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Finite Type Hypersurfaces of a Sphere

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§1. Introduction.

Let M be an n-dimensional compact submanifold of an m-dimensional Euclidean space $\boldsymbol{R}^m$ and Δ the Laplacian of M (with respect to the induced metric) acting on smooth functions on M. We denote by x the position vector of M in $\boldsymbol{R}^m$. Then we have the following spectral decomposition of x;

$$x=x_0+\sum_{t\geq 1} x_t \qquad \Delta x_t=\lambda_t x_t \qquad (\text{in } L^2\text{-sense})\,. \tag{1.1}$$

If there are exactly k nonzero x_t's ($t\geq 1$) in the decomposition (1.1), then the submanifold M is said to be of *k-type*. Here x_0 in (1.1) is exactly the center of mass in $\boldsymbol{R}^m$. A submanifold M of a hypersphere S^{m-1} of $\boldsymbol{R}^m$ is said to be *mass-symmetric* in S^{m-1} if the center of mass of M in $\boldsymbol{R}^m$ is the center of the hypersphere S^{m-1} in $\boldsymbol{R}^m$.

In terms of these notions, a well-known result of Takahashi (cf. [6]) says that a submanifold M in $\boldsymbol{R}^m$ is of 1-type if and only if M is a minimal submanifold of a hypersphere S^{m-1} of $\boldsymbol{R}^m$. Furthermore, a minimal submanifold of a hypersphere S^{m-1} in $\boldsymbol{R}^m$ is mass-symmetric in S^{m-1}. On the other hand, in [3], mass-symmetric, 2-type hypersurfaces of S^{m-1} are characterized. In [1], it is proved that a compact 2-type surface in S^3 is mass-symmetric.

In this paper, we will show that many 2-type hypersurfaces of a hypersphere S^{n+1} are mass-symmetric and that mass-symmetric, 2-type hypersurfaces of S^{n+1} have no umbilic point. More precisely, we will prove the following.

THEOREM 1. *Let $x: M\to S^{n+1}$ be a compact hypersurface of a hypersphere S^{n+1} in $\boldsymbol{R}^{n+2}$. If M is of 2-type (i.e., $x=x_0+x_p+x_q$) and*

$$(\lambda_p+\lambda_q)-\frac{9n+16}{(3n+2)^2}\lambda_p\lambda_q\geq n\,,$$

then M is mass-symmetric (i.e., $x_0=0$).

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THEOREM 2. *Let M be a compact and mass-symmetric hypersurface of a hypersphere S^{n+1} in $\boldsymbol{R}^{n+2}$. If M is of 2-type, then M has no umbilic point.*

In [2], it is proved that a compact 2-type hypersurface of S^{n+1} is mass-symmetric if and only if it has constant mean curvature. Therefore, Theorem 1 implies that many compact 2-type hypersurfaces of S^{n+1} have constant mean curvatures. But this does not occur when we consider 3-type hypersurfaces of S^{n+1}. More exactly, we will obtain the following.

THEOREM 3. *There is no compact hypersurface of constant mean curvature in S^{n+1} which is of 3-type.*

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§ 2. Preliminaries.

Let M be an n-dimensional compact hypersurface of the unit hypersphere $S^{n+1}(1)$ of $\boldsymbol{R}^{n+2}$ centered at the origin. Denote by ∇, D and D' the Riemannian connection of M, the normal connection of M in $\boldsymbol{R}^{n+2}$ and the normal connection of M in $S^{n+1}(1)$, respectively. Let h, A and H (respectively, h', A' and H') denote the second fundamental form, the Weingarten map, and the mean curvature vector of M in $\boldsymbol{R}^{n+2}$ (respectively, those quantities of M in $S^{n+1}(1)$).

Let $e_1, \cdots, e_n, \xi$ be an orthonormal local frame field such that $e_1, \cdots, e_n$ are tangent to M and ξ is normal to M in $S^{n+1}(1)$. Let $\Delta^{D'}$ denote the Laplacian associated with D'.

Then we have the following useful formula.

LEMMA A ([4]). *Let M be a hypersurface of $S^{n+1}(1)$ in $\boldsymbol{R}^{n+2}$. Then we have*

$$\Delta H = \Delta^{D'} H' + \frac{n}{2}\operatorname{grad}(\alpha^2) + 2 \operatorname{tr} A_{D'H'} + \|h\|^2 H' - n\alpha^2 x\,,$$

where $\alpha = |H|$, $\operatorname{tr} A_{D'H'} = \sum_{i=1}^{n} A_{D'_{e_i}H'} e_i$.

The following results are known.

THEOREM A ([3]). *Let M be a compact submanifold of $\boldsymbol{R}^m$. Then M is of finite type if and only if there exists a non-trivial polynomial P such that $P(\Delta)H = 0$ (or $P(\Delta)(x - x_0) = 0$), where H is the mean curvature vector.*

THEOREM B ([3]). *Let M be a finite type submanifold of $\boldsymbol{R}^m$. Denote by $P_m(t)$ a monic polynomial of least degree with $P_m(\Delta)H = 0$. Then we have*

(a) *the polymonial $P_m(t)$ is unique,*

(b) *if Q is a polynomial with $Q(\Delta)H = 0$, then $P_m(t)$ is a factor of Q, and*

(c) *M is of k-type if and only if* deg $P_m = k$.

THEOREM C ([3]). *Let M be a compact, mass-symmetric and 2-type hypersurface of* $S^{n+1}(r)$, *where r is the radius. Then*

(1) *the mean curvature* α *of M in* $\boldsymbol{R}^{n+2}$ *is given by*

$$\alpha^2 = \frac{1}{n}(\lambda_p + \lambda_q) - \left(\frac{r}{n}\right)^2 \lambda_p \lambda_q ,$$

(2) *the scalar curvature* τ *of M is given by*

$$\tau = (n-1)(\lambda_p + \lambda_q) - r^2 \lambda_p \lambda_q , \quad \textit{and}$$

(3) *the length of the second fundamental form h of M in* $\boldsymbol{R}^{n+2}$ *is given by*

$$\|h\|^2 = \lambda_p + \lambda_q .$$

The following theorems are proved.

THEOREM D ([4]). *Let M be a compact hypersurface of* S^{n+1} *such that M is not a small hypersphere of* S^{n+1}. *Then M is mass-symmetric and of 2-type if and only if M has nonzero constant mean curvature and constant scalar curvature.*

We also need the following.

THEOREM E ([3] and [5]). *If M is a compact 2-type hypersurface of a unit hypersphere* $S^{n+1}(1)$ *in* $\boldsymbol{R}^{n+2}$, *then we have*

$$\lambda_p < n < \lambda_q .$$

§3. Proof of Theorem 1.

Let M be a hypersurface of a unit hypersphere $S^{n+1}(1)$ in $\boldsymbol{R}^{n+2}$. Then, from Theorems A and B, we have

$$\Delta H = bH + c(x - x_0) , \qquad b = \lambda_p + \lambda_q , \quad c = \frac{\lambda_p \lambda_q}{n} .$$

By using $H = H' - x$, we get

$$\Delta H = bH' + (c-b)x - cx_0 . \tag{3.1}$$

On the other hand, from Lemma A, we have

$$\Delta H = \Delta^{D'} H' + \frac{n}{2} \operatorname{grad} \alpha^2 + 2 \operatorname{tr} A_{D'H'} + \|h\|^2 H' - n\alpha^2 x . \tag{3.2}$$

We put $H' = \alpha' \xi$. Then, by a direct computation, we get

$$\Delta^{D'} H' = (\Delta \alpha') \xi .$$

Hence, from (3.1) and (3.2), we have

$$\Delta\alpha' + (\|h\|^2 - b)\alpha' = -c\langle x_0, \xi\rangle, \tag{3.3}$$

$$c\langle x_0, x\rangle = n\alpha^2 + c - b, \tag{3.4}$$

from which, for any vector field X tangent to M, we get

$$cX\langle x_0, x\rangle = c\langle x_0, X\rangle = nX(\alpha^2). \tag{3.5}$$

We use (3.1) and (3.5) to obtain

$$\langle \Delta H, X\rangle = -c\langle x_0, X\rangle = -nX(\alpha^2). \tag{3.6}$$

Therefore, from (3.2) and (3.6), we have

$$\operatorname{tr} A_{D'H'} = -\frac{3n}{4}\operatorname{grad}\alpha^2.$$

By a direct computation, we get

$$\operatorname{tr} A_{D'H'} = A_\xi \operatorname{grad}\alpha'.$$

These, together with $\alpha^2 = \alpha'^2 + 1$, yield

$$A_\xi \operatorname{grad}\alpha' = -\frac{3}{2}n\alpha' \operatorname{grad}\alpha'. \tag{3.7}$$

Let $E_1, \cdots, E_n$ be orthonormal principal directions of A_ξ with principal curvatures $\mu_1, \cdots, \mu_n$, respectively.

Then (3.7) gives

$$(2\mu_i + 3n\alpha')E_i(\alpha') = 0, \qquad i = 1, \cdots, n. \tag{3.8}$$

We give the following general lemma on 2-type hypersurfaces of $S^{n+1}(1)$.

LEMMA 1. *Let M be a compact 2-type hypersurface of $S^{n+1}(1)$ in $\boldsymbol{R}^{n+2}$. Then we have*

$$\int_M (\|h\|^2 - b)\alpha'^2 dV + \int_M |\operatorname{grad}\alpha'|^2 dV + c|x_0|^2 \operatorname{vol} M = 0. \tag{3.9}$$

PROOF. Let M be a 2-type hypersurface of $S^{n+1}(1)$. By a direct computation, we get

$$\Delta\alpha'^2 = 2\alpha'\Delta\alpha' - 2|\operatorname{grad}\alpha'|^2,$$

which, together with Hopf's lemma, yields

$$\int_M \alpha'\Delta\alpha' dV = \int_M |\operatorname{grad}\alpha'|^2 dV. \tag{3.10}$$

On the other hand, we have

$$\int_M \langle x_0, x\rangle dV = \int_M \langle x_0, x_0 + x_p + x_q\rangle dV = |x_0|^2 \operatorname{vol} M, \tag{3.11}$$

which implies

$$\int_M \langle x_0, H\rangle dV = \int_M \langle x_0, H'\rangle dV - |x_0|^2 \operatorname{vol} M. \tag{3.12}$$

By combining $\Delta x = -nH$ with Hopf's lemma, we obtain

$$\int_M \langle x_0, H\rangle dV = -\frac{1}{n}\int_M \langle x_0, \Delta x\rangle dV = 0. \tag{3.13}$$

It follows from (3.12) and (3.13) that

$$\int_M \langle x_0, H'\rangle dV = |x_0|^2 \operatorname{vol} M. \tag{3.14}$$

From (3.3), (3.10) and (3.14), we find (3.9).

Now, we assume that M is not mass-symmetric. Then by using (3.4), M has non-constant mean curvature. Furthermore, we have the following Lemma (See [2].).

LEMMA B. *Let M be a 2-type hypersurface of $S^{n+1}(1)$ in R^{n+2}. Then either M has constant mean curvature or $U = \{u \in M \mid \operatorname{grad} \alpha^2 \neq 0$ at $u\}$ is dense in M.*

Consequently, the open subset U is dense in M. Then, from (3.7), we know that $\operatorname{grad}\alpha'$ is a principal direction with principal curvature $-\frac{3}{2}n\alpha'$ on U. We put

$$\nabla_{E_i} E_j = \sum_k \omega_j^k(E_i) E_k, \qquad i, j, k = 1, \cdots, n.$$

Then, from Codazzi's equation, we see that

$$(\mu_i - \mu_j)\omega_i^j(E_i) = E_j(\mu_i), \qquad i \neq j. \tag{3.15}$$

By using (3.8) and (3.15), we may find that the multiplicity of $\mu_1 = -\frac{3}{2}n\alpha'$ is one (For further details, refer to [2].). Therefore, we get

$$\|h\|^2 - n = \sum_i \mu_i^2 = \frac{9}{4}n^2\alpha'^2 + \sum_{i=2}^n \mu_i^2.$$

On the other hand, we have

$$\begin{aligned}(n-1)\sum_{i=2}^n \mu_i^2 &\geq \left(\sum_{i=2}^n \mu_i\right)^2 \\ &= (n\alpha' - \mu_1)^2 = \frac{25}{4}n^2\alpha'^2.\end{aligned}$$

Thus we obtain

$$\|h\|^2-n\geq\frac{9}{4}n^2\alpha'^2+\frac{25}{4(n-1)}n^2\alpha'^2=\frac{9n+16}{4(n-1)}n^2\alpha'^2, \tag{3.16}$$

from which it follows

$$\int_M(\|h\|^2-b)\alpha'^2dV\geq\frac{9n+16}{4(n-1)}n^2\int_M\alpha'^4dV+(n-b)\int_M\alpha'^2dV. \tag{3.17}$$

From (3.4), (3.11) and $\alpha^2=\alpha'^2+1$, we get

$$c|x_0|^2\operatorname{vol}M=n\int_M\alpha'^2dV+(n+c-b)\operatorname{vol}M. \tag{3.18}$$

Expanding the left-hand-side of $\{n\alpha'^2+(n+c-b)\}^2\geq 0$ and integrating it on M with use of (3.18), we get

$$\begin{aligned} n^2\int_M\alpha'^4dV&\geq(b-n-c)\left\{2n\int_M\alpha'^2dV+(n+c-b)\operatorname{vol}M\right\}\\ &=(b-n-c)\{2c|x_0|^2+(b-n-c)\}\operatorname{vol}M. \end{aligned} \tag{3.19}$$

From (3.17), (3.18) and (3.19), we see that

$$\begin{aligned} \int_M(\|h\|^2-b)\alpha'^2dV&\geq\frac{(3n+2)^2}{4(n-1)}\left\{b-n-\frac{9n+16}{(3n+2)^2}nc\right\}\frac{1}{n}(c|x_0|^2+b-n-c)\operatorname{vol}M\\ &\quad+\frac{9n+16}{4(n-1)}(b-n-c)c|x_0|^2\operatorname{vol}M. \end{aligned} \tag{3.20}$$

Theorem E gives

$$b-n-c=\frac{1}{n}(n-\lambda_p)(\lambda_q-n)>0. \tag{3.21}$$

By combining (3.20) and (3.21) with the hypothesis of Theorem 1, we may find

$$\int_M(\|h\|^2-b)\alpha'^2dV>0,$$

which is a contradiction in consideration of Lemma 1.

§4. Proof of Theorem 2.

We use the same notation as in §3. If $p\in M$ is an umbilic point, then we have

$$n\alpha'^2=\|h\|^2-n,\qquad \text{at}\quad p. \tag{4.1}$$

Since M is a compact, mass-symmetric and 2-type hypersurface of $S^{n+1}(1)$, Theorem C gives

$$(4.2) \qquad n\alpha'^2 = b - n - c\,, \qquad \|h\|^2 = b\,.$$

Comparing (4.1) with (4.2), we get

$$c = 0\,.$$

This is a contradiction in consideration of $c > 0$. Theorem 2 is thereby proved.

§5. Proof of Theorem 3.

Let M be a hypersurface of a hypersphere $S^{n+1}(1)$ in $\boldsymbol{R}^{n+2}$ which is of 3-type and has constant mean curvature α'. Then, Lemma A gives

$$(5.1) \qquad \Delta H = \|h\|^2 H' - n\alpha^2 x\,.$$

By a direct computation, (5.1) yields

$$(5.2) \qquad \begin{aligned} \Delta^2 H = &(\Delta\|h\|^2 + \|h\|^4 - n\|h\|^2 + n^2\alpha^2)H' \\ &-(n\alpha'^2\|h\|^2 + n^2\alpha^2)x + 2\alpha' A_\xi \,\mathrm{grad}\,\|h\|^2\,. \end{aligned}$$

On the other hand, from Theorems A and B, there exist nonzero constants c_1, c_2 and c_3 such that

$$(5.3) \qquad \Delta^2 H = c_1 \Delta H + c_2 H + c_3(x - x_0)\,.$$

Substituting (5.1) and $H = H' - x$ in (5.3), we have

$$(5.4) \qquad \Delta^2 H = (c_1\|h\|^2 + c_2)H' + (-c_1 n\alpha^2 - c_2 + c_3)x - c_3 x_0\,.$$

From (5.2) and (5.4), we find

$$(5.5) \qquad \alpha'(\Delta\|h\|^2 + \|h\|^4 - n\|h\|^2 + n^2\alpha^2) = \alpha'(c_1\|h\|^2 + c_2) - c_3\langle x_0, \xi\rangle\,,$$

$$(5.6) \qquad n\alpha'^2\|h\|^2 + n^2\alpha^2 = c_1 n\alpha^2 + c_2 - c_3 + c_3\langle x_0, x\rangle\,.$$

Applying the Laplacian to (5.6), we get

$$(5.7) \qquad \alpha'^2\Delta\|h\|^2 = -c_3\alpha'\langle x_0, \xi\rangle + c_3\langle x_0, x\rangle\,.$$

By using (5.5) and (5.7), we have

$$\alpha'^2(\|h\|^4 - n\|h\|^2 + n^2\alpha^2) = \alpha'^2(c_1\|h\|^2 + c_2) - c_3\langle x_0, x\rangle\,,$$

which, together with (5.6), implies

$$\alpha'^2\|h\|^4 - c_1\alpha'^2\|h\|^2 + n\alpha^4 - c_1 n\alpha^2 - c_2\alpha^2 + c_3 = 0\,.$$

Since M is of 3-type, α'^2 is non-zero and hence, we conclude that h has constant length.

By the Gauss equation, M has constant scalar curvature. Therefore, by applying Theorem D, we obtain a contradiction.

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