

Homogeneous Siegel Domains of Nonpositive Holomorphic Bisectional Curvature

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Introduction.

The notion of quasi-symmetric Siegel domains was introduced by Satake [10] in an algebraic manner. D'Atri and Dorfmeister [2] proved that an irreducible homogeneous Siegel domain D is quasi-symmetric if and only if the Bergman metric of D induces a symmetric metric on the canonical tube domain associated with D . They proved in [3] that every quasi-symmetric Siegel domain has nonpositive holomorphic bisectional curvature relative to its Bergman metric. Besides these, few other differential geometric characterizations of quasi-symmetric Siegel domains are known. It is interesting to see whether a homogeneous Siegel domain D of nonpositive holomorphic bisectional curvature in the Bergman metric is necessarily quasi-symmetric. In this paper, we will prove it affirmatively in the following special cases:

- (1) The holomorphic sectional curvature restricted on a certain submanifold of D satisfies a pinching condition;
- (2) D is a domain over a dual square cone due to Xu [14];
- (3) The rank of D is less than or equal to three;
- (4) The dimension of D is less than or equal to ten.

We note that the same problem for the sectional curvature has been solved by D'Atri and Miatello [4].

The main results are proved in §4.

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§ 1. Normal j -algebras and Bergman metrics.

In this section, we describe briefly some of the basic facts on the structure of normal j -algebras which we need later. For more details, the reader is referred to [1], [4]–[9].

1.1. Let \mathfrak{g} be a solvable Lie algebra over the field \mathbf{R} of real numbers. We suppose that \mathfrak{g} has an almost complex structure j and a linear form ω satisfying the following conditions:

$$(1.1) \quad \begin{aligned} [jx, jy] &= j[jx, y] + j[x, jy] + [x, y] \quad \text{for all } x, y \in \mathfrak{g}; \\ \omega([jx, jy]) &= \omega([x, y]) \quad \text{for all } x, y \in \mathfrak{g}; \\ \omega([jx, x]) &> 0 \quad \text{for } x \neq 0 \in \mathfrak{g}. \end{aligned}$$

If the adjoint representation of \mathfrak{g} is triangular in $\mathfrak{gl}(\mathfrak{g})$, then the pair (\mathfrak{g}, j) is called a *normal j -algebra*.

It is known that homogeneous Siegel domains in complex vector spaces and normal j -algebras are in one to one correspondence up to holomorphic equivalence and j -isomorphism, respectively. Let us recall this correspondence briefly. Let D be a homogeneous Siegel domain. Then there exists a maximal \mathbf{R} -triangular subalgebra (the Iwasawa algebra) \mathfrak{g} in the Lie algebra of the affine automorphism group of D . Since the Lie group generated by \mathfrak{g} acts simply transitively on D , the tangent space of D at a base point can be identified with \mathfrak{g} as vector spaces. By this identification and the complex structure of the domain D , we can define an almost complex structure j on the Lie algebra \mathfrak{g} . Taking the linear form ω on \mathfrak{g} defined by

$$(1.2) \quad \omega(x) = \frac{1}{2} \text{trace}(\text{ad}(jx) - j \text{ad}(x)),$$

we have a normal j -algebra (\mathfrak{g}, j) . By this correspondence, the Bergman metric $\langle \cdot, \cdot \rangle$ of D at the base point is given on \mathfrak{g} as follows:

$$(1.3) \quad \langle x, y \rangle = \omega([jx, y]) \quad \text{for all } x, y \in \mathfrak{g} \quad ([8]).$$

1.2. Let D be a homogeneous Siegel domain and (\mathfrak{g}, j) the normal j -algebra corresponding to D . Let \mathfrak{a} be the orthogonal complement of $[\mathfrak{g}, \mathfrak{g}]$ in \mathfrak{g} with respect to the Bergman metric $\langle \cdot, \cdot \rangle$. Then \mathfrak{a} is an abelian subalgebra. For a linear form α on \mathfrak{a} , we define the root space $\mathfrak{g}(\alpha)$ by

$$\mathfrak{g}(\alpha) = \{x \in [\mathfrak{g}, \mathfrak{g}] ; [h, x] = \alpha(h)x \text{ for } h \in \mathfrak{a}\}.$$

Let $\dim \mathfrak{a} = r$. Then it is known by [9] that there exists a system $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r\}$ of linear independent roots satisfying the following (1.4)–(1.6):

$$(1.4) \quad \text{All roots are of the form } \frac{1}{2}\varepsilon_k, \varepsilon_k \quad (1 \leq k \leq r), \frac{1}{2}(\varepsilon_i \pm \varepsilon_k) \quad (1 \leq i < k \leq r) \\ \text{(but not all these need be roots) and}$$

$$\mathfrak{g} = \mathfrak{a} + \sum_k \mathfrak{g}(\varepsilon_k) + \sum_{i < k} \mathfrak{g}(\frac{1}{2}(\varepsilon_i + \varepsilon_k)) + \sum_{i < k} \mathfrak{g}(\frac{1}{2}(\varepsilon_i - \varepsilon_k)) + \sum_k \mathfrak{g}(\frac{1}{2}\varepsilon_k)$$

is the orthogonal direct sum with respect to the Bergman metric.

(1.5)
$$\begin{aligned} j(\mathfrak{g}(\varepsilon_k)) &= \mathfrak{a}; & j(\mathfrak{g}(\frac{1}{2}(\varepsilon_i + \varepsilon_k))) &= \mathfrak{g}(\frac{1}{2}(\varepsilon_i - \varepsilon_k)) \quad (1 \leq i < k \leq r); \\ j(\mathfrak{g}(\frac{1}{2}\varepsilon_k)) &= \mathfrak{g}(\frac{1}{2}\varepsilon_k). \end{aligned}$$

(1.6) Let us take a pair (i, k) of indices $i < k$ satisfying $\dim \mathfrak{g}(\frac{1}{2}(\varepsilon_i + \varepsilon_k)) \neq 0$. Then for any $a_{ik} \neq 0 \in \mathfrak{g}(\frac{1}{2}(\varepsilon_i + \varepsilon_k))$, the following linear mappings are injective:

$$\begin{aligned} x \in \mathfrak{g}(\frac{1}{2}(\varepsilon_k + \varepsilon_i)) &\longrightarrow [ja_{ik}, x] \in \mathfrak{g}(\frac{1}{2}(\varepsilon_i + \varepsilon_i)) & (k < i); \\ y \in \mathfrak{g}(\frac{1}{2}(\varepsilon_s + \varepsilon_i)) &\longrightarrow [jy, a_{ik}] \in \mathfrak{g}(\frac{1}{2}(\varepsilon_s + \varepsilon_k)) & (s < i); \\ u \in \mathfrak{g}(\frac{1}{2}\varepsilon_k) &\longrightarrow [ja_{ik}, u] \in \mathfrak{g}(\frac{1}{2}\varepsilon_i). \end{aligned}$$

Let us define numbers $\{n_{ik}\}, \{m_k\}, \{n_k\}$ by

(1.7)
$$\begin{aligned} n_{ik} &= n_{ki} = \dim \mathfrak{g}(\frac{1}{2}(\varepsilon_i + \varepsilon_k)) & (i < k); \\ m_k &= \dim \mathfrak{g}(\frac{1}{2}\varepsilon_k); \\ n_k &= 1 + \frac{1}{4}m_k + \frac{1}{2} \sum_{i < k} n_{ik} + \frac{1}{2} \sum_{k < i} n_{ki}. \end{aligned}$$

Then (1.6) implies the following:

(1.8) If $n_{ik} \neq 0$ (resp. $n_{kl} \neq 0$) for a triplet (i, k, l) of indices $i < k < l$, then $n_{kl} \leq n_{il}$ (resp. $n_{ik} \leq n_{il}$).

(1.9) If $n_{ik} \neq 0$ for a pair (i, k) of indices $i < k$, then $m_k \leq m_i$.

Since $\dim \mathfrak{g}(\varepsilon_k) = 1$ ($1 \leq k \leq r$), we take $E_k \in \mathfrak{g}(\varepsilon_k)$ which satisfies $\varepsilon_i(jE_k) = \delta_{ik}$. By using (1.2)–(1.5), we have

(1.10)
$$\langle E_k, E_k \rangle = \omega([jE_k, E_k]) = \omega(E_k) = n_k.$$

We define the *rank* of a domain D by the dimension of \mathfrak{a} . A homogeneous Siegel domain D is said to be *irreducible* if the domain D does not admit the direct product decomposition into homogeneous Siegel domains of lower dimension. By the notation $i \sim k$ for two indices i and k , we mean that $n_{ik} \neq 0$. It is known that a domain D is irreducible if and only if the normal j -algebra (\mathfrak{g}, j) corresponding to D satisfies the following connectedness condition on the indices; for any indices $1 \leq i < k \leq r$, there exists a sequence i_1, i_2, \dots, i_s of indices such that $i_1 = i, i_s = k$, and $i_1 \sim i_2 \sim \dots \sim i_s$ ([5], [6]).

1.3. Now we explain how we compute the curvature of the Bergman metric of a

Siegel domain D in terms of the normal j -algebra (\mathfrak{g}, j) corresponding to D ([7]).

Let ∇ be the invariant connection induced by the Bergman metric of D . Then ∇ is identified with a bilinear mapping $\nabla: (x, y) \in \mathfrak{g} \times \mathfrak{g} \rightarrow \nabla(x, y) \in \mathfrak{g}$ which satisfies the following conditions:

$$(1.11) \quad \begin{aligned} 2\langle \nabla(x, y), z \rangle &= \langle [z, x], y \rangle + \langle [z, y], x \rangle + \langle [x, y], z \rangle; \\ \nabla(x, jy) &= j\nabla(x, y) \end{aligned}$$

for all $x, y, z \in \mathfrak{g}$.

We denote by R the curvature tensor of the Bergman metric. Then R is given by the following formula:

$$\begin{aligned} R: (x, y, z) \in \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} &\longrightarrow R(x, y)z \in \mathfrak{g}, \\ R(x, y)z &= \nabla(x, \nabla(y, z)) - \nabla(y, \nabla(x, z)) - \nabla([x, y], z) \end{aligned}$$

for all $x, y, z \in \mathfrak{g}$.

The holomorphic bisectional curvature HBSC is defined by

$$\text{HBSC}(x, y) = \langle R(x, jx)jy, y \rangle$$

for $x, y \in \mathfrak{g}$, $\langle x, x \rangle = \langle y, y \rangle = 1$. From the definition of the holomorphic sectional curvature HSC, it follows that $\text{HSC}(x) = \text{HBSC}(x, x)$ for $x \in \mathfrak{g}$, $\langle x, x \rangle = 1$. By using ∇ , we get an explicit formula of HBSC as follows;

$$(1.12) \quad \text{HBSC}(x, y) = 2\langle j\nabla(x, y), \nabla(jx, y) \rangle + \langle \nabla([x, jx], y), jy \rangle.$$

The subalgebra $\mathfrak{a} + ja$ is totally geodesic, i.e., $\nabla(\mathfrak{a} + ja, \mathfrak{a} + ja) = \mathfrak{a} + ja$. We define a submanifold A in D by the orbit of the subgroup $\exp(\mathfrak{a} + ja)$ through the base point. Then the submanifold A is totally geodesic and holomorphically equivalent to the product of r upper-half planes.

§ 2. Curvature on the submanifold A of D .

We denote by r the rank of a homogeneous Siegel domain D . Since a domain of rank one is holomorphically equivalent to an open unit ball and it is symmetric, we assume that r is greater than one.

First we compute the holomorphic bisectional curvature $\text{HBSC}(x, y)$ for unit vectors $x, y \in \mathfrak{a} + ja$. It is easy to get the following formulas:

$$(2.1) \quad \nabla(jE_k, \mathfrak{g}) = 0;$$

$$(2.2) \quad \nabla(E_i, E_k) = j[jE_i, E_k] = \delta_{ik}jE_k.$$

Let us take $x = \sum_k (a_k E_k + b_k jE_k)$ and $y = \sum_k (c_k E_k + d_k jE_k)$ in $\mathfrak{a} + ja$ satisfying $\langle x, x \rangle = \sum_k n_k (a_k^2 + b_k^2) = 1$ and $\langle y, y \rangle = \sum_k n_k (c_k^2 + d_k^2) = 1$, respectively. Then, using the

formulas (2.1) and (2.2), we have

$$\nabla(x, y) = \sum_k a_k(-d_k E_k + c_k j E_k) \quad \text{and} \quad \nabla(jx, y) = \sum_k b_k(d_k E_k - c_k j E_k).$$

Hence, $\langle j\nabla(x, y), \nabla(jx, y) \rangle = 0$. Since $[x, jx] = -\sum_k (a_k^2 + b_k^2) E_k$, we have $\nabla([x, jx], y) = \sum_k (a_k^2 + b_k^2)(d_k E_k - c_k j E_k)$. Therefore,

$$\text{HBSC}(x, y) = \langle \nabla([x, jx], y), jy \rangle = -\sum_k n_k (a_k^2 + b_k^2)(c_k^2 + d_k^2).$$

Putting $x = y$ in HBSC, we have the holomorphic sectional curvature

$$\text{HSC}(x) = -\sum_k n_k (a_k^2 + b_k^2)^2.$$

In these formulas, calculating the maximum and the minimum of $\text{HBSC}(x, y)$ (resp. $\text{HSC}(x)$) under the constraint $\langle x, x \rangle = \langle y, y \rangle = 1$ (resp. $\langle x, x \rangle = 1$), we have the following:

LEMMA 1. *Let HBSC_A and HSC_A be the holomorphic bisectional curvature and the holomorphic sectional curvature restricted to the submanifold A of D , respectively. Then HBSC_A and HSC_A satisfy (i) $-1/L \leq \text{HBSC}_A \leq 0$ and (ii) $-1/L \leq \text{HSC}_A \leq -1/M$, where $L = \min_k \{n_k\}$ and $M = n_1 + n_2 + \dots + n_r$.*

We note that the inequalities in (ii) of the lemma stated above were proved by Zelow [15]. Using Lemma 1, we have the following easily:

LEMMA 2. *The following two conditions are equivalent:*

- (i) HSC_A is pinched as $-Kr \leq \text{HSC}_A \leq -K$ with $K > 0$;
- (ii) $n_1 = n_2 = \dots = n_r$.

It was proved by D'Atri and Miatello [4] that an irreducible domain D corresponding to a normal j -algebra (\mathfrak{g}, j) is quasi-symmetric if and only if the root spaces of \mathfrak{g} satisfy the following:

$$(2.3) \quad n_{ik} = n_{12} > 0 \quad (1 \leq i < k \leq r) \quad \text{and} \quad m_1 = m_2 = \dots = m_r.$$

By the definition of the numbers $\{n_k\}$, we have $n_1 = n_2 = \dots = n_r$ for an irreducible quasi-symmetric domain D , and hence, we have the pinching of HSC_A in Lemma 2.

REMARK. There exist irreducible domains which are not quasi-symmetric but satisfy the condition in Lemma 2. We can find examples of such domains in low dimension from the classification ([6], [12]).

§ 3. Nonpositivity of HBSC.

In this section, we try to get necessary conditions for the holomorphic bisectional curvature HBSC to be nonpositive. We put $B(x, y) = \langle R(x, jx)jy, y \rangle$ for $x, y \in \mathfrak{g}$. Then HBSC is nonpositive if and only if $B(x, y) \leq 0$ for all $x, y \in \mathfrak{g}$. We denote general root

vectors in $\mathfrak{g}(\frac{1}{2}(\varepsilon_k + \varepsilon_l))$ by $x_{kl}, y_{kl}, z_{kl}, \dots, (k < l)$.

3.1. By making use of (1.1), (1.4), and (1.5), we have the bracket relation as follows:

$$(3.1) \quad [jx_{ik}, E_l] = \delta_{kl}x_{ik} \quad (i < k).$$

$$(3.2) \quad [jx_{ik}, y_{ik}] = \frac{\langle x_{ik}, y_{ik} \rangle}{n_i} E_i \quad (i < k).$$

$$(3.3) \quad [jx_{kl}, y_{il}] = [jy_{il}, x_{kl}] \quad (i < k < l).$$

Moreover, by (1.3), we get

$$(3.4) \quad \langle [jx_{ik}, y_{kl}], z_{il} \rangle = \langle [jy_{kl}, z_{il}], x_{ik} \rangle \quad (i < k < l).$$

For the connection ∇ , we have the following formulas easily by (1.1), (1.4), (1.5), and (1.11) (cf. [1]):

$$(3.5) \quad \nabla(x_{ik}, E_l) = \nabla(E_l, x_{ik}) = \frac{1}{2}(\delta_{il} + \delta_{kl})jx_{ik} \quad (i < k).$$

$$(3.6) \quad \nabla(x_{ik}, y_{ik}) = \frac{\langle x_{ik}, y_{ik} \rangle}{2n_i} jE_i + \frac{\langle x_{ik}, y_{ik} \rangle}{2n_k} jE_k \quad (i < k).$$

$$(3.7) \quad \nabla(x_{ik}, y_{kl}) = \frac{1}{2}j[jx_{ik}, y_{kl}] \quad (i < k < l).$$

$$(3.8) \quad \nabla(z_{il}, y_{kl}) = \frac{1}{2}j[jz_{il}, y_{kl}] \quad (i < k < l).$$

$$(3.9) \quad \langle \nabla(x_{ik}, z_{il}), jy_{kl} \rangle = \frac{1}{2} \langle [jy_{kl}, z_{il}], x_{ik} \rangle \quad (i < k < l).$$

$$(3.10) \quad \nabla(jx_{ik}, E_l) = \frac{1}{2}(\delta_{kl} - \delta_{il})x_{ik} \quad (i < k).$$

$$(3.11) \quad \nabla(jx_{ik}, y_{ik}) = \frac{\langle x_{ik}, y_{ik} \rangle}{2n_i} E_i - \frac{\langle x_{ik}, y_{ik} \rangle}{2n_k} E_k \quad (i < k).$$

$$(3.12) \quad \nabla(jz_{il}, y_{kl}) = \nabla(jy_{kl}, z_{il}) = \frac{1}{2}[jz_{il}, y_{kl}] \quad (i < k < l).$$

3.2. We compute $B(x, y)$ for some test vectors $x, y \in \mathfrak{g}$ to see the sign of HBSC on some j -invariant planes. For indices $1 \leq i_1 < i_2 < \dots < i_m < l \leq r$, we take root vectors $x_{i_k l}$ and $y_{i_k l}$ in $\mathfrak{g}(\frac{1}{2}(\varepsilon_{i_k} + \varepsilon_l))$ ($1 \leq k \leq m$) and define two vectors x and y by

$$x = \sum_k E_{i_k} + E_l + \sum_k x_{i_k l} \quad \text{and} \quad y = \sum_k E_{i_k} + E_l + \sum_k y_{i_k l},$$

respectively. We now put

$$\langle x_{ikl}, x_{ikl} \rangle = 2\alpha_k, \quad \langle y_{ikl}, y_{ikl} \rangle = 2\beta_k, \quad \langle x_{ikl}, y_{ikl} \rangle = 2\gamma_k \quad (1 \leq k \leq m).$$

LEMMA 3. For the test vectors x and y defined above, the following formula of the holomorphic bisectonal curvature holds:

$$B(x, y) = \frac{2\left(\sum_k \gamma_k\right)^2}{n_l} - n_l - \sum_k \left(n_{ik} + 2\alpha_k + 2\beta_k + \frac{2\alpha_k\beta_k}{n_{ik}} + 8\gamma_k + \frac{2\gamma_k^2}{n_{ik}} \right) - \frac{1}{2} \sum_{s < t} (\| [jx_{is}, y_{it}] + [jy_{is}, x_{it}] \|^2 + 4\langle [jx_{is}, x_{it}], [jy_{is}, y_{it}] \rangle).$$

PROOF. We may assume that $i_k = k$ ($1 \leq k \leq m$) to simplify the notation. By using (2.2) and (3.5)–(3.8), we have easily the following:

$$\begin{aligned} \nabla(x, y) &= \sum_k \left(1 + \frac{\gamma_k}{n_k} \right) jE_k + \left(1 + \sum_k \frac{\gamma_k}{n_l} \right) jE_l + \sum_k (jx_{kl} + jy_{kl}) \\ &\quad + \frac{1}{2} \sum_{s < t} (j[jx_{sl}, y_{tl}] + j[jy_{sl}, x_{tl}]). \end{aligned}$$

By (2.1), (3.3), and (3.10)–(3.12), we get

$$\nabla(jx, y) = \sum_k \frac{\gamma_k}{n_k} E_k - \left(\sum_k \frac{\gamma_k}{n_l} \right) E_l + \frac{1}{2} \sum_{s < t} ([jx_{sl}, y_{tl}] + [jy_{sl}, x_{tl}]).$$

Hence,

$$\langle j\nabla(x, y), \nabla(jx, y) \rangle = -\sum_k \frac{\gamma_k^2}{n_k} + \frac{\left(\sum_k \gamma_k\right)^2}{n_l} - \frac{1}{4} \sum_{s < t} \| [jx_{sl}, y_{tl}] + [jy_{sl}, x_{tl}] \|^2.$$

On the other hand, (1.4) and (3.1)–(3.3) imply that

$$[x, jx] = -\sum_k \left(1 + \frac{2\alpha_k}{n_k} \right) E_k - E_l - 2\sum_k x_{kl} - 2\sum_{s < t} [jx_{sl}, x_{tl}].$$

By the same computation as in the above, we have

$$\begin{aligned} \nabla([x, jx], y) &= -\sum_k \left(1 + \frac{2\alpha_k}{n_k} + \frac{2\gamma_k}{n_k} \right) jE_k - \left(1 + 2\sum_k \frac{\gamma_k}{n_l} \right) jE_l \\ &\quad - 2\sum_k jx_{kl} - \sum_k \left(1 + \frac{\alpha_k}{n_k} \right) jy_{kl} - \sum_{s < t} (j[jx_{sl}, y_{tl}]) \end{aligned}$$

$$\begin{aligned}
& +j[jy_{st}, x_{tl}] - 2 \sum_{s < t} j[jx_{st}, x_{tl}] \\
& - 2 \sum_{s < t} \nabla([jx_{st}, x_{tl}], y_{st}) - 2 \sum_{s < t} \nabla([jx_{st}, x_{tl}], y_{tl}).
\end{aligned}$$

Using (3.4), (3.7), and (3.9), we get

$$\begin{aligned}
\langle \nabla([x, jx], y), jy \rangle &= -n_l - \sum_k (n_k + 2\alpha_k + 2\beta_k + 8\gamma_k) \\
& - 2 \sum_k \frac{\alpha_k \beta_k}{n_k} - 2 \sum_{s < t} \langle [jx_{st}, x_{tl}], [jy_{st}, y_{tl}] \rangle.
\end{aligned}$$

Substituting these formulas into (1.12), we have the formula of the lemma. q.e.d.

3.3. By making use of the explicit formula obtained in the above lemma, we have the following:

LEMMA 4. *If HBSC is nonpositive and $n_{ik} \neq 0$ for a pair (i, k) of indices $i < k$, then $n_i \leq n_k$.*

PROOF. Let us take test vectors;

$$x = E_i + E_k + x_{ik} \quad \text{and} \quad y = E_i + E_k + y_{ik}, \quad x_{ik}, y_{ik} \in \mathfrak{g}(\tfrac{1}{2}(\varepsilon_i + \varepsilon_k))$$

satisfying $\langle x_{ik}, x_{ik} \rangle = 2a$, $\langle y_{ik}, y_{ik} \rangle = 2b$, and $\langle x_{ik}, y_{ik} \rangle = 2c$. Then by putting $m = 1$, $i_1 = i$, $l = k$, $\alpha_1 = a$, $\beta_1 = b$, and $\gamma_1 = c$ in the formula of Lemma 3, we have

$$B(x, y) = 2c^2 \left(\frac{1}{n_k} - \frac{1}{n_i} \right) - \left(2a + 2b + 8c + n_i + n_k + \frac{2ab}{n_i} \right).$$

Now, putting $a = b = n_i$, $c = -n_i$, we have

$$B(x, y) = \frac{(2n_i + n_k)(n_i - n_k)}{n_k},$$

which implies that $n_i \leq n_k$.

q.e.d.

On the inequality between the numbers n_i and n_k in the lemma stated above, D'Atri [1] proved the same one and $n_i \leq 2n_k$ under the condition of nonpositive sectional curvature and of nonpositive holomorphic sectional curvature, respectively.

We now suppose that there exists a triplet (i, k, l) of indices $i < k < l$ satisfying the conditions $n_{kl} \neq 0$ and $n_{ik} < n_{il}$. Since the linear mapping: $x_{il} \in \mathfrak{g}(\tfrac{1}{2}(\varepsilon_i + \varepsilon_l)) \rightarrow [jx_{il}, a_{kl}] \in \mathfrak{g}(\tfrac{1}{2}(\varepsilon_i + \varepsilon_k))$ is not injective for any $a_{kl} \neq 0 \in \mathfrak{g}(\tfrac{1}{2}(\varepsilon_k + \varepsilon_l))$, there exists $a_{il} \neq 0 \in \mathfrak{g}(\tfrac{1}{2}(\varepsilon_i + \varepsilon_l))$ satisfying $[ja_{il}, a_{kl}] = 0$.

By using Lemma 3, we prove the following:

LEMMA 5. Let (i, k, l) be a triplet of indices $i < k < l$ satisfying the conditions: $n_{kl} \neq 0$ and $n_{ik} < n_{il}$. If HBSC is nonpositive, then $n_i + n_k \leq n_l$.

PROOF. Let us take $a_{il} \neq 0$ and $a_{kl} \neq 0$ satisfying $[ja_{il}, a_{kl}] = 0$. Then, putting $m = 2$, $i_1 = i$, $i_2 = k$, $x_{il} = -y_{il} = a_{il}$, and $x_{kl} = -y_{kl} = a_{kl}$ in the formula of Lemma 3, we have $\alpha_1 = \beta_1 = -\gamma_1$, $\alpha_2 = \beta_2 = -\gamma_2$, and

$$B(x, y) = \frac{2(\alpha_1 + \alpha_2)^2}{n_l} - n_l - \left(n_i + n_k - 4\alpha_1 - 4\alpha_2 + \frac{4\alpha_1^2}{n_i} + \frac{4\alpha_2^2}{n_k} \right).$$

Therefore, substituting n_i and n_k for α_1 and α_2 , respectively, we have

$$B(x, y) = \frac{\{2(n_i + n_k) + n_l\} \{(n_i + n_k) - n_l\}}{n_l},$$

which implies $n_i + n_k \leq n_l$.

q.e.d.

We will make use of the following lemma to distinguish the domains of nonpositive holomorphic bisectional curvature among the domains of low rank or of low dimension.

LEMMA 6. Let $\{i_1, i_2, \dots, i_m, l\}$ be the set of indices $1 \leq i_1 < i_2 < \dots < i_m < l \leq r$ satisfying the following conditions:

(i) $n_{i_k l} \neq 0$ ($1 \leq k \leq m$) and (ii) $n_{i_s i_t} = 0$ ($1 \leq s \neq t \leq m$).

If HBSC is nonpositive, then $n_{i_1} + n_{i_2} + \dots + n_{i_m} \leq n_l$.

PROOF. The condition (ii) implies that $[jx_{i_s l}, x_{i_t l}] = 0$ for all $x_{i_s l}$ and $x_{i_t l}$ ($1 \leq s < t \leq m$). Hence, by putting $\alpha_k = \beta_k = -\gamma_k = n_{i_k} > 0$ in the formula of Lemma 3, we have $B(x, y) = (2\sum_k n_{i_k} + n_l)(\sum_k n_{i_k} - n_l)/n_l$.

q.e.d.

§ 4. Characterization of quasi-symmetric domains.

In this section, we state the results obtained. By the curvature of a homogeneous Siegel domain D in the following theorems, we mean the one relative to the Bergman metric of D , exclusively.

4.1. First we prove the following:

THEOREM 1. Let D be an irreducible homogeneous Siegel domain of rank r and of nonpositive holomorphic bisectional curvature. If the holomorphic sectional curvature restricted on the submanifold A of D (defined in §1) takes the maximum $-K$ and the minimum $-rK$ with $K > 0$, then D is quasi-symmetric.

PROOF. As we saw in Lemma 2, the pinching condition for HSC_A implies the equalities $n_1 = n_2 = \dots = n_r$. By using this, we can prove that $n_{ik} = n_{12}$ for $1 \leq i < k \leq r$ and $m_i = m_1$ for $1 \leq i \leq r$. In fact, let us take an arbitrary pair (s, t) of indices $1 \leq s < t \leq r$ with $n_{st} \neq 0$. Then by Lemma 5, the equality $n_{is} = n_{it}$ holds for any index i with $i < s$. By

(1.8) and (1.9), we have $n_{ti} \leq n_{si}$ for $t < i$ and $m_t \leq m_s$. Using the both of Lemma 5 and (1.8), we have $n_{it} \leq n_{si}$ for $s < i < t$. On the other hand, by the definition of the numbers n_s and n_t in (1.7), we have

$$\begin{aligned} 0 &= n_s - n_t \\ &= \frac{1}{4}(m_s - m_t) + \frac{1}{2} \sum_{i < s} (n_{is} - n_{it}) + \frac{1}{2} \sum_{s < i < t} (n_{si} - n_{it}) + \frac{1}{2} \sum_{t < i} (n_{si} - n_{ti}). \end{aligned}$$

Every term of the right-hand side of this equality is nonnegative. Hence, we have $m_s = m_t$ and $n_{si} = n_{ti}$ for $i \neq s, t$. Since D is irreducible, the set $\{1, 2, \dots, r\}$ of indices satisfies the connectedness condition stated in §1, and hence, the numbers n_{ik} and m_i are constants independent of indices. Therefore, the condition (2.3) holds and the characterization by [4] stated in §2, shows that D is quasi-symmetric. q.e.d.

We note that the assumption on the holomorphic sectional curvature in the theorem stated above is satisfied for any irreducible quasi-symmetric Siegel domain, but as we see later, we can get the assertion of the theorem for domains of low rank or of low dimension without this assumption.

4.2. Following Xu [14], we call a homogeneous Siegel domain a domain over a cone of *dual square type* if the root space decomposition given by (1.4) satisfies the conditions:

$$n_{k,k+1} = n_{k,k+2} = \dots = n_{kr} > 0 \quad (k = 1, 2, \dots, r-1).$$

THEOREM 2. *Let D be a homogeneous Siegel domain over a cone of dual square type. If the holomorphic bisectional curvature of D is nonpositive, then D is quasi-symmetric.*

PROOF. We put $\lambda_k = n_{k,k+1}$ ($1 \leq k \leq r-1$) and $\lambda_r = 0$. Then by (1.8) and (1.9), we have $\lambda_{k+1} \leq \lambda_k$ and $m_{k+1} \leq m_k$ ($1 \leq k \leq r-1$). By (1.7),

$$n_k = 1 + \frac{1}{2}(\lambda_1 + \lambda_2 + \dots + \lambda_{k-1}) + \frac{1}{2}(r-k)\lambda_k + \frac{1}{4}m_k \quad (1 \leq k \leq r).$$

Hence,

$$n_k - n_{k+1} = \frac{1}{2}(r-k-1)(\lambda_k - \lambda_{k+1}) + \frac{1}{4}(m_k - m_{k+1}) \geq 0$$

for every k ($1 \leq k \leq r-1$). From this and Lemma 4, it follows that $n_1 = n_2 = \dots = n_r$, $\lambda_1 = \lambda_2 = \dots = \lambda_{r-1}$, and $m_1 = m_2 = \dots = m_r$. The domain D satisfies the condition (2.3) and hence, D is quasi-symmetric. q.e.d.

The irreducible decomposition of a homogeneous Siegel domain is the de Rham decomposition of a Kaehler manifold with respect to the Bergman metric ([5]). Let D

be a reducible homogeneous Siegel domain. Then D is quasi-symmetric if and only if every irreducible component of D is quasi-symmetric ([11]). Therefore, in order to prove that a domain D is quasi-symmetric, it suffices to prove it for every irreducible component of D .

COROLLARY. *Let D be a homogeneous Siegel domain over a self-dual cone. If the holomorphic bisectonal curvature of D is nonpositive, then D is quasi-symmetric.*

PROOF. As we noted above, we can assume that D is irreducible. Then D is a Siegel domain over an irreducible homogeneous self-dual cone ([5]). Any irreducible homogeneous self-dual cone is of dual square type ([13], [14]). Therefore, the assertion follows from Theorem 2. q.e.d.

4.3. By using the classification of homogeneous Siegel domains of low rank or of low dimension [6], [12], and the results obtained in the previous sections, we have the following:

THEOREM 3. *Let D be a homogeneous Siegel domain of rank ≤ 3 . If the holomorphic bisectonal curvature of D is nonpositive, then D is quasi-symmetric.*

PROOF. Let (g, j) be a normal j -algebra of rank r with $r \leq 3$. We can assume that the domain D corresponding to (g, j) is irreducible and $r \geq 2$. If $r = 2$, then by Lemma 4,

$$n_1 = 1 + \frac{1}{2}n_{12} + \frac{1}{4}m_1 \leq n_2 = 1 + \frac{1}{2}n_{12} + \frac{1}{4}m_2 \quad \text{and} \quad m_2 \leq m_1.$$

Therefore, this implies $m_1 = m_2$ and D is quasi-symmetric. Now we assume that $r = 3$. The irreducibility of the domain and the inequality (1.8) imply that the following three cases may occur: (i) $n_{12}n_{23} \neq 0$, $n_{12}, n_{23} \leq n_{13}$; (ii) $n_{23}n_{13} \neq 0$, $n_{12} = 0$; (iii) $n_{12}n_{13} \neq 0$, $n_{23} = 0$. In the case (i), by Lemma 4, we have

$$\begin{aligned} n_1 &= 1 + \frac{1}{2}n_{12} + \frac{1}{2}n_{13} + \frac{1}{4}m_1 \\ &\leq n_2 = 1 + \frac{1}{2}n_{12} + \frac{1}{2}n_{23} + \frac{1}{4}m_2 \\ &\leq n_3 = 1 + \frac{1}{2}n_{13} + \frac{1}{2}n_{23} + \frac{1}{4}m_3. \end{aligned}$$

If $n_{12} < n_{13}$, then by Lemma 5, we have $n_1 + n_2 \leq n_3$. This implies

$$1 + n_{12} + \frac{1}{4}m_1 + \frac{1}{4}m_2 \leq \frac{1}{4}m_3,$$

which contradicts $m_3 \leq m_1$. Hence, we have $n_{12} = n_{13} = n_{23}$ and $m_1 = m_2 = m_3$. The domain D satisfies the condition (2.3). Thus, D is quasi-symmetric. Now we consider

the case (ii). As we saw in the first case, by Lemma 5, we have $n_1 + n_2 \leq n_3$, and hence,

$$1 + \frac{1}{4}m_1 + \frac{1}{4}m_2 \leq \frac{1}{4}m_3.$$

Since $n_{23}n_{13} \neq 0$, we have $m_3 \leq m_1, m_2$ by (1.9). This is a contradiction and the case (ii) does not occur. In the case (iii), by Lemma 4, we have $n_1 \leq n_2, n_3$, and

$$n_{12} + n_{13} + \frac{1}{2}m_1 \leq n_{12} + \frac{1}{2}m_2, \quad n_{13} + \frac{1}{2}m_3.$$

This contradicts $m_2, m_3 \leq m_1$. Hence, the case (iii) also does not occur. q.e.d.

Applying Lemmas 4, 5, 6 and the theorems obtained above to every domain classified by [6] and [12], we have the following theorem. The proof is easy but tedious; so we omit it.

THEOREM 4. *Let D be a homogeneous Siegel domain of dimension ≤ 10 . If the holomorphic bisectonal curvature of D is nonpositive, then D is quasi-symmetric.*

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