \cdots, n)$ are not independent, and there are at most 2^{n-1} types of $\{\sum_{j \neq i} lk(K_i, K_j) \pmod 2\}$. Hence, there are exactly 2^{n-1} types of $\{\sum_{j \neq i} lk(K_i, K_j) \pmod 2\}$. The proof is completed.

3. Oriented patterns of Δ_{ij}-move.

There are abstractly eight oriented patterns of each Δ_{ij}-move. We can identify some of them by rotating and turning over the figures and there are essentially two oriented patterns of each Δ_{ij}-move like as in Fig. 9. We denote them by Δ°_{ij} and $\Delta^{\wedge}_{ij}$ like as in Fig. 9.

From now, we consider relationship among them, however some parts have been still open at the present time.

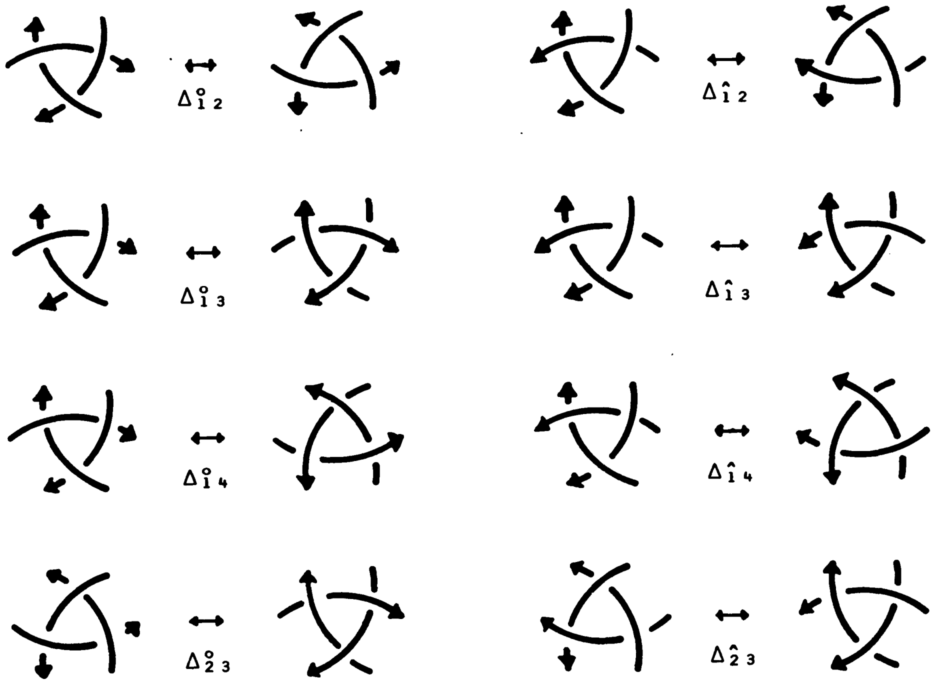

FIGURE 9.

PROPOSITION 5. (1) *$\Delta_{\mathring{1}4}$-, $\Delta_{\hat{1}4}$-, $\Delta_{\mathring{2}3}$-, and $\Delta_{\hat{2}3}$- moves are equivalent moves.*
(2) *$\Delta_{\hat{1}2}$- and $\Delta_{\mathring{1}3}$- moves are equivalent moves.*
(3) *A $\Delta_{\mathring{1}2}$-move can be realized by a finite sequence of $\Delta_{\hat{1}3}$- moves.*
(4) *A $\Delta_{\hat{1}3}$-move can be realized by a finite sequence of $\Delta_{\mathring{1}2}$- and $\Delta_{\hat{1}2}$-moves.*
(5) *A $\Delta_{\mathring{1}4}$-move can be realized by a finite sequence of $\Delta_{\mathring{1}2}$- and $\Delta_{\hat{1}2}$- moves.*

PROOF. (1): The proof is indicated in Fig. 1.1 (b) in [2]. Though it shows only one case, we can see all of them by the exchange of orientation of strings and the combination with Fig. 1.1 (c) in [2]. (2): For suitably oriented strings in Figs. 4 and 5, they indicate the proof. (3): Fig. 10 indicates that a $\Delta_{\mathring{1}2}$-move can be realized by two $\Delta_{\hat{1}3}$-moves. (4): For suitably oriented strings in Fig. 5, it indicates the proof. (5): For suitably oriented strings in Fig. 6, it indicates the proof. The proof is completed.

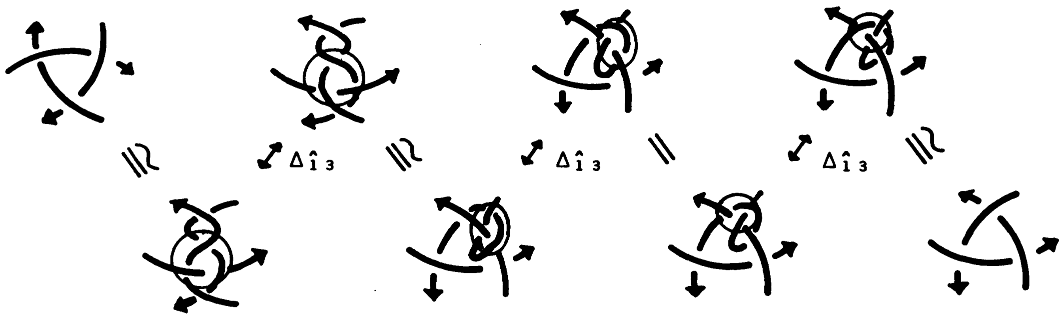

FIGURE 10.

After the work in [2], it is known that each one of Δ°_{14}-, $\Delta^{\wedge}_{14}$-, Δ°_{23}-, and $\Delta^{\wedge}_{23}$- moves is a kind of unknotting operation. At the present time, the author does not know whether Δ°_{12}-, $\Delta^{\wedge}_{12}$-, Δ°_{13}-, and $\Delta^{\wedge}_{13}$- moves are a kind of unknotting operation or not. It is still open.

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Present Address:
DEPARTMENT OF MATHEMATICS, KOBE UNIVERSITY
NADA-KU, KOBE 657, JAPAN