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Slant Submanifolds in Complex Euclidean Spaces

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Abstract. An immersion of a differentiable manifold into an almost Hermitian manifold is called a *general slant immersion* if it has constant Wirtinger angle ([3, 6]). A general slant immersion which is neither holomorphic nor totally real is called a *proper slant immersion*. In the first part of this article, we prove that every general slant immersion of a compact manifold into the complex Euclidean m-space C^m is totally real. This result generalizes the well-known fact that there exist no compact holomorphic submanifolds in any complex Euclidean space. In the second part, we classify proper slant surfaces in C^2 when they are contained in a hypersphere S^3, or contained in a hyperplane E^3, or when their Gauss maps have rank <2.

1. Introduction.

We follow the definitions and notations given in [1], [2], [3], and [6].

Let $E^{2m}=(R^{2m}, \langle\ ,\ \rangle)$ and $C^m=(E^{2m}, J_0)$ be the Euclidean $2m$-space and the complex Euclidean m-space, respectively, with the canonical inner product $\langle\ ,\ \rangle$ and the canonical (almost) complex structure J_0 given by

$$J_0(x_1, y_1, \cdots, x_m, y_m)=(-y_1, x_1, \cdots, -y_m, x_m)\,. \tag{1.1}$$

Denote by Ω_0 the Kaehler form of C^m, i.e.,

$$\Omega_0(X, Y)=\langle X, J_0Y\rangle\,, \qquad X, Y\in E^{2m}\,, \quad \Omega_0\in\bigwedge{}^2(E^{2m})^*\,. \tag{1.2}$$

By an α-*slant immersion* [3, 6] we mean a general slant immersion $x: M\to C^m$ with slant angle $\alpha\in[0, \pi/2]$, i.e.,

$$\angle(J_0(x_*X), x_*(T_pM))=\alpha \qquad \forall X\in T_pM-\{0\}\,, \quad \forall p\in M\,, \tag{1.3}$$

where $\angle$ denotes the angle in E^{2m} with respect to the inner product $\langle\ ,\ \rangle$. A totally real immersion [4] and a holomorphic immersion are nothing but a $\pi/2$-slant immersion and a 0-slant immersion, respectively. A proper slant immersion is an α-slant immersion with $\alpha\neq 0, \pi/2$.

In section 2 we prove the following

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THEOREM 1.1. *Let x be a general slant immersion of a differentiable manifold M into the complex Euclidean m-space C^m. If M is compact, then x is totally real.*

This theorem shows that there exist no compact proper slant submanifolds in C^m as in the case of holomorphic submanifolds. We note that one may construct compact proper slant surfaces in a (flat) complex torus via a proper slant plane in C^2 (cf. Example 3 of [3]). We also note that there exist many compact totally real submanifolds in C^m.

Next, we consider a slant immersion x from a noncompact oriented surface M into C^2. As in [6] we may extend the slant angle by defining it as

$$\alpha=\cos^{-1}(-\Omega_0(X, Y))\in[0, \pi]\,, \tag{1.4}$$

where $\{X, Y\}$ is a local positive orthonormal frame field on M.

Now we assume that the image of x is contained in a hypersphere S^3 of C^2. Since the slant angle is invariant under the parallel translations and homotheties in C^2, we can assume, without loss of generality, that S^3 is the unit hypersphere centered at the origin of C^2. Let η denote the unit outer normal vector of S^3 in C^2 and ξ the unit positive normal vector of $x(M)$ in S^3. It is known that S^3 is the Lie group of unit quaternions which can be regarded as a subgroup of $O(4)$ in a natural way (cf. [7, p. 142]). Let 1 denote the identity element of the Lie group S^3 given by

$$1=(1, 0, 0, 0)\in S^3\subset E^4\,. \tag{1.5}$$

We put

$$X_1=(0, 1, 0, 0)\,, \quad X_2=(0, 0, 1, 0)\,, \quad X_3=(0, 0, 0, 1)\in T_1S^3\,. \tag{1.6}$$

We denote by $\tilde{X}_i$, $i=1, 2, 3$, the left-invariant extensions of X_i, $i=1, 2, 3$, on S^3, respectively (cf. [7, p. 145]). Let $\phi: S^3\to S^3$ be the orientation-reversing isometry defined by

$$\phi(a, b, c, d)=(a, b, d, c)\,. \tag{1.7}$$

We define two maps g_+ and g_- from M into the unit sphere S^2 in T_1S^3 by

$$g_+(p)=(L_{\phi(x(p))*})^{-1}(\phi_*\xi(p))\,, \qquad g_-(p)=(L_{x(p)*})^{-1}(\xi(p)) \tag{1.8}$$

for $p\in M$. In fact, g_+ and g_- are the analogous of the classical Gauss map of a surface in E^3 in which the parallel translations in E^3 are replaced by the left-translations L_q on S^3. We also define a circle S^1_α for $\alpha\in[0, \pi]$ on the unit sphere S^2 in T_1S^3 by

$$S^1_\alpha=\{X\in T_1S^3 \mid \|X\|=1, \langle X, X_1\rangle=-\cos\alpha\}\,. \tag{1.9}$$

Finally, let J_0^- denote the complex structure on E^4 defined by

$$J_0^-(x, y, z, w)=(-y, x, w, -z)\,. \tag{1.10}$$

In section 3 we prove the following result which characterizes spherical slant surfaces in C^2.

PROPOSITION 1.2. *Let $x: M \to S^3 \subset E^4$ be an immersion of an oriented surface M. Then we have*

(i) *x is α-slant with respect to J_0 if and only if*

$$g_+(M) \subset S^1_\alpha \subset T_1 S^3 . \tag{1.11}$$

(ii) *x is α-salnt with respect to J_0^- if and only if*

$$g_-(M) \subset S^1_{\pi-\alpha} \subset T_1 S^3 . \tag{1.12}$$

This proposition provides us the spherical version of Proposition 4.1 of [6]. In order to describe slant surfaces in S^3 geometrically, we give the following

DEFINITION 1.3. Let $c(s)$ be a curve in S^3 parametrized by arclength and let

$$c'(s) = \sum_{i=1}^{3} f_i(s)\tilde{X}_i(c(s)) . \tag{1.3}$$

We call the curve $c(s)$ a *helix in S^3 with axis vector field $\tilde{X}_1$* if

$$f_1(s) = b , \quad f_2(s) = a \cos(ks + s_0) , \quad f_3(s) = a \sin(ks + s_0) \tag{1.14}$$

for some constants a, b, k, and s_0 satisfying

$$a^2 + b^2 = 1 . \tag{1.15}$$

We call the curve $c(s)$ a *generalized helix in S^3 with axis vector field $\tilde{X}_1$* if

$$\langle c'(s), \tilde{X}_1(c(s)) \rangle = \text{constant} . \tag{1.16}$$

The helices in S^3 defined above are the analogues of Euclidean helices in E^3.

DEFINITION 1.4. We call an immersion $x: D \to S^3$ of a domain D around the origin (0, 0) of $\boldsymbol{R}^2$ into S^3 a *helical cylinder in S^3* if

$$x(s, t) = \gamma(t) \cdot c(s) , \tag{1.18}$$

for some helix $c(s)$ in S^3 with axis $\tilde{X}_1$ satisfying $k = -2/b$ and $ab < 0$ and for some curve $\gamma(t)$ in S^3 which is either a geodesic or a curve of constant torsion 1 parametrized by arclength such that (i) $c(0) = \gamma(0)$, and (ii) the osculating planes of $c(s)$ and of $\gamma(t)$ coincide at $t = s = 0$. We note that the binormal of $c(s)$ is normal to $x(D)$ in S^3 by Lemma 4.2 in section 4 and [7, pp. 149–157]. Here we orient the curve c in such a way that the binormal of $c(s)$ is the positive unit normal of $x(D)$.

In section 4 we prove the following classification theorem for spherical slant surfaces.

THEOREM 1.5. *Let $x: M \to S^3 \subset \boldsymbol{C}^2 = (E^4, J_0)$ be a spherical immersion of an oriented surface M into the complex 2-plane $\boldsymbol{C}^2 = (E^4, J_0)$. Then x is a proper slant immersion if and only if $x(M)$ is locally of the form $\{\phi(\gamma(t) \cdot c(s))\}$ where ϕ is the isometry* *on* S^3 *defined by* (1.7) *and* $\{\gamma(t)\cdot c(s)\}$ *is a helical cylinder in* S^3 (cf. Definition 1.4).

For an immersion $x: M\to C^m$, the Gauss map ν of the immersion x is given by (cf. [5])

$$(1.18)\qquad \begin{gathered} \nu: M\to G(l, 2m)\equiv D_1(l, 2m)\subset S^{N-1}\subset \textstyle\bigwedge^l(E^{2m}),\\ \nu(p)=e_1(p)\wedge\cdots\wedge e_l(p), \qquad p\in M, \end{gathered}$$

where $l=\dim M$, $N=\binom{2m}{l}$, $D_1(l, 2m)$ is the set of all unit decomposable l-vectors in $\bigwedge^l E^{2m}$, identified with the real Grassmannian $G(l, 2m)$ in a natural way, and S^{N-1} is the unit hypersphere of $\bigwedge^l(E^{2m})$ centered at the origin, and $\{e_1, \cdots, e_{2m}\}$ is a local adapted orthonormal tangent frame along $x(M)$.

In section 5 we prove the following classification theorem.

THEOREM 1.6. *If* $x: M\to C^2=(E^4, J_0)$ *is a general slant immersion such that the rank of its Gauss map is less than* 2, *then the image* $x(M)$ *of* x *is a union of some flat ruled surfaces in* E^4. *Furthermore,*

(i) *A cylinder in* C^2 *is a general slant surface if and only if it is of the form* $\{c(s)+te\}$, *where* e *is a fixed unit vector and* $c(s)$ *is a* (*Euclidean*) *generalized helix with axis* J_0e *contained in a hyperplane of* E^4 *and with* e *as its hyperplane normal.*

(ii) *A cone in* C^2 *is a general slant surface if and only if, up to translations, it is of the form* $\{tc(s)\}$, *where* $(\phi\circ c)(s)$ *is a generalized helix in* S^3 *with axis* $\tilde{X}_1$ (cf. Definition 1.3).

(iii) *A tangential developable surface* $\{c(s)+(t-s)c'(s)\}$ *in* C^2 *is a general slant surface if and only if, up to rigid motions,* $(\phi\circ c')(s)$ *is a generalized helix in* S^3 *with axis* $\tilde{X}_1$.

In the last section we prove the following

THEOREM 1.7. *Let* $x: M\to C^2=(E^4, J_0)$ *be a proper slant immersion of an oriented surface* M *into* C^2. *If* $x(M)$ *is contained in a hyperplane* W *of* E^4, *then* x *is a doubly slant immersion* (*in the sense of* [6]) *and* $x(M)$ *is a union of some flat ruled surfaces in* W. *Furthermore,*

(i) *A cylinder in* W *is a proper slant surface with respect to a complex structure* J *on* E^4 *if and only if it is a portion of a 2-plane.*

(ii) *A cone in* W *is a proper slant surface with respect to a complex structure* J *on* E^4 *if and only if it is a circular cone.*

(iii) *A tangential developable surface in* W *is a proper slant surface with respect to a complex structure* J *on* E^4 *if and only if it is a tangential developable surface obtained from a generalized helix* c *in* W.

In the classifications of slant surfaces given in Theorems 1.5, 1.6, and 1.7, we avoid the messy argument of gluing.

2. Compact slant submanifolds.

The following two lemmas follow easily from direct computation.

LEMMA 2.1. *For $X_1, \cdots, X_{2n} \in E^{2m}$ $(n<m)$, we have*

$$(2.1)\quad (2n)!\Omega_0^n(X_1 \wedge \cdots \wedge X_{2n}) = \sum_{\sigma \in S_{2n}} \operatorname{sign}(\sigma)\Omega_0(X_{\sigma(1)}, X_{\sigma(2)}) \cdots \Omega_0(X_{\sigma(2n-1)}, X_{\sigma(2n)}),$$

where S_{2n} is the permutation group of order $2n$, sign *denotes the signature of permutations and $\Omega_0^n \in \bigwedge^{2n}(E^{2m})^* \equiv (\bigwedge^{2n} E^{2m})^*$.*

LEMMA 2.2. *Let $V \in G(l, 2m)$ and $\pi_V : E^{2m} \to V$ be the orthogonal projection. If V is α-slant in $C^m \equiv (E^{2m}, J_0)$ with $\alpha \neq \pi/2$, then the linear endomorphism J_V of V defined by*

$$(2.2)\quad J_V = (\sec \alpha)(\pi_V \circ J_0 |_V)$$

is a complex structure compatible with the inner product $\langle\, ,\, \rangle|_V$. In particular, l is even.

Let ζ_0 be the metrical dual of $(-\Omega_0)^n$ with respect to the inner product $\langle\, ,\, \rangle$ naturally defined on $\bigwedge^{2n} E^{2m}$, i.e.,

$$(2.3)\quad \langle \zeta_0, \eta \rangle = (-1)^n \Omega_0^n(\eta) \qquad \text{for} \quad \forall \eta \in \bigwedge^{2n} E^{2m},$$

then we have the following

LEMMA 2.3. *Let $V \in G(2n, 2m)$. If V is α-slant in C^m with $\alpha \neq \pi/2$, then*

$$(2.4)\quad \langle \zeta_0, V \rangle = \mu_n \cos^n \alpha,$$

where μ_n is a nonzero constant depending only on n.

PROOF. Let J_V be the complex structure on V defined by Lemma 2.2. For a unit vector $X \in V$, we put $Y = J_V X \in V$. Then we have

$$(2.5)\quad \Omega_0(X, J_V X) = \langle -J_V Y, J_0 Y \rangle = -\cos \alpha.$$

If $X, Z \in V$ and Z is perpendicular to $J_V X$, then

$$(2.6)\quad \Omega_0(X, Z) = \cos \alpha \langle X, J_V Z \rangle = 0.$$

Therefore, if we choose an orthonormal J_V-basis $\{e_1, \cdots, e_{2n}\}$ on V, i.e.,

$$(2.7)\quad e_{2k} = J_V e_{2k-1}, \qquad k = 1, \cdots, n,$$

and

$$(2.8)\quad V = e_1 \wedge \cdots \wedge e_{2n},$$

via the natural identification of $G(2n, 2m)$ with $D_1(2n, 2m)$, then we have

$$(2.9)\quad \Omega_0(e_a, e_b) = -\delta_{a^*b} \cos \alpha \qquad \text{for} \quad a < b,$$

where

$$(2k)^* = 2k-1\,, \quad (2k-1)^* = 2k \qquad \text{for} \quad k=1, \cdots, n\,. \tag{2.10}$$

By (2.8), Lemma 2.1, and (2.9) we find

$$\begin{aligned}
(2n)!\Omega_0^n(V) &= (2n)!\Omega_0^n(e_1 \wedge \cdots \wedge e_{2n}) \\
&= \sum_{\sigma \in S_{2n}} \operatorname{sign}(\sigma)\Omega_0(e_{\sigma(1)}, e_{\sigma(2)}) \cdots \Omega_0(e_{\sigma(2n-1)}, e_{\sigma(2n)}) \\
&= \sum_{a_1, \cdots, a_{2n}=1}^{2n} \delta^{12\cdots(2n)}_{a_1 \cdots a_{2n}} \Omega_0(e_{a_1}, e_{a_2}) \cdots \Omega_0(e_{a_{2n-1}}, e_{a_{2n}}) \\
&= \sum_{a_1, \cdots, a_n=1}^{2n} \delta^{12\cdots\cdots(2n)}_{a_1 a_1^* \cdots\, a_n a_n^*} \Omega_0(e_{a_1}, e_{a_1^*}) \cdots \Omega_0(e_{a_n}, e_{a_n^*}) \\
&= 2^n \sum_{a_1 < a_1^*} \cdots \sum_{a_n < a_n^*} \delta^{12\cdots\cdots(2n)}_{a_1 a_1^* \cdots\, a_n a_n^*} \Omega_0(e_{a_1}, e_{a_1^*}) \cdots \Omega_0(e_{a_n}, e_{a_n^*}) \\
&= 2^n(-\cos\alpha)^n \sum_{a_1 < a_1^*} \cdots \sum_{a_n < a_n^*} \delta^{12\cdots\cdots(2n)}_{a_1 a_1^* \cdots\, a_n a_n^*} \\
&= 2^n(-\cos\alpha)^n n!\,.
\end{aligned} \tag{2.11}$$

Hence, by (2.3) and (2.11) we obtain (2.4) with $\mu_n = 2^n n!/(2n)!$. Q.E.D.

PROOF OF THEOREM 1.1. Assume x is α-slant with $\alpha \neq \pi/2$. Then, by Lemma 2.2, l is even. Put $l=2n$. Since M is compact, it is known that the Gauss image $\nu(M)$ is mass-symmetric in S^{N-1} (cf. Lemma 3.1 of [5]). Therefore

$$\int_{p \in M} \langle \nu(p), \zeta \rangle dV_M = 0 \tag{2.12}$$

for any fixed $2n$-vector $\zeta \in \bigwedge^{2n}(E^{2m})$, where dV_M is the volume element of M with respect to the metric induced from the immersion x. Let $\zeta = \zeta_0$. Then Lemma 2.3 and (2.12) imply

$$\mu_n \operatorname{vol}(M) \cos^n \alpha = 0\,. \tag{2.13}$$

But this contradicts the assumption $\cos\alpha \neq 0$. Hence $\alpha = \pi/2$ and x is a totally real immersion. Q.E.D.

3. Characterization of spherical slant surfaces.

The main purpose of this section is to prove Proposition 1.2. To do so we recall that the left-translation L_p and the right-translation R_p on S^3 are isometries which are analogous to the parallel translations on E^3 and they are given by

$$ {}^t(L_pq)=\begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}, \tag{3.1} $$

$$ {}^t(R_pq)=\begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \tag{3.2} $$

for $p=(a, b, c, d)$, $q=(x, y, z, w)\in S^3\subset E^4$, where ${}^tA$ denotes the transpose of A.

Let η denote the unit outer normal of S^3 in E^4 and J_0 and J_0^- the complex structures on E^4 as defined by (1.1) and (1.10), respectively.

By direct computation we have the following

LEMMA 3.1. *For any $q\in S^3$, we have*

$$ (J_0\eta)(q)=R_{q*}X_1\,, \tag{3.3} $$

$$ (J_0^-\eta)(q)=L_{q*}X_1=\tilde{X}_1(q)\,. \tag{3.4} $$

Hence $J_0\eta$ and $J_0^-\eta$ are right- and left-invariant vector fields on S^3, respectively.

We denote by $\mathscr{J}$ the set of all complex structures on E^4 compatible with the inner product $\langle\ ,\ \rangle$. For each fixed $J\in\mathscr{J}$ an orthonormal basis $\{e_1, \cdots, e_4\}$ of E^4 is called a *J-basis* if $Je_1=e_2$, $Je_3=e_4$. Two J-bases have the same orientation. By using the natural orientation of E^4 we can divide $\mathscr{J}$ into two disjoint subsets:

$$ \mathscr{J}^+=\{J\in\mathscr{J}\mid J\text{-bases are positive}\}\,, $$

$$ \mathscr{J}^-=\{J\in\mathscr{J}\mid J\text{-bases are negative}\}\,. $$

Thus we have $\mathscr{J}=\mathscr{J}^+\cup\mathscr{J}^-$ (cf. section 3 of [6]).

LEMMA 3.2. *Let $W\in G(3,4)$ and $V\in G(2,4)$ such that $V\subset W$. Then V is α-slant with respect to a complex structure $J\in\mathscr{J}^+$ (respectively, $J\in\mathscr{J}^-$) if and only if*

$$ \langle\xi_V, J\eta_W\rangle=-\cos\alpha \qquad (\textit{respectively, } \langle\xi_V, J\eta_W\rangle=+\cos\alpha)\,, \tag{3.5} $$

where ξ_V and η_W are positive unit normal vectors of V in W and of W in E^4, respectively.

PROOF OF LEMMA 3.2. We choose an orthonormal J-basis $\{e_1, \cdots, e_4\}$ of E^4 such that

$$ e_1, e_2\in W\cap JW\,, \qquad e_4=Je_3=\eta_W\,. \tag{3.6} $$

We also choose a positive orthonormal basis $\{X_1, X_2\}$ of V. Let ζ_J be the 2-vector defined as the metrical dual of $-\Omega_J\in(\bigwedge^2E^4)^*$, i.e., $\langle\zeta_J, X\wedge Y\rangle=-\Omega_J(X, JY)$ for any

$X, Y \in E^4$. Then by formula (1.4) we see that the slant angle $\alpha_J(V)$ of V with respect to J satisfies

$$\cos \alpha_J(V) = \langle \zeta_J, X_1 \wedge X_2 \rangle = \langle e_1 \wedge e_2 + e_3 \wedge e_4, X_1 \wedge X_2 \rangle$$
$$= \langle e_1 \wedge e_2, X_1 \wedge X_2 \rangle = \langle \pm e_3, \xi_V \rangle = \mp \langle J\eta_W, \xi_V \rangle$$

for $J \in \mathscr{J}^{\pm}$. This proves the lemma. Q.E.D.

Let $x : M \to S^3 \subset E^4$ be a spherical immersion of an oriented surface M into S^3 and ξ the positive unit normal of $x(M)$ in S^3. Then we have

LEMMA 3.3. *The following three statements hold.*

(i) *x is α-slant with respect to J_0 if and only if*

$$\langle \xi(p), J_0 \eta(x(p)) \rangle = -\cos \alpha \qquad \textit{for} \quad \forall p \in M . \tag{3.7}$$

(ii) *x is α-slant with respect to J_0^- if and only if*

$$\langle \xi(p), \tilde{X}_1(x(p)) \rangle = +\cos \alpha \qquad \textit{for} \quad \forall p \in M . \tag{3.8}$$

(iii) *x is α-slant with respect to J_0 if and only if $\phi \circ x$ is α-slant with respect to J_0^-.*

PROOF OF LEMMA 3.3. Statement (i) follows from Lemma 3.2. Statement (ii) follows from Lemma 3.1 and Lemma 3.2. Finally, the last statement follows from statements (i) and (ii) and from the fact that ϕ is an isometric involution reversing the orientation of E^4. Q.E.D.

PROOF OF PROPOSITION 1.2. Proposition 1.2 follows from Lemma 3.1, Lemma 3.3 and the definitions of g_+ and g_-. Q.E.D.

Concerning the images of g_+ and g_- we give here the following two simple examples.

EXAMPLE 3.1. If $M = S^1 \times S^1$ is the flat torus in E^4 defined by

$$x(u, v) = \frac{1}{\sqrt{2}} (\cos u, \sin u, \cos v, \sin v) ,$$

then the images of g_+ and g_- are the great circle perpendicular to $X_1 = (0, 1, 0, 0)$.

EXAMPLE 3.2. If $M = S^2$ is the totally geodesic 2-sphere of S^3 parametrized by

$$x(u, v) = (\cos u \cos v, \sin u \cos v, \sin v, 0) ,$$

then

$$g_+(u, v) = (0, -\sin v, -\cos u \cos v, \sin u \cos v) ,$$

$$g_-(u, v) = (0, \sin v, \sin u \cos v, -\cos u \cos v) .$$

Hence, both g_+ and g_- are isometries.

4. Classification of spherical slant surfaces.

The main purpose of this section is to prove Theorem 1.5 which classifies spherical proper slant surfaces in $\boldsymbol{C}^2$. In order to do so we need several lemmas.

First, we note that curves in S^3 can be described in terms of the orthonormal left-invariant vector fields $\{\tilde{X}_1, \tilde{X}_2, \tilde{X}_3\}$ (cf. §1 or chapter 7 of [7]). Let I be an open interval containing 0 and $c: I \to S^3$ a curve parametrized by arclength s. Let $\boldsymbol{t}(s)$, $\boldsymbol{n}(s)$, $\boldsymbol{b}(s)$, $\kappa(s)$, and $\tau(s)$ be the unit tangent vector, the unit principal normal vector, the unit binormal vector, the curvature, and the torsion of c in S^3, respectively. We put

$$\boldsymbol{t}(s) = \sum_{i=1}^{3} f_i(s)\tilde{X}_i(c(s)) . \tag{4.1}$$

Then

$$(f_1(s))^2 + (f_2(s))^2 + (f_3(s))^2 = 1 . \tag{4.2}$$

Conversely, we have the following

LEMMA 4.1. *Let $f_i(s)$, $i=1, 2, 3$, be differentiable functions on I satisfying* (4.2). *Then, for any point $p_0 \in S^3$, there exists a curve $c(s)$ in S^3 defined on an open subinterval I' of I containing* 0 *and satisfying* (4.1) *and $c(0) = p_0$.*

PROOF. Considering the curve $L_{p_0}^{-1} \circ c$ instead if necessary, we can assume without loss of generality that $p_0 = 1$. The solution of the following system of the first order linear differential equations

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} x & -y & -z & -w \\ y & x & -w & z \\ z & w & x & -y \\ w & -z & y & x \end{pmatrix} \begin{pmatrix} 0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} \tag{4.3}$$

with the initial condition $(x(0), y(0), z(0), w(0)) = (1, 0, 0, 0)$ satisfies $xx' + yy' + zz' + ww' = 0$ and the curve $c(s) = (x(s), y(s), z(s), w(s))$ is in fact the desired one. Q.E.D.

Lemma 4.1 guarantees the existence of helices in S^3.

LEMMA 4.2. *The following two statements are equivalent*:

(i) *The curve $c(s)$ is a helix in S^3 with axis vector $\tilde{X}_1$ of the form*

$$f_1(s) = b , \tag{4.4}$$

$$f_2(s) = a \cos\left(-\frac{2}{b}s + s_0\right) , \tag{4.5}$$

$$f_3(s) = a \sin\left(-\frac{2}{b}s + s_0\right) , \tag{4.6}$$

(4.7) $$a^2+b^2=1\,,\qquad ab<0\,.$$

(ii) *The curve $c(s)$ satisfies*

(4.8) $$\tau(s)\equiv -1\,,$$

(4.9) $$\langle \boldsymbol{b}(s), \tilde{X}_1(c(s))\rangle \equiv a\,,$$

(4.10) $$a\neq \pm 1, 0\,.$$

PROOF. (ii) $\Rightarrow$ (i): We apply the theory of curves in S^3 mentioned in [7, pp. 145–148].

Suppose c is a curve in S^3 parametrized by arclength and let the unit tangent vector $\boldsymbol{t}$ of c be given by (4.1). Let $g_1=f_2f'_3-f_3f'_2$, $g_2=f_3f'_1-f_1f'_3$, $g_3=f_1f'_2-f_2f'_1$. By Frenet-Serret formulas and (4.8) we have

(4.11) $$\left(\frac{g_i}{\kappa}\right)'=\frac{2f'_i}{\kappa}\,,\qquad i=1, 2, 3\,.$$

By using (4.9) and the identity $\boldsymbol{b}=\boldsymbol{t}\times\boldsymbol{n}$, we may obtain

(4.12) $$a=\frac{g_1}{\kappa}\,.$$

Hence, we find $2f'_1/\kappa=a'=0$. Let b denote f_1 which is a constant. Then, from (4.12) and $\boldsymbol{b}=\boldsymbol{t}\times\boldsymbol{n}$, we may find

(4.13) $$\boldsymbol{b}=a\tilde{X}_1-\left(\frac{bf'_3}{\kappa}\right)\tilde{X}_2+\left(\frac{bf'_2}{\kappa}\right)\tilde{X}_3\,,$$

(4.14) $$\kappa^2=(f'_2)^2+(f'_3)^2\,,$$

(4.15) $$\boldsymbol{n}=\left(\frac{f'_2}{\kappa}\right)\tilde{X}_2+\left(\frac{f'_3}{\kappa}\right)\tilde{X}_3\,.$$

Since $\|\boldsymbol{b}\|=1$, $\|\boldsymbol{n}\|=1$, (4.13) and (4.15) imply $a^2+b^2=1$. Thus, from $\|\boldsymbol{t}\|=1$ and (4.1) we get $f_2^2+f_3^2=a^2$. So we may put

(4.16) $$f_2=a\cos\theta\,,\quad f_3=a\sin\theta\,,\quad \theta=\theta(s)\,.$$

Thus by applying the definition of g_1, g_2, g_3 we have

(4.17) $$g_1=a\kappa\,,\quad g_2=-bf'_3\,,\quad g_3=bf'_2\,.$$

By using (4.10), (4.14), (4.16), and $\tau\neq 0$, we get

(4.18) $$\kappa=|a\theta'|\neq 0\,.$$

From (4.1), (4.16), (4.17) and (4.18) we find $(b\theta'+2)\sin\theta=0$. Since $\sin\theta(s)$ has only isolated zeros by (4.18), $b\theta'+2=0$. Thus, $b\neq 0$. So $\theta=-(2/b)s+s_0$, $s_0=\text{const}$. Hence, by (4.12) and $\kappa>0$, we get $ab<0$.

(i) $\Rightarrow$ (ii) follows from straight-forward computation. Q.E.D.

LEMMA 4.3. *A helical cylinder* $x(M)=\{\gamma(t)\cdot c(s)\}$ *in* S^3 *is a proper slant surface with respect to* J_0^- *with the slant angle equal to* $\cos^{-1}a$, *where* a *is the constant given by* (1.14) *of Definition* 1.3.

PROOF. Let ξ be the positive unit normal of $x(M)$ in S^3 and $\boldsymbol{b}$ the binormal vector of c in S^3. Then we have

$$\xi(\gamma(t)\cdot c(s))=L_{\gamma(t)*}\boldsymbol{b}(s)\,. \tag{4.19}$$

Lemma 4.3 then follows from Lemma 3.3 and Lemma 4.2. Q.E.D.

LEMMA 4.4. *For any point* $p_0\in S^3$ *and any oriented* 2-*plane* $P_0\subset T_{p_0}S^3\subset E^4$ *which is proper slant with respect to* J_0^-, *there exist helical cylinders in* S^3 *passing through* p_0 *and whose tangent planes at* p_0 *are* P_0.

PROOF. Let ξ be the positive unit normal of P_0 in $T_{p_0}S^3$ and α the slant angle of P_0 with respect to J_0^-. Put

$$a=\cos\alpha\ (\neq 0,\ \pm 1)\,,\qquad b=\pm(1-a^2)^{1/2}\,,$$

where $\pm$ is chosen so that $ab<0$. Pick $s_0\in[0, 2\pi)$ such that

$$\cos s_0=-\frac{1}{b}\langle\xi, \tilde{X}_2(p_0)\rangle\,,\qquad \sin s_0=-\frac{1}{b}\langle\xi, \tilde{X}_3(p_0)\rangle\,.$$

We define f_i by (4.4)–(4.6). Then they satisfy (4.2) and we can choose a curve $c(s)$ satisfying the conditions mentioned in Lemma 4.1.

Let $\gamma(t)$ be either a geodesic in S^3 satisfying

$$\gamma(0)=p_0\,,\quad \gamma'(0)\in P_0\,,\quad \gamma'(0)\neq c'(0)\,,$$

or a curve in S^3 satisfying this condition and the condition $\tau\equiv 1$ (see Theorem 3 of [7, p. 35] for the existence of such curves). Then we can verify that $\{\gamma(t)\cdot c(s)\}$ is a desired surface. Q.E.D.

PROOF OF THEOREM 1.5. First, we note that the isometry ϕ of S^3 has the following properties:

$$\phi(p\cdot q)=\phi(q)\cdot\phi(p)\,,\qquad \text{for}\quad \forall p, q\in S^3\,, \tag{4.20}$$

$$\begin{aligned}&X\in\mathscr{X}(S^3)\text{ is left-(respectively, right-)invariant}\\ &\Longleftrightarrow \phi_*X\text{ is right-(respectively, left-)invariant}\,,\end{aligned} \tag{4.21}$$

$$\tau_{\phi\circ c}=-\tau_c\qquad \text{for a curve } c \text{ in } S^3\,, \tag{4.22}$$

(4.23) $\quad \boldsymbol{b}$ is the binormal of a curve c in $S^3 \Leftrightarrow -\phi_* \boldsymbol{b}$ is the binormal of $\phi \circ c$ in S^3,

where τ_c denotes the torsion of the curve c in S^3.

Let α be the slant angle of $x(M)$ with respect to J_0. Since $x(M)$ is spherical, the normal curvature R^D of the slant immersion x vanishes. Thus, by Lemma 4.1 of [3], M is a flat surface in S^3. Therefore, $x(M)$ is locally a flat translation surface $x(M)=\{c(s)\cdot\gamma(t)\}$ (cf. [7, pp. 149–157]), where c and γ are curves in S^3 parametrized by arclength satisfying one of the following conditions:

(i) $$\tau_c \equiv +1 \quad \text{and} \quad \tau_\gamma \equiv -1 ,$$

(ii) $$\tau_c \equiv +1 \quad \text{and} \quad \gamma \text{ is a geodesic} ,$$

(ii′) $$c \text{ is a geodesic and } \tau_\gamma \equiv -1 ,$$

(iii) $$c \text{ and } \gamma \text{ are distinct geodesics} .$$

Cases (i) *and* (ii): Let $\boldsymbol{b}$ be the binormal of c. With a suitable choice of orientations, $\boldsymbol{b}$ is the positive unit normal of $x(M)$ in S^3. By Lemma 3.3, Lemma 4.2, (4.22), and (4.23), $\phi \circ c$ is a helix in S^3 with a and b in (4.4)–(4.7) determined by

(4.24) $$a=\cos\alpha , \quad b=\pm\sin\alpha , \quad ab<0 ,$$

and either $\tau_{\phi\circ\gamma} \equiv +1$ or $\phi\circ\gamma$ is a geodesic. So, by (4.20), $(\phi\circ x)(M)$ is a helical cylinder in S^3.

The converse is given by Lemma 4.3. Moreover, Lemma 4.4 guarantees the existence of such surfaces.

Next, we want to show that both cases (ii′) and (iii) do not occur. Without loss of generality we can assume

(4.25) $$c(0)=\gamma(0)=1\in S^3 ,$$

because Lemmas 3.1 and 3.3 imply that the slantness of a surface in S^3 with respect to J_0 is right-invariant, i.e., if x is α-slant with respect to J_0, so is $R_q \circ x$ for any $q\in S^3$, and hence we can replace x, c and γ by $R_{c(0)\cdot\gamma(0)}\circ x$, $R_{c(0)}\circ c$ and $L_{c(0)}\circ R_{\gamma(0)^{-1}\cdot c(0)^{-1}}\circ\gamma$, respectively, if necessary.

Case (ii′): Let $\boldsymbol{b}$ be the binormal of γ. We can choose the orientation so that

(4.26) $$\xi(c(s)\cdot\gamma(t))=L_{c(s)*}\boldsymbol{b}(t) , \qquad \text{for} \quad \forall s, t .$$

So, by Lemmas 3.1 and 3.3, and (4.25), we have

(4.27) $$\langle L_{c(s)*}\boldsymbol{b}(0), R_{c(s)*}X_1\rangle = -\cos\alpha \qquad \text{for} \quad \forall s .$$

Put

(4.28) $$c'(0)=(0, a_1, a_2, a_3) , \quad \boldsymbol{b}(0)=(0, b_1, b_2, b_3)\in T_1S^3\subset E^4 .$$

Then

$$c(s)=(\cos s,\ a_1 \sin s,\ a_2 \sin s,\ a_3 \sin s)\,. \tag{4.29}$$

Putting $s=0$ in (4.27), we find

$$b_1=-\cos\alpha\neq 0,\ \pm 1\,, \tag{4.30}$$

since $x(M)$ is properly slant. On the other hand, by (3.1), (3.2), (4.20), and (4.21), we have

$$\langle L_{c(s)*}b(0),\ R_{c(s)*}X_1\rangle=b_1\cos 2s+(-a_3b_2+a_2b_3)\sin 2s\,. \tag{4.31}$$

So, from (4.27) and (4.31), we get $b_1=0$ which contradicts (4.30). Consequently, case (ii′) cannot occur.

Case (iii): Let $\{c(s)\cdot\gamma(t)\}$ be defined by using two distinct geodesics c and γ and assume

$$x: I_1\times I_2\to S^3\,;\qquad (s,t)\mapsto c(s)\cdot\gamma(t) \tag{4.32}$$

is properly slant. Since the geodesics c and γ can be extended for all $s, t\in \boldsymbol{R}$, we can extend the immersion x to a global mapping:

$$y: \boldsymbol{R}^2\to S^3\,;\qquad (s,t)\mapsto c(s)\cdot\gamma(t)\,. \tag{4.33}$$

Now, we claim that y is in fact an immersion and properly slant. To see this, we recall (4.25) and put

$$c'(0)=(0,a_1,a_2,a_3)\,,\quad \gamma'(0)=(0,b_1,b_2,b_3)\in T_1S^3\,. \tag{4.34}$$

Let $\tilde{X}$, $\tilde{Y}$ be the vector fields along $y(\boldsymbol{R}^2)$ defined by

$$\tilde{X}(s,t)=R_{\gamma(t)*}c'(s)\,,\quad \tilde{Y}(s,t)=L_{c(s)*}\gamma'(t)\,. \tag{4.35}$$

Then it follows from (3.1) and (3.2) that

(4.36) $\langle\tilde{X}(s,t),\ \tilde{Y}(s,t)\rangle$ is a polynomial of $\sin s$, $\cos s$, $\sin t$ and $\cos t$.

On the other hand, since s-curves and t-curves on $x(I_1\times I_2)$ intersect at a constant angle (cf. [7, p. 157]),

$$\langle\tilde{X}(s,t),\ \tilde{Y}(s,t)\rangle=\text{const.}\neq\pm 1\,,\qquad \text{for}\quad \forall(s,t)\in I_1\times I_2\,. \tag{4.37}$$

From (4.36), we see that (4.37) holds for all $(s,t)\in\boldsymbol{R}^2$ and hence y is an immersion. Since

$$\xi(c(s)\cdot\gamma(t))=\|\tilde{X}(s,t)\times\tilde{Y}(s,t)\|^{-1}(\tilde{X}(s,t)\times\tilde{Y}(s,t))\,,$$

where $\times$ denotes the usual vector product in $T_{c(s)\cdot\gamma(t)}S^3$ determined by the metric and the orientation, so, by (3.1), (3.2) and (3.3), we know that $\langle\xi(c(s)\cdot\gamma(t)),\ J_0\eta(c(s)\cdot\gamma(t))\rangle$ is a polynomial of $\sin s$, $\cos s$, $\sin t$ and $\cos t$. By Lemma 3.3 we conclude that this polynomial is a constant on $I_1\times I_2$ and hence y is a proper slant immersion defined globally on $\boldsymbol{R}^2$. Now, by the double periodicity, y induces a proper slant immersion:

$$\tilde{y}: T^2=(\boldsymbol{R}/2\pi\boldsymbol{Z})\times(\boldsymbol{R}/2\pi\boldsymbol{Z})\to \boldsymbol{C}^2=(E^4, J_0)$$

of a torus into $\boldsymbol{C}^2$, which contradicts Theorem 1.1. Conseqeuntly, case (iii) cannot occur. This completes the proof of the theorem. Q.E.D.

5. Slant surfaces with *rank*(v)<2.

In this section we prove Theorem 1.6.

Assume $x: M\to\boldsymbol{C}^2=(E^4, J_0)$ is a general slant immersion of an oriented surface M into $\boldsymbol{C}^2$ with Gauss map v.

Let $*$ be the Hodge star operator $*: \bigwedge^2 E^4\to\bigwedge^2 E^4$ induced from the natural orientation and the canonical inner product of E^4. Denote by $\bigwedge^2_+ E^4$ and $\bigwedge^2_- E^4$ the eigenspaces of $*$ with eigenvalues $+1$ and -1, respectively. Then it is well-known that both eigenspaces are of dimension 3. We denote by S^2_+ and S^2_- the 2-spheres centered at the origin with radius $1/\sqrt{2}$ in $\bigwedge^2_+ E^4$ and $\bigwedge^2_- E^4$, respectively. Then we have $D_1(2, 4)=S^2_+\times S^2_-$. Let

$$\pi_+: D_1(2, 4)\to S^2_+\,, \qquad \pi_-: D_1(2, 4)\to S^2_-$$

denote the natural projections. We define two maps v_+ and v_- respectively by

$$v_+=\pi_+\circ v \quad\text{and}\quad v_-=\pi_-\circ v\,.$$

Suppose that the slant immersion $x: M\to\boldsymbol{C}^2=(E^4, J_0)$ satisfies *rank*(v)<2. Then we have *rank*($v_\pm$)<2. Hence, $x(M)$ is a flat surface in E^4 (cf. Lemma 6.2 of [6]). Furthermore, we have

LEMMA 5.1. *If x is a general slant immersion with rank(v)<2, then $x(M)$ is a union of flat ruled surfaces in E^4.*

PROOF. Since the normal curvature $R^D\equiv 0$, we can choose $\{e_1, e_2\}$ such that the second fundamental form $\{h^r_{ij}\}$ is simultaneously diagonalized, i.e., we have

$$(5.1)\qquad (h^3_{ij})=\begin{pmatrix} b & 0\\ 0 & c\end{pmatrix},\qquad (h^4_{ij})=\begin{pmatrix} d & 0\\ 0 & e\end{pmatrix}.$$

Put

$$(5.2)\quad M_1=\{p\in M \mid H(p)\neq 0\},\quad M_0=\text{Interior of }(M-M_1),\quad M=M_0\cup\partial M_1\cup M_1\,,$$

where H is the mean curvature vector.

Since $x(M)$ is flat and $H=0$ on M_0, $x(M_0)$ is a union of portions of 2-planes in E^4 with the same slant angle.

On M_1, we put $e_3=H/\|H\|$. Since *rank*(v)<2, we have $bc=0$ and $d=e=0$. We may choose $\{e_1, e_2\}$ such that $b\neq 0$, $c=d=e=0$ on M_1. From these we may prove that the integral curves of x_*e_2 are open portions of straight lines and therefore $x(M_1)$ is a union of flat ruled surfaces. Consequently, $x(M)$ is a union of flat ruled surfaces possibly glued along ∂M_1. Q.E.D.

We recall that a flat ruled surface in E^4 is "in general" a cylinder, a cone or a tangential developable surface (cf. [7]).

PROOF OF THEOREM 1.6. The first part of the theorem is given in Lemma 5.1. Now, we prove the remaining part of the theorem.

Case (i): If $x(M)$ is a slant cylinder, then we may assume that $x(M)$ is of the form:

$$x(M)=\{c(s)+te\}\,, \tag{5.3}$$

where e is a fixed unit vector in E^4 and $c(s)$ is a curve parametrized by arclength which lies in the orthogonal complement (up to sign), say $W\in G(3, 4)$, of e. Since $\{c'(s), e\}$ is a positive orthonormal basis of $T_{c(s)+te}M$, $\cos\alpha=\langle c'(s), -J_0e\rangle$ by (1.4). Hence, $c(s)$ is a generalized helix in $W(\equiv E^3)$ whose tangents make a constant angle α with $-J_0e\in W$.

Case (ii): If $x(M)$ is a slant cone, then, without loss of generality, we may assume that the vertex of the cone is the origin of E^4. So we can write

$$x(M)=\{tc(s)\}\,, \tag{5.4}$$

where $c(s)$ is a curve in S^3 parametrized by arclength. Since $\{c'(s), \eta(c(s))\}$ is a positive orthonormal basis of $T_{tc(s)}M$, $\cos\alpha=\langle c'(s), -J_0\eta(c(s))\rangle$ for all s. Thus, by Lemmas 3.1 and 3.3, we conclude that $(\phi\circ c)(s)$ is a generalized helix in S^3 with axis $\tilde{X}_1$ (cf. Definition 1.3).

Case (iii): If $x(M)$ is a slant tangential developable surface, the surface has the form:

$$x(M)=\{c(s)+(t-s)c'(s)\}\,, \tag{5.5}$$

where $c(s)$ is a curve parametrized by arclength. We put

$$\boldsymbol{v}_1(s)=c'(s)\,,\quad \kappa_1(s)=\|\boldsymbol{v}_1'(s)\|\,,\quad \boldsymbol{v}_2(s)=\left(\frac{1}{\kappa_1(s)}\right)\boldsymbol{v}_1'(s)\,. \tag{5.6}$$

Note that $\kappa_1\neq 0$, since $c(s)$ generates a tangential developable surface. $\{\boldsymbol{v}_2(s), \boldsymbol{v}_1(s)\}$ forms a positive orthonormal basis of $T_{c(s)+(t-s)c'(s)}M$, and so we have

$$\cos\alpha=\langle \boldsymbol{v}_1'(s)/\|\boldsymbol{v}_1'(s)\|,\ -J_0\boldsymbol{v}_1(s)\rangle$$

for all s. If we consider $\boldsymbol{v}_1(s)$ as a curve in S^3, then (5.6) means that

$$\cos\alpha=\langle \boldsymbol{t}(s),\ -J_0\eta(\boldsymbol{v}_1(s))\rangle\,,$$

where $\boldsymbol{t}(s)$ is the unit tangent of $\boldsymbol{v}_1(s)$. So, as in case (ii), $(\phi\circ\boldsymbol{v}_1)(s)$ is a generalized helix in S^3 with axis $\tilde{X}_1$.

It is easy to verify that in each of the cases (i)–(iii), the converse is also true.

Q.E.D.

6. Slant surfaces in E^3.

In this section we assume that the slant surfaces are contained in a hyperplane W of E^4.

LEMMA 6.1. *Let $x: M \to C^2 = (E^4, J_0)$ be a general slant immersion of an oriented surface. If M is contained in some $W \in G(3, 4)$, then rank$(\nu) < 2$ and the immersion x is doubly slant with the same slant angle.*

PROOF. As in the proof of Lemma 3.2, we choose a positive orthonormal J_0-basis $\{e_1, e_2, e_3, e_4\}$ such that $e_1, e_2 \in W \cap J_0 W$, $e_4 = J_0 e_3 = \eta_W$, where η_W is the positive unit normal vector of the hyperplane W in E^4. We put

$$G_W = G(2, 4) \cap \bigwedge{}^2 W \subset \bigwedge{}^2 E^4 . \tag{6.1}$$

Then G_W is the unit 2-sphere in the 3-dimensional Euclidean space $\bigwedge^2 W$. For $\alpha \in [0, \pi]$ we put

$$G_{W,\alpha} = G_{J_0,\alpha} \cap G_W , \tag{6.2}$$

where $G_{J_0,\alpha}$ is the set of all 2-planes in E^4 with slant angle α with respect to J_0. We recall that a 2-plane V was identified with a unit decomposable 2-vector $e_1 \wedge e_2$ in $\bigwedge^2 E^4$ with $\{e_1, e_2\}$ as a positively oriented orthonormal basis of V. From the proof of Lemma 3.2, we see that $G_{W,\alpha}$ is the circle on $G_W = S^2 \subset \bigwedge^2 W$ defined by $G_{W,\alpha} = \{V \in G_W \mid \langle V, e_1 \wedge e_2 \rangle = \cos\alpha\}$.

For each $J \in \mathscr{J}$, we denote by ζ_J the 2-vector which is the metrical dual of $-\Omega_J$ as defined in section 3. Let $\zeta: \mathscr{J} \to \bigwedge^2 E^4$ be the mapping defined by $\zeta(J) = \zeta_J$. Then ζ gives rise to two bijections (cf. Lemma 3.1 of [6]):

$$\zeta^+ : \mathscr{J}^+ \to S^2_+(\sqrt{2}) \quad \text{and} \quad \zeta^- : \mathscr{J}^- \to S^2_-(\sqrt{2}) .$$

For each 2-plane $V \in G(2, 4)$ we define two complex structures $J_V^+ \in \mathscr{J}^+$ and $J_V^- \in \mathscr{J}^-$ by

$$J_V^+ = (\zeta^+)^{-1}(2\pi_+(V)) \quad \text{and} \quad J_V^- = (\zeta^-)^{-1}(2\pi_-(V)) .$$

Let $J_1 = J^-_{e_1 \wedge e_2}$. Then we have

$$\pi_+(G_{W,\alpha}) = S^+_{J_0,\alpha} \subset S^2_+ , \qquad \pi_-(G_{W,\alpha}) = S^-_{J_1,\alpha} \subset S^2_- , \tag{6.3}$$

where $S^\pm_{J,\alpha}$ are the circles (possibly singletons) on $S^2_\pm$, respectively, consisting of all 2-vectors on $S^2_\pm$ which make constant angle α with ζ_J (cf. [6]). If x is α-slant with respect to J_0 and $x(M) \subset W$, then $\nu(M) \subset G_{W,\alpha}$. Therefore, rank$(\nu) < 2$ and, by (6.3), x is also α-slant with respect to J_1. This proves the lemma. Q.E.D.

We note here that if we identify $\bigwedge^2 W$ with the Euclidean 3-space $E^3 \equiv W$ (W spanned by $\{e_1, e_2, e_3\}$) via the isometry $X \wedge Y \to X \times Y$, then $\nu: M \to G_W \subset \bigwedge^2 W$ is nothing but the classical Gauss map $g: M\to S^2\subset E^3$. Since $e_1\times e_2=e_3=-J_0\eta_W$, x is α-slant if and only if

$$g(M)\subset S^1_\alpha=\{Z\in S^2 \mid \langle Z, -J_0\eta_W\rangle=\cos\alpha\}\subset S^2\subset W. \tag{6.4}$$

PROOF OF THEOREM 1.7. Assume $x: M\to C^2=(E^4, J_0)$ is a proper slant immersion of an oriented surface M such that $x(M)$ is contained in some $W\in G(3,4)$. The first part of Theorem 1.7 is given by Lemma 6.1. For the remaining part it suffices to check the three cases of Theorem 1.6.

Suppose x is properly slant with slant angle α. Denote by ξ the local unit normal of $x(M)$ in W. We put

$$e_1=t\xi/\|t\xi\|\,,\quad e_2=(\sec\alpha)Pe_1\,,\quad e_3=(\csc\alpha)Fe_1\,,\quad e_4=(\csc\alpha)Fe_2\,, \tag{6.5}$$

where PX and FX denote the tangential and the normal components of J_0X, respectively, and $t\xi$ is the tangential component of $J_0\xi$. Then $\{e_1,\cdots,e_4\}$ is an adapted orthonormal frame along $x(M)$ and it satisfies

$$e_3=\text{unit normal of } x(M) \text{ in } W\,,\qquad e_4\in W^\perp\,, \tag{6.6}$$

$$te_3=-(\sin\alpha)e_1\,,\quad te_4=-(\sin\alpha)e_2\,,\quad fe_3=-(\cos\alpha)e_4\,,\quad fe_4=(\cos\alpha)e_3\,, \tag{6.7}$$

where fe_3 is the normal component of J_0e_3. Since e_4 is a constant vector in E^4, Lemma 3.1 of [3] implies that the second fundamental form (h^r_{ij}) is of the following form:

$$(h^3_{ij})=\begin{pmatrix} b & 0\\ 0 & 0\end{pmatrix},\qquad (h^4_{ij})=0\,, \tag{6.8}$$

which shows that our frame $\{e_1,\cdots,e_4\}$ coincides with that chosen in the proof of Lemma 5.1 (up to orientations). Since J_0e_4 is also a constant vector in E^4, from (6.7) we have

$$-\sin\alpha\nabla_Xe_2-\cos\alpha A_{e_3}X=0\,,\qquad \text{for}\quad X\in TM\,. \tag{6.9}$$

Hence, we get

$$\omega^1_2(e_1)=-b\cot\alpha\,,\qquad e_2b=b^2\cot\alpha\,. \tag{6.10}$$

Case (i): In this case, the curve $c(s)$ of (5.3) lies in a 2-plane $W'=\{e\}^\perp\cap W\subset W$ which is perpendicular to e. So, $x(M)$ is totally real with respect to the complex structures $\pm J^{\pm}_{W'}$ defined above. If $\nu_+(M)$ is not a singleton, then J_0 is one of the complex structures $\pm J^{\pm}_{W'}$ according to Lemma 4.2 and Theorem 4.3 of [6]. Hence we get $\alpha=\pi/2$, which contradicts the assumption. So, $\nu_+(M)$ is a singleton and hence $x(M)$ is minimal (cf. Theorem 4.3 of [6]). Thus, by (6.8), $x(M)$ is an open portion of an α-slant 2-plane.

In Cases (ii) and (iii), we may assume $M=M_1$ according to the remark after Theorem 1.7.

Case (ii): In this case the curve $c(s)$ in (5.4) lies in the unit 2-sphere $S^2=S^3\cap W$.

Choose $\{e_1, \cdots, e_4\}$ according to (6.5) and let $\boldsymbol{t}$, $\boldsymbol{n}$, $\boldsymbol{b}$, κ, and τ be the unit tangent, unit principal normal, the unit binormal, the curvature, and the torsion of $c(s)$ in $W=E^3$, respectively. We want to show that $\tau\equiv 0$.

Since

$$(6.11)\qquad e_1(s,t)=\boldsymbol{t}(s)=\frac{1}{t}\frac{\partial}{\partial s}, \quad e_2(s,t)=c(s)=\frac{\partial}{\partial t}, \quad e_3(s,t)=e_1(s,t)\times e_2(s,t),$$

where $\times$ is the vector product in W, we have

$$(6.12)\qquad b=-\left(\frac{\kappa}{t}\right)\langle \boldsymbol{b}, c\rangle .$$

From $\|c\|=1$, we get

$$(6.13)\qquad \kappa\langle \boldsymbol{n}, c\rangle=-1 .$$

Differentiating (6.13) with respect to s, we get

$$(6.14)\qquad \kappa^2\tau\langle \boldsymbol{b}, c\rangle=\kappa' .$$

From (6.12) we obtain

$$(6.15)\qquad -t\tau\kappa b=\kappa' .$$

Differentiating (6.15) with respect to t and using (6.10) and (6.15), we obtain

$$(6.16)\qquad \kappa'(\tau\kappa\tan\alpha-\kappa')=0 .$$

By (6.12), (6.14) and $\langle \boldsymbol{t}, c\rangle=0$, we find

$$(6.17)\qquad \kappa^2\tau c=-\kappa\tau\boldsymbol{n}+\kappa'\boldsymbol{b} .$$

Since $\|c\|=1$, we also get

$$(6.18)\qquad \tau^2\kappa^4=\tau^2\kappa^2+(\kappa')^2 .$$

If $\kappa'(s_0)=0$ at a point $s=s_0$, then, by (6.15), we have $\tau(s_0)=0$, since $b(s,t)\neq 0$ by assumption and $\kappa(s)\neq 0$ because $c(s)$ is spherical.

If $\kappa'(s_0)\neq 0$, we choose a neighborhood U of s_0 on which κ' never vanishes. By (6.16), (6.18) and $\kappa\neq 0$, we get

$$(6.19)\qquad (\tau(s))^2\{(\kappa(s))^2-1-\tan^2\alpha\}=0 \qquad \text{for} \quad \forall s\in U .$$

If $\tau(s_0)\neq 0$ in addition, we choose another neighborhood U' of s_0 contained in U on which τ never vanishes. Then, by (6.19), we get

$$(\kappa(s))^2-1-\tan^2\alpha=0$$

for all s in U'. By continuity wet get $\kappa(s)=\textit{constant}$ on U' which contradicts $\kappa'(s)\neq 0$ on U'. So, again we have $\tau(s_0)=0$. Therefore, $\tau\equiv 0$, which means that $c(s)$ is a circle on

S^2 and thus $x(M)$ is a circular cone. According to the remark after the proof of Lemma 6.1, the axis of the cone is given by $-J_0e_4$.

Case (iii): We assume the surface is given by (5.5) and $\{e_1, \cdots, e_4\}$, $\boldsymbol{t}$, $\boldsymbol{n}$, $\boldsymbol{b}$, κ, and τ are given as in case (ii). We have

$$e_1(s, t)=\boldsymbol{n}(s)=\frac{1}{(t-s)\kappa}\frac{\partial}{\partial s}, \quad e_2(s, t)=\boldsymbol{t}(s)=\frac{\partial}{\partial t}, \quad e_3(s, t)=e_1\times e_2=-\boldsymbol{b}(s). \tag{6.20}$$

Hence

$$\tilde{\nabla}_{e_1}e_1=-\frac{1}{(t-s)}e_2-\frac{\tau}{(t-s)\kappa}e_3. \tag{6.21}$$

So, by (6.8), we find

$$b=-\frac{\tau}{(t-s)\kappa}. \tag{6.22}$$

By (6.20) and (6.22), we obtain

$$e_2b=\frac{\tau}{\kappa(t-s)^2}.$$

This formula together with (6.10) and (6.21) implies that $\tau/\kappa=\tan\alpha$ is a constant. This means that the curve $c(s)$ is a generalized helix in W. The axis of the helix is $-J_0e_4$.

In each of the cases (i)–(iii), the converse is easy to verify. For example, if $x(M)$ is a circular cone with the axis vector e in a 3-plane W perpendicular to a unit vector η in E^4, then, by picking a complex structure J such that $J=J^+_{e\wedge\eta}$, $x(M)$ is properly slant with respect to J.

This completes the proof of the theorem. Q.E.D.

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