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Local Rings of Cohen-Macaulay F-Rational Rings Are F-Rational

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1. Introduction.

Let p be a prime number and let R be a commutative Noetherian ring of $\operatorname{ch} R=p$. We put $R^0=R\setminus\bigcup_{\mathfrak{p}\in\operatorname{Min} R}\mathfrak{p}$. Then for each ideal I of R the tight closure I^* of I is defined as follows:

$$I^*:=\{x\in R \mid {}^{\exists}c\in R^0 \text{ such that } c\cdot x^{p^e}\in I^{[p^e]} \text{ for all } e\gg 0\},$$

where $I^{[p^e]}$ denotes the ideal of R generated by the elements i^{p^e} $(i\in I)$. Notice that I^* is an ideal of R and

$$I\subset I^*\subset \bar{I},$$

where $\bar{I}$ denotes the integral closure of I.

The notion of tight closure was introduced by Hochster and Huneke [3] and they are now developing a marvellous theory on tight closures. For example using it they gave a beautiful new proof of the Briançon-Skoda theorem in characteristic p. See [4] for the detail.

The purpose of the present paper is to prove the following

THEOREM (1.1). *Let R be a Cohen-Macaulay local ring of* $\operatorname{ch} R=p$ *and suppose that $Q^*=Q$ for some parameter ideal Q of R. Then for any* $\mathfrak{p}\in\operatorname{Spec} R$ *and for any parameter ideal J of $R_{\mathfrak{p}}$ we have $J^*=J$ in $R_{\mathfrak{p}}$.*

We say that a Noetherian local ring R of $\operatorname{ch} R=p$ is *F-rational* if $Q^*=Q$ for any parameter ideal Q of R (cf. [1]). With this terminology our theorem (1.1) guarantees that every local ring of a Cohen-Macaulay F-rational local ring is again F-rational. The ring R is called *F-regular* if $I^*=I$ in $R_{\mathfrak{p}}$ for any $\mathfrak{p}\in\operatorname{Spec} R$ and for any ideal I of $R_{\mathfrak{p}}$. When R is a Gorenstein local ring, it is proved in [3, Proposition 5.1] that $I^*=I$ for any ideal I of R once $Q^*=Q$ for some parameter ideal Q of R. Therefore as an immediate consequence of Theorem (1.1) we get

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COROLLAY (1.2). *Let R be a Gorenstein local ring of* $\operatorname{ch} R=p$ *and suppose that* $Q^*=Q$ *for some parameter ideal Q of R. Then R is F-regular.*

2. Proof of Theorem (1.1).

Let R be a Noetherian ring of $\operatorname{ch} R=p$. The aim of this section is to prove Theorem (1.1). We begin with the following

LEMMA (2.1). *Let $f_1, f_2, \cdots, f_r$ $(r\geq 1)$ be a regular sequence in R. Let $I=(f_1, f_2, \cdots, f_{r-1})R$ and $S=R[1/f_r]$. Then we have*

$$I^*S=(IS)^* .$$

PROOF. Let $x\in R$ and assume that $x/1\in(IS)^*$. We want to show that $x\in I^*$. First of all choose $c\in R$ so that $c/1\in S^0$ and $(c/1)\cdot(x/1)^{p^e}\in I^{[p^e]}S$ for all $e\gg 0$. Notice that we may assume $c\in R^0$. In fact, suppose $c\notin R^0$ and put $\mathscr{F}=\{\mathfrak{p}\in\operatorname{Min} R \mid \mathfrak{p}\not\ni c\}$. Choose $d\in\bigcap_{\mathfrak{p}\in\mathscr{F}}\mathfrak{p}$ so that $d\notin\bigcup_{\mathfrak{p}\in\operatorname{Min} R\setminus\mathscr{F}}\mathfrak{p}$. Then $d/1$ is nilpotent in S and so replacing d by a suitable power of it, we may assume that $d/1=0$ in S. Then $c+d\in R^0$ and $c/1=(c+d)/1$ in S; thus we can take c inside of R^0.

Now let $e\gg 0$ be an integer with $(c/1)\cdot(x/1)^{p^e}\in I^{[p^e]}S$. Then $f_r^k\cdot(cx^{p^e})\in I^{[p^e]}=(f_1^{p^e}, f_2^{p^e}, \cdots, f_{r-1}^{p^e})R$ for some $k>0$ and so we have that $c\cdot x^{p^e}\in I^{[p^e]}$ because $f_1^{p^e}, \cdots, f_{r-1}^{p^e}, f_r^k$ is an R-regular sequence. Hence $x\in I^*$ and we have $(IS)^*\subset I^*S$. As the opposite inclusion is obvious, this completes the proof of (2.1).

The next result is a generalization of [4, (4.14) Proposition]. They proved it in the case where $\#\operatorname{Ass}_R R/I=1$.

LEMMA (2.2). *Let I be an ideal of R such that $\operatorname{Ass}_R R/I\subset\operatorname{Max} R$. Then*

$$I^*R_{\mathfrak{p}}=(IR_{\mathfrak{p}})^*$$

for any $\mathfrak{p}\in\operatorname{Ass}_R R/I$.

PROOF. Let $\mathscr{F}=\operatorname{Ass}_R R/I$ and let $I=\bigcap_{\mathfrak{p}\in\mathscr{F}}I(\mathfrak{p})$ denote a primary decomposition of I with $\sqrt{I(\mathfrak{p})}=\mathfrak{p}$ for each $\mathfrak{p}\in\mathscr{F}$. Then we have that

$$I^*\subset\bigcap_{\mathfrak{p}\in\mathscr{F}}I(\mathfrak{p})^*=\prod_{\mathfrak{p}\in\mathscr{F}}I(\mathfrak{p})^* ,$$

because $I(\mathfrak{p})^*$ is again a $\mathfrak{p}$-primary ideal of R. Let $\{x_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathscr{F}}$ be a family of elements of R such that $x_{\mathfrak{p}}\in I(\mathfrak{p})^*$ for each $\mathfrak{p}\in\mathscr{F}$. Choose $c_{\mathfrak{p}}\in R^0$ so that $c_{\mathfrak{p}}\cdot x_{\mathfrak{p}}^{p^e}\in I(\mathfrak{p})^{[p^e]}$ for all $e\gg 0$. Then since

$$\left(\prod_{\mathfrak{p}\in\mathscr{F}}c_{\mathfrak{p}}\right)\cdot\left(\prod_{\mathfrak{p}\in\mathscr{F}}x_{\mathfrak{p}}^{p^e}\right)\in\prod_{\mathfrak{p}\in\mathscr{F}}I(\mathfrak{p})^{[p^e]}=\left(\prod_{\mathfrak{p}\in\mathscr{F}}I(\mathfrak{p})\right)^{[p^e]},$$

we see that $\prod_{\mathfrak{p}\in\mathscr{F}}x_{\mathfrak{p}}\in(\prod_{\mathfrak{p}\in\mathscr{F}}I(\mathfrak{p}))^*=I^*$. Hence $\prod_{\mathfrak{p}\in\mathscr{F}}I(\mathfrak{p})^*\subset I^*$ and so we get

$$I^* = \bigcap_{\mathfrak{p} \in \mathscr{F}} I(\mathfrak{p})^* .$$

Now let $\mathfrak{p} \in \mathscr{F}$. Then since $I^* R_\mathfrak{p} = I(\mathfrak{p})^* R_\mathfrak{p}$ and since $I(\mathfrak{p})^* R_\mathfrak{p} = (I(\mathfrak{p}) R_\mathfrak{p})^*$ by [4, (4.14) Proposition], we have $I^* R_\mathfrak{p} = (I(\mathfrak{p}) R_\mathfrak{p})^* = (IR_\mathfrak{p})^*$. Hence the result follows.

COROLLARY (2.3). *Suppose that R is a Cohen-Macaulay local ring of* $\dim R = d \geq 1$ *and let $f_1, f_2, \cdots, f_{d-1}$ be a subsystem of parameters of R. Let $I = (f_1, f_2, \cdots, f_{d-1})R$. Then we have*

$$I^* R_\mathfrak{p} = (IR_\mathfrak{p})^*$$

for any $\mathfrak{p} \in \operatorname{Spec} R$.

PROOF. Let $\mathfrak{m}$ be the maximal ideal of R. We may assume that $I \subset \mathfrak{p} \subsetneqq \mathfrak{m}$. Hence $\dim R/\mathfrak{p} = 1$. Choose $f_d \in R$ so that $f_1, \cdots, f_{d-1}, f_d$ forms a system of parameters of R and let $S = R[1/f_d]$. Then by (2.1) we get $I^* S = (IS)^*$. Notice that $\mathfrak{p}S$ is a maximal ideal of S, because $\dim R/\mathfrak{p} = 1$ and $f_d \notin \mathfrak{p}$. By the same reason we find $\operatorname{Ass}_S S/IS \subset \operatorname{Max} S$ and so it follows from (2.2) that $(IS)^* \cdot S_{\mathfrak{p}S} = ((IS) \cdot S_{\mathfrak{p}S})^*$. Hence we get $I^* R_\mathfrak{p} = (IR_\mathfrak{p})^*$ as $I^* S = (IS)^*$.

We note the following striking result of Fedder and Watanabe [1].

PROPOSITION (2.4) ([1, Proposition 2.2]). *Let R be a Cohen-Macaulay local ring and assume that $Q^* = Q$ for some parameter ideal Q of R. Then R is F-rational.*

PROOF OF THEOREM (1.1). Let $f_1, f_2, \cdots, f_d$ be a system of parameters of R and put $Q_k = (f_1, f_2, \cdots, f_k)R$ for $0 \leq k \leq d$. Then because R is F-rational by (2.4) and because $Q_k \subset Q_k + (f_{k+1}^n, \cdots, f_d^n)R$, we see

$$\begin{aligned} Q_k^* &\subset [Q_k + (f_{k+1}^n, \cdots, f_d^n)R]^* \\ &= Q_k + (f_{k+1}^n, \cdots, f_d^n)R \end{aligned}$$

for any integer $n \geq 1$. Hence $Q_k^* = Q_k$ for all $0 \leq k \leq d$.

Now let $\mathfrak{p} \in \operatorname{Spec} R$ of $\dim R/\mathfrak{p} = 1$ and choose a subsystem $f_1, f_2, \cdots, f_{d-1}$ of parameters of R inside of $\mathfrak{p}$. We put $I = (f_1, f_2, \cdots, f_{d-1})R$. Then by (2.3) we see $I^* R_\mathfrak{p} = (IR_\mathfrak{p})^*$ in $R_\mathfrak{p}$. Consequently we have $(IR_\mathfrak{p})^* = IR_\mathfrak{p}$ because $I^* = I$ as we have checked above. Since $IR_\mathfrak{p}$ is a parameter ideal of $R_\mathfrak{p}$, we finally find by (2.4) that $R_\mathfrak{p}$ is F-rational. Thus by the induction on $\dim R$, we complete the proof of Theorem (1.1).

REMARK (2.5). A generalization of Theorem (1.1) and its consequences will be given in the subsequent joint paper [2].

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