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Curvature Functions for the Sphere in Pseudohermitian Geometry

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Dedicated to Professor Tadashi Nagano on his sixtieth birthday

1. Introduction.

In Jerison and Lee's work on the CR Yamabe problem [JL], they consider the following equation of prescribing pseudohermitian scalar curvature $(2(n+1)/n)R$ under the choice of contact forms in a fixed CR structure:

$$\Delta_b u + \frac{n}{2(n+1)} R_0 u - R u^{(n+2)/n} = 0\,, \qquad u>0 \tag{1.1}$$

with $R \equiv$ constant, R_0 is a given pseudohermitian scalar curvature, where the sub-laplacian operator Δ_b is the real part of Kohn's $\square_b$ acting on functions. (See §2 for the definition.)

Let S^{2n+1} be the unit sphere in C^{n+1} equipped with the canonical pseudohermitian structure having pseudohermitian scalar curvature $n(n+1)/2$ (see §2). In this paper, we study the problem of prescribing arbitrary R on S^{2n+1} with $R_0 = n(n+1)/2$ in (1.1). In fact, the equation we consider reads

$$\Delta_b u + \frac{n^2}{4} u - R u^a = 0\,, \qquad u>0 \tag{1.2}$$

on S^{2n+1}, where $a>1$ is a constant. Our canonical pseudohermitian structure is determined by a certain contact form θ. Let L_θ denote the associated Levi form. The volume form $\theta \wedge (d\theta)^n$ is denoted by dv_θ. The gradient operator relative to the metric $\langle\ ,\ \rangle = (1/4)\theta^2 + L_\theta$ is denoted by ∇. In §3, we obtain an integrability condition as follows.

THEOREM A. *If u is a positive solution of* (1.2), *then*

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$$(1.3)\qquad \int_{S^{2n+1}} u^{a+1}\langle \nabla R, \nabla f\rangle dv_\theta = \frac{1}{2}(n+2-na)\int_{S^{2n+1}} u^{a+1}Rf\,dv_\theta$$

for all bigraded spherical harmonic f of type (1, 0) *or* (0, 1) *or their linear combinations on* C^{n+1}.

It is now clear that from (1.3), we have

COROLLARY. *There are no positive solutions of* (1.2) *for $R=f$ if $a \geq (n+2)/n$. And the same conclusion holds for $R=\text{const}+f$ if $a=(n+2)/n$.*

The equation (1.2) with the critical exponent $a=(n+2)/n$ appears to be (1.1) in our setting.

For the proof of Theorem A, we may think that the similar idea as in [KW] for the Riemannian case should work at a first glance. This is partly right. Indeed, the analogous divergence formula for integrating by parts still holds in pseudohermitian geometry. However, Δ_b is not elliptic (but subelliptic) and there appears a certain characteristic direction in the tangent space, which needs special care. Actually, following a standard argument, we arrive at (3.3) on the right-hand side of which there is an "extra" u_0-term. To see how to deal with this term, we carry out a variational argument to get the equality (3.11). Comparing (3.3) with (3.11) gives us a hope of relating the u_0-term in (3.3) to the second term in the integrand of (3.11), which involves both u_0 and $\nabla_b u$. Through the later computations, the hope comes true while it provides a clue leading to a proof of (1.3). (See §3 for more details.)

When a equals the value $(n+2)/n$ of geometric interest, the left-hand side of (1.3) has a certain geometric interpretation. Along this line, Theorem A is extended to certain compact pseudohermitian manifolds. Let $\text{Aut}^0_{\text{CR}}(M)$ denote the identity component of the CR automorphism group on a given CR manifold M. Let $\pi_1(M)$ denote the fundamental group of M. In §4, we prove

THEOREM B. *Let (M, θ) be a compact pseudohermitian manifold with its pseudohermitian scalar curvature R_θ. Suppose $\text{Aut}^0_{\text{CR}}(M)$ is compact or $\pi_1(M)$ is finite. Then for any CR vector field X, we have*

$$\int_M XR_\theta dv_\theta = 0\,.$$

The proof of Theorem B is based on an analogous idea of Bourguignon ([B]). Note that the real or imaginary part of gradient (with respect to $\langle\ ,\ \rangle$) of a bigraded spherical harmonic of type (1, 0) or (0, 1) is a CR vector field.

According to Webster [W2] p. 63, if $\text{Aut}^0_{\text{CR}}(M)$ is non-compact, then either M is CR-equivalent to S^{2n+1} or every closed non-compact one-parameter subgroup on M has no fixed points. We are wondering if the conclusion of Theorem B holds in the latter case.

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2. Some calculus in pseudohermitian geometry.

First we follow [L1] to give a brief description of pseudohermitian structures. Let M be a smooth, oriented, compact $(2n+1)$-dimensional manifold. A CR structure on M is an n-dimensional complex subbundle $T_{1,0}$ of the complexified tangent bundle CTM satisfying $T_{1,0} \cap T_{0,1} = \{0\}$, where $T_{0,1} = \bar{T}_{1,0}$. We will always assume that the CR structure is integrable, that is, $T_{1,0}$ satisfies the Frobenius condition $[T_{1,0}, T_{1,0}] \subset T_{1,0}$. Set $H = \mathrm{Re}(T_{1,0} \oplus T_{0,1})$. We single out a real nonvanishing 1-form θ annihilating H. A choice of θ is called a pseudohermitian structure on M. The Levi form of θ is defined by

$$L_\theta(V, \bar{W}) = d\theta(V \wedge J\bar{W})$$

for $V, W \in T_{1,0}$. L_θ extends by complex linearity to a symmetric form on CH, real on H, which is also denoted by L_θ. If L_θ is positive definite, M is said to be strictly pseudoconvex. We will also assume throughout that M is strictly pseudoconvex.

Now let (M, θ) be a pseudohermitian manifold. The characteristic vector field of θ is the unique vector field T such that $T \lrcorner \theta = 1$, $T \lrcorner d\theta = 0$. Let $\{\theta^1, \cdots, \theta^n\}$ be 1-forms in CT^*M, vanishing on $T_{0,1}$. We call $\{\theta^\alpha\}$ an admissible coframe if their restrictions to $T_{1,0}$ form a basis for $T^*_{1,0}$, and $T \lrcorner \theta^\alpha = 0$ for $\alpha = 1, \cdots, n$. With respect to an admissible coframe, we have

$$d\theta = ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}} \quad \text{(Hereafter the summation convention is used.)}$$

for some positive definite hermitian matrix of functions $(h_{\alpha\bar{\beta}})$. Let $\{Z_1, \cdots, Z_n\}$ be the frame for $T_{1,0}$ dual to $\{\theta^\alpha\}$. Then

$$L_\theta(V, \bar{W}) = \frac{1}{2} h_{\alpha\bar{\beta}} V^\alpha W^{\bar{\beta}}$$

for $V = V^\alpha Z_\alpha$, $W = W^\alpha Z_\alpha \in T_{1,0}$.

It is well known that there exists a unique pseudohermitian-invariant affine connection on (M, θ), satisfying certain geometric properties ([T1], [T2], [W1], [W2]). Let D denote the associated covariant differentiation. For a (smooth) function f on M, write $f_\alpha = Z_\alpha f$, $f_{\bar{\alpha}} = Z_{\bar{\alpha}} f$, $f_0 = Tf$, so that $Df = df = f_\alpha\theta^\alpha + f_{\bar{\alpha}}\theta^{\bar{\alpha}} + f_0\theta$. The second covariant differential $D^2 f$ of f in directions (Z_α, Z_β) $((Z_\alpha, T), (T, T)$, etc., respectively) will be denoted by $f_{\alpha\beta}$ $(f_{\alpha 0}, f_{00}$, etc., respectively). It follows that

$$f_{\alpha\bar{\beta}} - f_{\bar{\beta}\alpha} = ih_{\alpha\bar{\beta}} f_0\,, \qquad f_{\alpha\beta} - f_{\beta\alpha} = 0\,. \tag{2.1}$$

Let $(h^{\alpha\bar{\beta}})$ be the inverse matrix of $(h_{\alpha\bar{\beta}})$. As usual, $h_{\alpha\bar{\beta}}$ and $h^{\alpha\bar{\beta}}$ are used to raise or lower indices. Now we define the subgradient operator ∇'_b of type $(1, 0)$ (∇''_b of type

(0, 1), respectively) by

$$\nabla'_b f = f^{\alpha} Z_{\alpha}$$

($\nabla''_b f = f^{\bar{\alpha}} Z_{\bar{\alpha}}$, respectively) where $f^{\alpha} = h^{\alpha\bar{\beta}} f_{\bar{\beta}}$ ($f^{\bar{\alpha}} = h^{\beta\bar{\alpha}} f_{\beta}$, respectively). Of course, if f is real, $f^{\bar{\alpha}} = \overline{(f^{\alpha})}$. And then the subgradient operator ∇_b is defined by the sum of ∇'_b and ∇''_b. Similarly, the subdivergence operator div'_b of type (1, 0) (div''_b of type (0, 1), respectively) is given by

$$\operatorname{div}'_b V = \frac{1}{2} V^{\alpha}{}_{,\alpha} \quad \text{and} \quad \operatorname{div}'_b \bar{V} = 0$$

($\operatorname{div}''_b \bar{W} = \frac{1}{2} W^{\bar{\alpha}}{}_{,\bar{\alpha}}$ and $\operatorname{div}''_b W = 0$, respectively) for $V = V^{\alpha} Z_{\alpha} \in T_{1,0}$ ($\bar{W} = W^{\bar{\alpha}} Z_{\bar{\alpha}} \in T_{0,1}$, respectively) where, as usual, we denote covariant derivatives of a tensor by indices separated by a comma. Thus the subdivergence operator div_b is given by the sum of div'_b and div''_b. It follows that

$$\operatorname{div}_b X = \frac{1}{2} (V^{\alpha}{}_{,\alpha} + W^{\bar{\alpha}}{}_{,\bar{\alpha}})$$

for $X = V + \bar{W}$.

To simplify the notation, we also denote $L_{\theta}(X, Y)$ by $X \cdot Y$. The following formulas:

$$\operatorname{div}'_b(fX) = \nabla''_b f \cdot X + f \operatorname{div}'_b X,$$
$$\operatorname{div}''_b(fX) = \nabla'_b f \cdot X + f \operatorname{div}''_b X, \quad \text{and}$$
$$\operatorname{div}_b(fX) = \nabla_b f \cdot X + f \operatorname{div}_b X$$

for $X \in CH$ hold.

The sublaplacian operator Δ'_b (Δ''_b, Δ_b, respectively) acting on functions is now defined by

$$\Delta'_b f = -2 \operatorname{div}'_b(\nabla'_b f)$$

($\Delta''_b f = -2 \operatorname{div}''_b(\nabla''_b f)$, $\Delta_b = \Delta'_b + \Delta''_b$, respectively).

It follows that $\Delta_b f = -2 \operatorname{div}_b(\nabla_b f) = -(f^{\alpha}{}_{\alpha} + f^{\bar{\alpha}}{}_{\bar{\alpha}}) = -(f_{\alpha}{}^{\alpha} + f_{\bar{\alpha}}{}^{\bar{\alpha}})$ which agrees with the formula in [L1]. We will often use the following divergence formula ([L2]) for integrating by parts:

$$\int_M \operatorname{div}'_b V \theta \wedge (d\theta)^n = \frac{1}{2} \int_M V^{\alpha}{}_{,\alpha} \theta \wedge (d\theta)^n = 0$$

for $V = V^{\alpha} Z_{\alpha}$. (Of course, similar formulas hold for div''_b and div_b.) By (2.1), we also have

$$\int_M Tf \theta \wedge (d\theta)^n = 0 .$$

Our canonical pseudohermitian structure on the unit sphere S^{2n+1} in C^{n+1} with the induced CR structure is given by $\theta=i(\sigma-\bar{\sigma})$, $\sigma=\sum_{j=1}^{n+1} z_j d\bar{z}_j$ for $(z_1, \cdots, z_{n+1})\in C^{n+1}$. Hereafter, for S^{2n+1}, θ is taken to be the above one. For what we need in the next section, we have to know the extrinsic expressions of ∇'_b, ∇''_b and Δ_b for S^{2n+1} and compute $\Delta_b f$ and the Hessian of f for a bigraded spherical harmonic f of type (1, 0) or (0, 1) at least. We write $\partial_j=\partial/\partial z_j$, $\bar{\partial}_j=\partial/\partial\bar{z}_j$. Set $Z_{jk}=\bar{z}_j\partial_k-\bar{z}_k\partial_j$. It is not difficult to derive that

$$\nabla'_b f=\frac{1}{2}\sum_{1\le j<k\le n+1}(\bar{Z}_{jk}f)Z_{jk} \tag{2.2}$$

for f on S^{2n+1}. One way to derive (2.2) is to apply the canonical isomorphism φ between $T^*_{0,1}$ and $T_{1,0}$ induced by the Levi metric L_θ to Geller's formula ([G]) for $\bar{\partial}_b$:

$$\bar{\partial}_b f=\sum_{1\le j<k\le n+1}(\bar{Z}_{jk}f)\bar{\theta}_{jk}$$

where $\bar{\theta}_{jk}=\bar{z}_j d\bar{z}_k-\bar{z}_k d\bar{z}_j$. Note that $(1/2)\theta^2+L_\theta$ coincides with the induced metric in [G] on S^{2n+1}. Therefore $d\bar{z}_k$ and $\theta^{\bar{\alpha}}$ are mapped to ∂_k and $2Z_\alpha$ respectively under φ (we have taken $h_{\alpha\bar{\beta}}=\delta_{\alpha\beta}$ here). According to the formula of Geller ([G] p. 420 for $\alpha=n$) and Lee ([L1], p. 414) for $\Box_b$ acting on functions, we have

$$\Delta_b f=\left(-\frac{1}{2}\right)\sum_{1\le j<k\le n+1}(Z_{jk}\bar{Z}_{jk}+\bar{Z}_{jk}Z_{jk})f\ .$$

It follows that $\Delta_b z_j=(n/2)z_j$ and

$$\Delta_b f=(n/2)f \tag{2.3}$$

for all bigraded spherical harmonic f of type (1, 0) or (0, 1) or their linear combinations. (A bigraded spherical harmonic of type (p, q) on C^{n+1} is a harmonic polynomial which is a linear combination of terms of the form $z^\alpha\bar{z}^\beta$, α, β multi-indices with $|\alpha|=p$, $|\beta|=q$.)

Next we compute H_f, the Hessian of f, in the direction $\nabla_b u$. First observe that $D_Z\bar{W}=[Z, \bar{W}]_{T_{0,1}}$, the orthogonal projection of $[Z, \bar{W}]$ onto $T_{0,1}$ for Z, $W\in T_{1,0}$ ([T1], p. 31). It follows that

$$D_{Z_{jk}}\bar{Z}_{lm}=(\delta_{kl}\bar{z}_j-\delta_{jl}\bar{z}_k)(\bar{\partial}_m-z_m\sigma^\#)+(\delta_{jm}\bar{z}_k-\delta_{km}\bar{z}_j)(\bar{\partial}_l-z_l\sigma^\#) \tag{2.4}$$

where $\sigma^\#=\sum_{j=1}^{n+1}\bar{z}_j\bar{\partial}_j$. Therefore writing $\nabla'_b u=(1/2)\sum(\bar{Z}_{jk}u)Z_{jk}$ and $\nabla''_b u=(1/2)\sum(Z_{lm}u)\bar{Z}_{lm}$ by (2.2), we have for $f=\bar{z}_\alpha$

(2.5.1) $\quad H_f(\nabla'_b u, \nabla''_b u)$

$$=\frac{1}{4}H_f(Z_{jk}, \bar{Z}_{lm})(\bar{Z}_{jk}u)(Z_{lm}u)\quad \text{(summation convention)}$$

$$=\frac{1}{4}[(Z_{jk}\bar{Z}_{lm}-D_{Z_{jk}}\bar{Z}_{lm})f](\bar{Z}_{jk}u)(Z_{lm}u)$$

$$=-\frac{1}{4}f[z_l(\bar{z}_j\delta_{km}-\bar{z}_k\delta_{jm})+z_m(\bar{z}_k\delta_{jl}-\bar{z}_j\delta_{kl})](\bar{Z}_{jk}u)(Z_{lm}u) \quad \text{(by (2.4))}$$

$$=-\frac{1}{4}fL_\theta(Z_{jk}, \bar{Z}_{lm})(\bar{Z}_{jk}u)(Z_{lm}u) \quad \text{(Note that } L_\theta=2dz_j\wedge d\bar{z}_j.\text{)}$$

$$=(-f)L_\theta(\nabla'_b u, \nabla''_b u), \quad \text{and}$$

(2.5.2) $$H_f(\nabla'_b u, \nabla''_b u)=0 \quad \text{for} \quad f=z_\alpha \quad \text{by the type reason}.$$

On the other hand, we observe that $D_X Y$ for $X, Y\in T_{1,0}$ is uniquely determined by the condition:

$$L_\theta(D_X Y, \bar{W})=XL_\theta(Y, \bar{W})-L_\theta(Y, [X, \bar{W}]_{T_{0,1}})$$

for $\bar{W}\in T_{0,1}$ (e.g. [T1] p. 31). A straightforward computation shows that the right-hand side of the above formula vanishes for $X=Z_{pq}$, $Y=Z_{jk}$, $\bar{W}=\bar{Z}_{lm}$. It follows that $D_{Z_{pq}}Z_{jk}=0$ since $\{\bar{Z}_{lm}\}$ spans $T_{0,1}$. Then it is easy to see that $H_f(Z_{jk}, Z_{lm})=0$ and

(2.6) $$H_f(\nabla'_b u, \nabla'_b u)=H_f(\nabla''_b u, \nabla''_b u)=0$$

for either $f=z_\alpha$ or $f=\bar{z}_\alpha$. Now, for the same f and u real,

(2.7)
$$\begin{aligned}H_f(\nabla_b u, \nabla_b u)&=H_f(\nabla'_b u, \nabla''_b u)+H_f(\nabla''_b u, \nabla'_b u) \quad \text{(by (2.6))}\\&=H_f(\nabla'_b u, \nabla''_b u)+\overline{H_{\bar{f}}(\nabla'_b u, \nabla''_b u)}\\&=-\frac{1}{2}fL_\theta(\nabla_b u, \nabla_b u) \quad \text{(by (2.5.1) and (2.5.2))}.\end{aligned}$$

3. An integrability condition: Proof of Theorem A.

By letting $Y=\nabla_b u$ in the following identity:

$$\nabla_b(\nabla_b u\cdot\nabla_b f)\cdot Y=\frac{1}{4}[H_u(Y, \nabla_b f)+H_f(Y, \nabla_b u)],$$

we obtain

(3.1) $$2(\nabla_b u\cdot\nabla_b f)\Delta_b u\equiv H_u(\nabla_b u, \nabla_b f)+H_f(\nabla_b u, \nabla_b u)$$

where the symbol "$\equiv$" denotes equality modulo terms which are subdivergences. Set $|\nabla_b u|_\theta^2=\nabla_b u\cdot\nabla_b u$ for u real. Since, for $A, B=1, \cdots, n, \bar{1}, \cdots, \bar{n}$, $\alpha, \beta=1, \cdots, n$,

$$2u_{AB}f^Au^B=(u_Bf^Au^B)_A+2(u_{AB}-u_{BA})f^Au^B-u_Bf^A{}_Au^B$$
$$\equiv 2\Delta_b f|\nabla_b u|_\theta^2+2iu_0h_{\alpha\bar\beta}(f^\alpha u^{\bar\beta}-u^\alpha f^{\bar\beta})$$

by the commutation relations (2.1), we obtain

$$H_u(\nabla_b u\cdot\nabla_b f)\equiv\Delta_b f|\nabla_b u|_\theta^2+2iu_0(\nabla_b' f\cdot\nabla_b'' u-\nabla_b'' f\cdot\nabla_b' u) \tag{3.2}$$

for u real and f complex.

Remember that we are working on S^{2n+1} with $\theta=i(\sigma-\bar\sigma)$ (see §2). Therefore T equals $(i/2)(\zeta_j\partial_j-\bar\zeta_j\bar\partial_j)$ if the ambient space C^{n+1} has coordinates $\zeta_1,\cdots,\zeta_{n+1}$.

Substituting (2.3), (2.7) in (3.1), (3.2) for $f=z_\alpha$ or $\bar z_\alpha$ gives

$$4(\nabla_b u\cdot\nabla_b f)\Delta_b u\equiv(n-1)f|\nabla_b u|_\theta^2+4iu_0(\nabla_b' f\cdot\nabla_b'' u-\nabla_b'' f\cdot\nabla_b' u)\,. \tag{3.3}$$

To get an idea of how to deal with the u_0-term, we carry out a variational argument which has an analogue in the Riemannian case ([R], [KW]).

Consider now u to be a solution of the equation

$$\Delta_b u=q(x,u) \tag{3.4}$$

on S^{2n+1} where x denotes a point of S^{2n+1}. Let

$$Q(x,u)=\int_0^u q(x,s)ds\,.$$

Then u is a critical point of the functional

$$F(u)=\int_{S^{2n+1}}[|\nabla_b u|_\theta^2-Q(x,u)]dv_\theta \tag{3.5}$$

where $dv_\theta=\theta\wedge(d\theta)^n$. Let H^n denote the Heisenberg group whose underlying manifold is $C^n\times R$ with coordinates $(z,t)=(z_1,\cdots,z_n,t)$ (e.g. [JL]). Let $\psi_\lambda: H^n\to H^n$ be the dilation defined by $\psi_\lambda(z,t)=(\lambda z,\lambda^2 t)$, $\lambda>0$. Then $\{\psi_\lambda\}$ induces a family of CR automorphisms $\{\varphi_\lambda\}$ on S^{2n+1} under the Cayley transform ([JL]). Since $\varphi_1=\text{id}$, so

$$\left.\frac{dF(u\circ\varphi_\lambda)}{d\lambda}\right|_{\lambda=1}=0\,. \tag{3.6}$$

If the Cayley transform reads

$$w=i\left(\frac{1-\zeta_{n+1}}{1+\zeta_{n+1}}\right),\qquad z_k=\frac{\zeta_k}{1+\zeta_{n+1}},\quad k=1,\cdots,n$$

for $(z_1,\cdots,z_n,\operatorname{Re}w)\in H^n$, $\operatorname{Im}w=\sum_{j=1}^n|z_j|^2$, and $\zeta=(\zeta_1,\cdots,\zeta_{n+1})\in S^{2n+1}\subset C^{n+1}$ with $(0,\cdots,-1)$ deleted, then $\tilde\zeta=\varphi_\lambda(\zeta)$ is given by

$$\begin{cases} \bar{\zeta}_k = 2i\lambda\zeta_k/[i(1+\lambda^2)+i(1-\lambda^2)\zeta_{n+1}], & k=1,\cdots,n \\ \bar{\zeta}_{n+1} = [i(1-\lambda^2)+i(1+\lambda^2)\zeta_{n+1}]/[i(1+\lambda^2)+i(1-\lambda^2)\zeta_{n+1}]. \end{cases}$$

It follows that

$$\left.\frac{d}{d\lambda}\varphi_\lambda\right|_{\lambda=1} = \sum_{\alpha=1}^{n+1}(\zeta_{n+1}\zeta_\alpha - \delta^\alpha_{n+1})\frac{\partial}{\partial\zeta_\alpha} + \text{conjugate}. \tag{3.7}$$

On the other hand, using the extrinsic expression (2.2) for ∇'_b (hence ∇''_b), we easily obtain

$$2\nabla_b\zeta_{n+1} + 4(T\zeta_{n+1})T = \frac{\partial}{\partial\bar{\zeta}_{n+1}} - \zeta_{n+1}\zeta_\alpha\frac{\partial}{\partial\zeta_\alpha}. \tag{3.8}$$

Comparing (3.7) and (3.8) gives

$$\left.\frac{d}{d\lambda}\varphi_\lambda\right|_{\lambda=1} = -2\nabla_b h - 4h_0 T \tag{3.9}$$

for $h = \zeta_{n+1} + \bar{\zeta}_{n+1}$. Therefore

$$\begin{aligned} \left.\frac{d}{d\lambda}(u\circ\varphi_\lambda)\right|_{\lambda=1} &= \left(\left.\frac{d}{d\lambda}\varphi_\lambda\right|_{\lambda=1}\right)u \\ &= -4(\nabla_b h\cdot\nabla_b u + h_0 u_0). \end{aligned} \tag{3.10}$$

Substituting (3.5) in (3.6) and using (3.10), we have

$$\begin{aligned} 0 &= \left.\frac{d}{d\lambda}F(u\circ\varphi_\lambda)\right|_{\lambda=1} \\ &= \int_{S^{2n+1}} 2\nabla_b\left.\frac{d}{d\lambda}(u\circ\varphi_\lambda)\right|_{\lambda=1}\cdot\nabla_b u - q(x,u)\left.\frac{d}{d\lambda}(u\circ\varphi_\lambda)\right|_{\lambda=1} \\ &= \int_{S^{2n+1}} -8\nabla_b(\nabla_b h\cdot\nabla_b u)\cdot\nabla_b u - 8\nabla_b(h_0u_0)\cdot\nabla_b u + 4q\nabla_b h\cdot\nabla_b u + 4qh_0u_0. \end{aligned} \tag{3.11}$$

The second term in the integrand of (3.11) is expected to deal with the u_0 term in (3.3). So we compute $\nabla_b(f_0u_0)\cdot\nabla_b u$ modulo terms of subdivergences: for $f = \zeta_{n+1}$ (hereafter through (3.19))

$$\begin{aligned} 2\nabla_b(f_0u_0)\cdot\nabla_b u &\equiv u\Delta_b(f_0u_0) \\ &= (n/2)uu_0f_0 + uf_0q_0 - iuf_\alpha u_0{}^\alpha \end{aligned} \tag{3.12}$$

by (2.3), (3.4) and the commutation relation $u_{0\alpha} = u_{\alpha 0}$. (See [L2]. Note that the torsion for our sphere vanishes.) Similar computation shows that

$$(3.13)\qquad \begin{aligned} 2iu_0(\nabla'_b f\cdot\nabla''_b u-\nabla''_b f\cdot\nabla'_b u) &= iu_0(f^\alpha u_\alpha - f^{\bar\alpha}u_{\bar\alpha}) \\ &\equiv iuu_0(f^{\bar\alpha}{}_{\bar\alpha}-f^\alpha{}_\alpha)+iu(f^{\bar\alpha}u_{0\bar\alpha}-f^\alpha u_{0\alpha}) \\ &= -nuu_0f_0+iuf_\alpha u_0{}^\alpha \end{aligned}$$

by (2.1). Adding (3.12) and (3.13) gives

$$(3.14)\qquad 2iu_0(\nabla'_b f\cdot\nabla''_b u-\nabla''_b f\cdot\nabla'_b u)\equiv -(n/2)uu_0f_0+uf_0q_0-u_0f_0q$$

since the left-hand side of (3.12) $\equiv f_0u_0q$ by (3.4).

Let Q_{bx} denote the ∇_b of Q in the variable x while u is considered to be fixed. Then

$$(3.15)\qquad \begin{aligned} q\nabla_b u\cdot\nabla_b f &= (\nabla_b Q-Q_{bx})\cdot\nabla_b f \quad \text{(by the chain rule)} \\ &\equiv (1/2)Q\Delta_b f - Q_{bx}\cdot\nabla_b f \\ &= (n/4)Qf-Q_{bx}\cdot\nabla_b f \end{aligned}$$

by (2.3). One more $\nabla_b u$ term to be estimated:

$$(3.16)\qquad \begin{aligned} f|\nabla_b u|_\theta^2 &= -(1/4)f\Delta_b(u^2)+(1/2)fu\Delta_b u \\ &\equiv -(n/8)fu^2+(1/2)fuq \end{aligned}$$

by (2.3) and (3.4). For our purpose, assume that $q(x, u)$ has the form

$$(3.17)\qquad \lambda u+R(x)u^a$$

where both λ and a are constants with $a>1$ and R is a (smooth) real function on S^{2n+1}. Now we estimate terms involving u_0 on the right-hand side of (3.14):

$$(3.18)\qquad \begin{cases} uu_0f_0 = (1/2)(u^2)_0f_0\equiv(-1/2)u^2f_{00}=(1/8)u^2f \\ uf_0q_0 \equiv -u_0f_0q-uf_{00}q=-u_0f_0q+(1/4)ufq \\ u_0f_0q = [(1/2)\lambda u^2+(1/(a+1))Ru^{a+1}]_0f_0-(1/(a+1))R_0f_0u^{a+1} \\ \qquad\equiv (1/8)\lambda u^2f+(1/4(a+1))Ru^{a+1}f-(1/(a+1))R_0f_0u^{a+1} \end{cases}$$

by $f_{00}=(-1/4)f$. Now substituting (3.14), (3.15) (by (3.4)) and (3.16) in (3.3) and using (3.17), (3.18), we finally obtain

$$\begin{aligned} &(4/(a+1))(\nabla_b R\cdot\nabla_b f+R_0f_0)u^{a+1} \\ &\equiv (n^2/8+(1/2)\lambda)u^2f+[(n+1)/(a+1)-n/2]Ru^{a+1}f. \end{aligned}$$

Letting $\lambda=-n^2/4$ in the above identity and integrating give

$$(3.19)\qquad \int_{S^{2n+1}} u^{a+1}(\nabla_b R\cdot\nabla_b f+R_0f_0)dv_\theta=\frac{1}{8}(n+2-na)\int_{S^{2n+1}} u^{a+1}Rf\,dv_\theta\,.$$

Let ∇ be the gradient operator relative to the metric $\langle\ ,\ \rangle=(1/4)\theta^2+L_\theta$. Then $\nabla g=2\nabla_b g+4g_0T$ for a function g. It follows that

(3.20) $$4(\nabla_b R\cdot\nabla_b f+R_0 f_0)=\langle\nabla R, \nabla f\rangle .$$

By substituting (3.20) in (3.19) and using symmetry, (1.3) holds for $f=\zeta_\alpha$, $\alpha=1, \cdots, n+1$. Taking conjugation and observing the linearity of (1.3) in f complete the proof of Theorem A.

REMARK 1. Our pseudohermitian sphere (S^{2n+1}, θ) has constant pseudohermitian scalar curvature $n(n+1)/2$. If another contact form $\tilde{\theta}$ changes according to $\tilde{\theta}=u^{2/n}\theta$, $u>0$, then its associated pseudohermitian scalar curvature $R_{\tilde{\theta}}$ and u satisfy the following equation:

$$\Delta_b u+\frac{n^2}{4}u-\frac{n}{2(n+1)}R_{\tilde{\theta}}u^{(n+2)/n}=0 .$$

REMARK 2. When $a=(n+2)/n$, the volume form transforms like $dv_{\tilde{\theta}}=u^{a+1}dv_\theta$. Therefore (3.19) can be rewritten as

(3.21) $$\int_{S^{2n+1}} XR_{\tilde{\theta}}dv_{\tilde{\theta}}=0$$

by (3.20) where $X=\nabla f=2\nabla_b f+4f_0 T$. By (3.9),

$$X=-\frac{d}{d\lambda}\varphi_\lambda\bigg|_{\lambda=1}$$

for $f=\zeta_{n+1}+\overline{\zeta_{n+1}}$ (hence $\operatorname{Re}\zeta_\alpha$ or $\operatorname{Im}\zeta_\alpha$, $\alpha=1, \cdots, n+1$ with the suitable choice of Cayley transform) is a CR vector field. It is in the form of (3.21), which we generalize to certain pseudohermitian manifolds in the next section.

4. Geometric interpretation and generalization: Proof of Theorem B.

Let M be a compact oriented CR manifold of dimension $2n+1$. Set $H=\operatorname{Re}(T_{1,0}\oplus T_{0,1})$ as before (see §2). Let $\hat{\theta}$ be a non-vanishing real (smooth) 1-form annihilating H. Let Ω be the set of all $\theta=f\hat{\theta}$ where f is a smooth positive function on M. The space Ω is a Frechet manifold modelled on the Frechet space $C^\infty(M)$ through the correspondence $\theta\rightarrow\varphi$ where $\theta=e^\varphi\hat{\theta}$. Thus $T_\theta\Omega=\{\varphi\in C^\infty(M)\}$. A 1-form ω on Ω is defined by

$$\omega_\theta=R_\theta dv_\theta : \varphi\rightarrow\int_M \varphi R_\theta dv_\theta .$$

Let $G=\operatorname{Aut}^0_{CR}(M)$. G acts on Ω by $g\theta=(g^{-1})^*\theta$ where $g\in G$. It follows that $g_* : T_\theta\Omega\rightarrow T_{g\theta}\Omega$ is given by $g_*(\varphi)=(g^{-1})^*\varphi$.

LEMMA. *ω is a G-invariant closed form.*

PROOF.
$$(g^*\omega_{g\theta})(\varphi)=\omega_{g\theta}(g_*\varphi)=\int_M (g^{-1})^*\varphi R_{g\theta}dv_{g\theta}$$
$$=\int_M (g^{-1})^*\varphi(g^{-1})^*(R_\theta dv_\theta)$$
$$=\int_M \varphi R_\theta dv_\theta=\omega_\theta(\varphi)\,.$$

Hence ω is G-invariant.

Let $\theta_{j,t}=e^{t\varphi_j}\theta$, $j=1, 2$. Then

$$d\omega_\theta(\varphi_1, \varphi_2)=\int_M \varphi_2\frac{d}{dt}(R_{\theta_{1,t}}dv_{\theta_{1,t}})\big|_{t=0}-\int_M \varphi_1\frac{d}{dt}(R_{\theta_{2,t}}dv_{\theta_{2,t}})\big|_{t=0}\,. \tag{4.1}$$

By definition, $dv_{\theta_{j,t}}=\theta_{j,t}\wedge(d\theta_{j,t})^n=e^{(n+1)t\varphi_j}dv_\theta$. And

$$R_{\theta_{j,t}}=[\exp(-1-n/2)t\varphi_j][(2+2/n)\Delta_b\exp(nt\varphi_j/2)+R_\theta\exp(nt\varphi_j/2)]$$

([L1]). Therefore

$$\frac{d}{dt}(R_{\theta_{j,t}}dv_{\theta_{j,t}})\big|_{t=0}=n\varphi_jR_\theta+(n+1)\Delta_b\varphi_j\,. \tag{4.2}$$

Substituting (4.2) in (4.1), we obtain

$$d\omega_\theta(\varphi_1, \varphi_2)=0$$

by self-adjointness of Δ_b. Hence ω is closed. Q.E.D.

A CR vector field X on M induces a vector field φ_x on Ω by $L_x\theta=\varphi_{x,\theta}\theta$. It follows that

$$L_xdv_\theta=(n+1)\varphi_{x,\theta}dv_\theta\,. \tag{4.3}$$

Let $i(\varphi_x)$ denote the operator of taking interior product in the direction φ_x. Applying the basic formula: $L_{\varphi_x}=i(\varphi_x)\circ d+d\circ i(\varphi_x)$ to ω and using the above lemma give that $\omega(\varphi_x)$ is constant on Ω. On the other hand, by definition,

$$\omega_\theta(\varphi_{x,\theta})=\int_M \varphi_{x,\theta}R_\theta dv_\theta$$
$$=\frac{1}{n+1}\int_M R_\theta L_xdv_\theta \quad \text{(by (4.3))}$$
$$=-\frac{1}{n+1}\int_M XR_\theta dv_\theta \quad \text{(by the divergence theorem)}\,.$$

We denote $\int_M XR_\theta dv_\theta$ by $\mu(\theta)$. Since $\mu(\theta)$ is constant, we only have to show that it vanishes for some specific θ. If G is compact, we can construct a G-invariant contact form $\tilde{\theta}$ by averaging the action:

$$\tilde{\theta}=\int_{g\in G} g^*\theta dg$$

for a given $\theta\in\Omega$ where dg denotes the Haar measure of G. (Note that if $\dim G=0$, we do not have any non-vanishing CR vector field at all.) Since $L_x\tilde{\theta}=0$, it follows that $L_x dv_{\tilde{\theta}}=0$ and $\mu(\tilde{\theta})=0$. If G is non-compact and $\pi_1(M)$ is finite, it can be shown that M is globally CR equivalent to S^{2n+1} ([W2] p. 55). So with respect to the standard contact form $\hat{\theta}$, $R_{\hat{\theta}}$ is constant. Hence $\mu(\hat{\theta})=0$. We have completed the proof of Theorem B.

Added in proof.

After this paper was submitted, the author gave a talk on results of this paper at the University of Washington, Seattle. Later Robin Graham pointed out that the conformal analogue of Theorem B can be proved directly by integrating by parts. (The author learned that Bourguignon also obtained this integrating-by-parts proof in a paper jointly with Ezin.) Inspired by Robin's argument, Jack Lee was able to give an integrating-by-parts proof of our Theorem B in full generality.

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