TOKYO J. MATH.
VOL. 15, NO. 2, 1992

On Wall Manifolds with Almost Free Z_{2^k} Actions

Tamio HARA

Science University of Tokyo
(Communicated by T. Nagano)

0. Introduction.

In order to understand the bordism classification of finite group actions on oriented manifolds, it is useful to consider some notion of manifolds with equivariant Wall structures. In [8], C. Kosniowski and E. Ossa studied the bordism theory $W_*(Z_2; All)$ of Wall manifolds with unrestricted involutions and determined completely the bordism theory $\Omega_*(Z_2; All)$ of oriented involutions, especially its torsion part as the image of the Bockstein homomorphism $\beta: W_*(Z_2; All) \to \Omega_*(Z_2; All)$. In this paper, we treat an almost free Z_{2^k} action on Wall manifold, i.e., one for which only the $Z_2 \subset Z_{2^k}$ may possibly fix points on manifold. From the viewpoint of action, such object is exactly Wall manifold with action of type $(Z_{2^k}, 1)$ in [13].

In section 1, we study the bordism theory $W_*(Z_{2^k}; Af)$ of these objects. By the map which ignores Wall structures, the theories $W_*(Z_{2^k}; Free)$ and $W_*(Z_{2^k}; Af, Free)$ are derived from the corresponding unoriented theories as usual (Propositions 1.4 and 1.8). In particular, we have that $W_*(Z_{2^k}; Af, Free)$ is the sum of three parts; the images $Im(t)$ of two kinds of extensions from Z_2 actions and another part $\bar{L}_*$. Using these results, we obtain the exact sequence for the triple $(Af, Free, \emptyset)$ (Proposition 1.11), and the W_*-module structure of $W_*(Z_{2^k}; Af)$ (Theorem 1.19). There the classes $\{V(0, 2n+2)\}$ (Definition 1.17) are useful to describe the part K_t which lies in $Im(t) \subset W_*(Z_{2^k}; Af, Free)$, while the part L_* is isomorphic to $\bar{L}_*$ naturally.

In section 2, we describe the image $\mathscr{T}$ of the map β: $W_*(Z_{2^k}; Af) \to \Omega_*(Z_{2^k}; Af)$; the bordism module of orientation preserving almost free Z_{2^k} actions, and describe the torsion part of order 2 (Theorem 2.3). As an application, we study the image of $I_*: \Omega_*(Z_4; Free) \to \Omega_*(Z_4; Af)$; the forgetful homomorphism by using the result of principal Z_{2^k} actions in [5] (Theorem 2.9).

The author would like to thank the referee for his many valuable comments.

Received January 13, 1992
Revised August 7, 1992

1. Wall manifolds with almost free Z_{2^k} actions.

Let G be a finite abelian group, and let a pair of G space (X, A, σ) be fixed. Then

DEFINITION 1.1 (cf. [6] [13]). *A Wall manifold with G action in* (X, A, σ) is a 4-tuple (M, φ, α, f) where:

(i) M is a compact smooth unoriented manifold with G action $\varphi: G \times M \to M$,

(ii) a Wall structure map $\alpha: M \to \boldsymbol{RP}(1) \subseteq \boldsymbol{RP}(\infty)$ (i.e., one which classifies the determinant bundle $\det\tau_M$ of the tangent bundle τ_M of M) which is equivariant with respect to φ, i.e., $\alpha \circ \varphi(g, -) = \alpha$ for each $g \in G$,

(iii) a bundle map $\bar{\alpha}$ covering α such that $\bar{\alpha} \circ \det d_{\varphi(g,-)} = \bar{\alpha}$ for each $g \in G$ where $\det d_{\varphi(g,-)}: \det\tau_M \to \det\tau_M$ is the map induced by $\varphi(g, -)$, and

(iv) $f: (M, \partial M, \varphi) \to (X, A, \sigma)$ is an equivariant map.

We identify (M, φ, α, f) and $(M', \varphi', \alpha', f')$ if and only if there is a diffeomorphism $M \approx M'$ which is equivariant under α and α', f and f', $\varphi(g, -)$ and $\varphi'(g, -)$ for each $g \in G$.

EXAMPLE 1.2. Let (M, φ) be an orientation-preserving G action on an oriented manifold M. Then (M, φ) may be a Wall manifold with the trivial structure $\alpha = 1$. When such M admits an orientation-reversing involution R such that $\varphi(g, -) \circ R = R \circ \varphi(g, -)$ for each $g \in G$, $S^1 \times_R M = S^1 \times M / -1 \times R$ is a Wall manifold with the non-trivial structure map $\alpha([z, w]) = [z] \in \boldsymbol{RP}(1)$ for each $z \in S^1$ and $m \in M$, and the induced G action $\mathrm{id} \times \varphi$. We treat these types after all, and may omit the map α if no confusion can arise.

Suppose that $\mathscr{F}' \subset \mathscr{F}$ are families in G. We say that such action (M, φ) is $(\mathscr{F}, \mathscr{F}')$-*free* if for each $x \in M$ the isotropy subgroup $G_x \in \mathscr{F}$ and if for each $x \in \partial M$, $G_x \in \mathscr{F}'$. Using these objects gives, in the usual way, *a singular bordism group* $W_n(G; \mathscr{F}, \mathscr{F}')(X, A, \sigma)$ *of n-dimensional Wall manifolds of* $(\mathscr{F}, \mathscr{F}')$*-free G actions in* (X, A, σ), in which every element has order 2 (cf. [13]). Thus we have a graded abelian group

$$W_*(G; \mathscr{F}, \mathscr{F}')(X, A, \sigma) = \sum_{n \geq 0} W_n(G; \mathscr{F}, \mathscr{F}')(X, A, \sigma),$$

which has a natural module structure over the Wall cobordism ring W_* (cf. [12; p. 163]). We may denote that $W_*(G; \mathscr{F}, \mathscr{F}')(\mathrm{pt}, \mathrm{pt}, \mathrm{id}) = W_*(G; \mathscr{F}, \mathscr{F}')$ and $W_*(G; \mathscr{F}, \emptyset)$ $(X, \emptyset, \sigma) = W_*(G; \mathscr{F})(X, \sigma)$ as usual.

Let $G_k = Z_{2^k}$ denote the cyclic group of order 2^k, $k \geq 2$ and T its generator. Let Af be the family $\{Z_2, \{1\}\}$ of subgroup of G_k, then we say that a G_k action $T: M \to M$ is *almost free* if it is Af-free. We always denote by (S^{2n+1}, T), the standard G_k action on the $(2n+1)$-sphere.

DEFINITION 1.3. For $k \geq 1$, let e and s: $W_*(Z_2; \mathscr{F}, \mathscr{F}') \to W_*(G_k; \mathscr{F}, \mathscr{F}')$ be the extension maps defined by $e([M, A]) = [G_k \times_{Z_2} M, T \times \mathrm{id}]$ and $s([M, A]) = [S^1 \times_{Z_2} M, T \times \mathrm{id}]$ for each class $[M, A] \in W_*(Z_2; \mathscr{F}, \mathscr{F}')$ respectively. The extensions e_2 and s_2

from $\mathfrak{N}_*(Z_2; \mathscr{F}, \mathscr{F}')$ are similarly defined in the unoriented case. When $k=1$, we denote the map s by d for convenience.

We first consider the principal G_k actions.

PROPOSITION 1.4. *For* $k\geq 1$,

(i) $W_*(G_k; Free)$ *is freely generated over* W_* *by the classes* $e(X(2n))$ *and* $[S^{2n+1}, T]$ *for* $n\geq 0$, *where* $X(2n)=[S^1\times_R S^{2n-1}, A=\mathrm{id}\times -1]$ *with the reflection* R: $S^{2n-1}\to S^{2n-1}$ *defined by* $R(x_0, x_1, \cdots, x_{2n-1})=(-x_0, x_1, \cdots, x_{2n-1})$ *if* $n\geq 1$, *and* $X(0)=[Z_2, -1]$.

(ii) *The generators* $\{[S^{2n+1}, T]\}$ *may be replaced by* $\{s(X(2n))\}$ *for* $n\geq 0$ *and the following relation holds*:

$$[S^{2n+1}, T]=\sum_{j=0}^{m}[CP(2j)]s(X(2n-4j)) \tag{1.5}$$

for $n=2m$ *or* $n=2m+1$ *with* $m\geq 0$.

PROOF. Let F: $W_*(G_k; Free)\subsetneq \mathfrak{N}_*(G_k; Free)$ be the embedding which forgets Wall structures (cf. [12: p. 163]). Since $e_2([S^{2n}, -1])$ and $[S^{2n+1}, T]$ generate $\mathfrak{N}_*(G_k; Free)$ freely (cf. [5; Prop. 1.7]) and the relation $[S^{2n}, -1]=\sum_{j=0}^{n}[RP(2n-2j)]X(2j)$ holds (cf. [14; Prop. 3.1]), these implies the result (i) in the unoriented case hence in $W_*(G_k; Free)$ via the map F. Next, we have that

$$\begin{aligned}
[S^{2n+1}, T] &= \sum_{p=0}^{n}[RP(2p)]s([S^{2n-2p}, -1]) && \text{by [5; Theorem 1.17, Lemma 1.10 (ii)]}\\
&= \sum_{0\leq p+q\leq n}[RP(2p)][RP(2q)]s(X(2n-2p-2q)) && \text{by [14; Prop. 3.1]}\\
&= \sum_{i=0}^{n}\Big(\sum_{p+q=i}[RP(2p)][RP(2q)]\Big)s(X(2n-2i))\\
&= \sum_{j=0}^{[n/2]}[RP(2j)]^2 s(X(2n-4j))\\
&= \sum_{j=0}^{[n/2]}[CP(2j)]s(X(2n-4j)) && \text{by [16; Lemma 7]}.
\end{aligned}$$

Hence the above relation (1.5) holds. See [4: Theorem 2.5] about the class $X(2n)$ represented by the Wall manifold as mentioned above. q.e.d.

Next we study the relative theory $W_*(G_k; Af, Free)$ by using a standard fixed point construction. Suppose that (M, T, α) is an n-dimensional Wall manifold with $(Af, Free)$-G_k action. We note that if F^{n-i} is the $(n-i)$-dimensional fixed point set of Z_2 in M, then $G_{k-1}\cong G_k/Z_2$ acts freely on F^{n-i} while Z_2 acts on its normal bundle ν_i by multiplication by -1. Here $\det\tau_M|F\cong\det\nu_i\otimes\det\tau_F$ with Z_2 acting as $(-1)^i$ in the fibers. Thus the fixed point set F has even codimension by the definition 1.1 (iii). For such $\nu_{2j}\to F^{n-2j}$, let $\gamma_{2j}=\gamma_{2j}(C^\infty, k)\to BO_{2j}=BO_{2j}(C^\infty, k)$ be its classifying space and universal bundle (cf. [2]). Now let $g_{2j}\colon F^{n-2j}\to BO_{2j}$ be the classifying map. By taking the determinant bundles of v_{2j} and γ_{2j} respectively, we have the map $\det g^*_{2j}\colon (D(\det v_{2j}), S(\det v_{2j}))\to (D(\det\gamma_{2j}), S(\det\gamma_{2j}))$ naturally where $D(-)$ and $S(-)$ denote the associated disk and sphere bundles respectively. Such $D_{2j}=D(\det v_{2j})$ possesses the Wall structure $\bar{\alpha}=\alpha\circ i_*\colon \det\tau_D\cong i_*\det\tau_M\to \boldsymbol{RP}(1)$ for the embedding $i\colon F^{n-2j}\subset M$, and the induced $\det\bar{T}\in G_{k-1}\cong G_k/\boldsymbol{Z}_2$ acts on it freely. Hence we have an isomorphism:

$$(1.6)\qquad h\colon \boldsymbol{W}_*(G_k;\ Af,\ Free)\cong\sum_{j=0}^{[*/2]}\boldsymbol{W}_{*-2j+1}((D(\det\gamma_{2j}), S(\det\gamma_{2j}))\times_{G_{k-1}}EG_{k-1})$$

assigning to $[M, T, \alpha]$ the sum of pairs (p_{2j}, q_{2j}) where $p_{2j}=\det g^*_{2j}$ and q_{2j} classifies the orbit $D_{2j}\to D_{2j}/G_{k-1}$. Let $Y_{i,k}=(D(\det\gamma_i), S(\det\gamma_i))\times_{G_{k-1}}EG_{k-1}\to X_{i,k}=BO_i\times_{G_{k-1}}EG_{k-1}$ in general, then the above h induces the embedding $F\colon \boldsymbol{W}_*(G_k;\ Af,\ Free)\subsetneq \mathfrak{N}_*(G_k;\ Af,\ Free)\cong\sum_{i=0}^{*}\mathfrak{N}_{*-i}(X_{i,k})$ through the maps $\boldsymbol{W}_{*-2j+1}(Y_{2j,k})\subsetneq\mathfrak{N}_{*-2j+1}(Y_{2j,k})\cong\mathfrak{N}_{*-2j}(X_{2j,k})$ in the usual way.

Definition 1.7 (cf. [8; Sec. 4]). For each $n\geq 0$, let ξ_{2n} be the normal bundle of $\boldsymbol{RP}(2n)$ in $\boldsymbol{RP}(2n+1)$ equipped with the orientation on its total space induced from the standard orientation on $\boldsymbol{RP}(2n+1)$. Now ξ_{2n} admits an involution $R\colon \xi_{2n}\to\xi_{2n}$ obtained by reflection in the fiber. Since this involution changes the orientation of its total space, we put $\xi_{2n+1}=S^1\times_R\xi_{2n}$ with non-trivial Wall structure.

Then $\mathfrak{N}_*(\boldsymbol{Z}_2;\ All,\ Free)$ is the free $\mathfrak{N}_*$-module with basis $\xi_J=\xi_{j(1)}\times\cdots\times\xi_{j(n)}$, $J=(j(1),\cdots,j(n))$ with $j(1)\geq\cdots\geq j(n)\geq 0$ and $\boldsymbol{W}_*(\boldsymbol{Z}_2;\ All,\ Free)$ is the free $\boldsymbol{W}_*$-module with basis ξ_J, $J=(j(1),\cdots,j(2n))$ with even length by considering the embedding $\boldsymbol{W}_*(\boldsymbol{Z}_2;\ All,\ Free)\subsetneq\mathfrak{N}_*(\boldsymbol{Z}_2;\ All,\ Free)$ (cf. [8; Theorem 4.2]).

Using these results, we have

Proposition 1.8. *For $k\geq 2$, $\boldsymbol{W}_*(G_k;\ Af,\ Free)\cong Im(e)\oplus Im(s)\oplus\bar{L}_*$ as $\boldsymbol{W}_*$-modules, where e and s are two extensions from $\boldsymbol{W}_*(\boldsymbol{Z}_2;\ All,\ Free)$ and $\bar{L}_*$ is freely generated by the following;*

(i) *$Q(2p+1, 2K)=[S^{2p+1}\times D(\eta_{2K}), \bar{T}\times T]$ and*

(ii) *$Q(2p, 2K)=\bar{e}([X(2p)\times D(\eta_{2K}), A\times i])$, where $p\geq 1$ and $\eta_{2K}=\eta_{2k(1)}\times\cdots\times\eta_{2k(n)}\to \boldsymbol{CP}(2K)=\boldsymbol{CP}(2k(1))\times\cdots\times\boldsymbol{CP}(2k(n))$, the product of the canonical complex line bundles over the complex projective spaces $\boldsymbol{CP}(2k(j))$ for each $2K=(2k(1),\cdots 2k(n))$ with $k(1)\geq\cdots\geq k(n)\geq 0$. The group G_k acts on S^{2p+1} by $\bar{T}\in G_k/\boldsymbol{Z}_2\cong G_{k-1}$ and each fiber of η_{2K} by T naturally. Further the map $\bar{e}$ is the extension from $\boldsymbol{W}_*(\boldsymbol{Z}_4;\ Af,\ Free)$ defined by $\bar{e}([M, i])=[G_k\times_{\boldsymbol{Z}_4}M, T\times\mathrm{id}]$ where $i=\sqrt{-1}\in\boldsymbol{Z}_4$.*

Proof. We consider the embedding $F\colon \boldsymbol{W}_*(G_k;\ Af,\ Free)\subsetneq\mathfrak{N}_*(G_k;\ Af,\ Free)$ through the isomorphism h (cf. 1.6). We see that $\sum_j\mathfrak{N}_{*-2j}(X_{2j,k})\cong Im(e_2)\oplus Im(s_2)\oplus\bar{L}_{*,2}$ where $Im(e_2)$ (or $Im(s_2)$) is generated by $e_2(\xi_J)$ (or $s_2(\xi_J)$) for J with even length respectively, and $\bar{L}_{*,2}$ is given in [2; Prop. 3.12] as the sum of parts (iii) and (iv) there. We note that $F(Q(2p, 2K))$ differs from the element of type (iii) by $\mathfrak{N}_*$-decomposables by the relation in the proof of the proposition 1.4, while $Q(2p+1, 2K)$ is appeared in the part (iv). Thus $\bar{L}_{*,2}$ is generated by the classes $\{Q(q, 2K) \mid q \geq 2\}$ over $\mathfrak{N}_*$, and so is $\bar{L}_*$ over W_* by restricting the coefficient ring $\mathfrak{N}_*$ to W_*. Note that the classes $\{Q(q, 2K) \mid q \geq 2\}$ is linearly independent over W_* (cf. [2], [7; Theorem 3.3.5]). We see that $Q(0, 2K) = e(\eta_{2K}) \in Im(e)$ and $Q(1, 2K) = s(\eta_{2K}) \in Im(s)$. q.e.d.

Let d: $W_*(Z_2; All, Free) \rightarrow W_{*+1}(Z_2; All, Free)$ be the map mentioned in the definition 1.3. Note that $d^2 = d \circ d = 0$ and we have

LEMMA 1.9. *For the polynomial generators* $\{\xi_m\}$ *of* $W_*(Z_2; All, Free)$ *in the definition* 1.7, *the following properties hold:*

(i) $d\xi_{2n} = \xi_{2n+1}$, $d\xi_{2n+1} = 0$ *by definition and* d *acts on* $\xi_J = \xi_{j(1)} \cdots \xi_{j(2n)}$ *by the derivation in general,*

(ii) *the homology* H_* *of the complex* $(W_*(Z_2; All, Free), d)$ *is isomorphic to the free* W_**-algebra on the squares* $\xi_{2i}{}^2$ (cf. [1; Lemma 7]), *and*

(iii) *the sequence*

$$W_*(Z_2; All, Free) \xrightarrow{d} W_*(Z_2; All, Free) \xrightarrow{t} W_*(G_k; Af, Free)$$

is exact for $t = e$ *or* s.

The above properties hold in the unoriented case hence in our case via the embedding $W_*(Z_2; All, Free) \subsetneq \mathfrak{N}_*(Z_2; All, Free)$ as usual. In particular, the property (i) is obtained from [1; Theorem 3] and [14; Prop. 3.3]. Further the exactness of the sequences (d, t) in (iii) are proved in [2; Prop. 5.4] simultaneously. In other words, this means that $[S^1, T] \otimes_{G_k} -$: $Im(e) \cong Im(s)$ in $W_*(G_k; Af, Free)$.

LEMMA 1.10 (cf. [8; Theorem 6.2]). *As a set of generators of* W_*, *we can choose as follows;* $w_4 = [CP(2)]$ *and for* $n > 4$ $w_n = [RP(\xi_{I_n})]$; *the projective space bundle associated to* ξ_{I_n} *for some sequence* $I_n = (a, b, 0, 0)$ *with* $a + b + 3 = n$. *Let* E_* *be the ideal of those Wall manifolds* M *which has even Euler characteristic. Then* E_* *is generated by* $\{w_n \mid n > 4\}$, *and* $W_*/E_* \cong Z_2[[CP(2)]]$, *a polynomial ring generated by* $w_4 = [CP(2)]$.

Using this coefficient ring, we have

PROPOSITION 1.11. *The long exact sequence for the triple* $(Af, Free, \varnothing)$ *induces the following one*;

(i) $$0 \longrightarrow \mathscr{P} \xrightarrow{i_*} W_*(G_k; Af) \xrightarrow{j_*} W_*(G_k; Af, Free) \xrightarrow{\partial} Im\partial \longrightarrow 0,$$

where $\mathscr{P} \cong W_*/E_*\{\{[G_k, T], [S^1, T]\}\}$, *the free* W_*/E_**-module generated by the classes* $\{\cdots\}$, *and*

(ii) $Im\partial \cong W_*\{\{e(X(2n+2)), s(X(2n+2)) \mid n \geq 0\}\} \oplus E_*\{\{[G_k, T], [S^1, T]\}\}$.

COROLLARY 1.12. *The kernel of* F *is isomorphic to* $\mathscr{P}$ *for the forgetting* *homomorphism* F: $W_*(G_k; Af) \to \mathfrak{N}_*(G_k; Af)$.

PROOFS OF PROPOSITION 1.11 AND COROLLARY 1.12. When $G_1 = Z_2$, there is the exact sequence;

$$(1.13) \qquad 0 \longrightarrow W_*/E_*\{\{[Z_2, -1]\}\} \xrightarrow{i_*} W_*(Z_2; All) \xrightarrow{j_*} W_*(Z_2; All, Free) \xrightarrow{\partial} \tilde{W}_*(BZ_2) \oplus E_* \longrightarrow 0,$$

where $\tilde{W}_*(BZ_2)$ is the kernel of the augmentation map $\varepsilon_*: W_*(Z_2; Free) \to W_*$ (cf. [8; Corollary 7.5]), and this is freely generated by $X(2n+2)$ and $[S^{2n+1}, -1]$ for $n \geq 0$ by the proposition 1.4. Using the fact that $W_*(G_k; Af, Free) \cong Im(e) \oplus Im(s) \oplus \bar{L}_*$ as mentioned in the proposition 1.8, we have that $\partial(Im(t)) \cong W_*\{\{t(X(2n+2)) \mid n \geq 0\}\} \oplus E_*\{\{t(X(0))\}\}$ ($t = e$ or s) and $\partial(\bar{L}_*) = \{0\}$. The former is derived from the commutative diagram $(\partial | Im(t)) \circ t = t \circ \partial$ starting from $W_*(Z_2; All, Free)$ and the fact that $t([S^{2n+1}, -1]) = 0$ in $W_*(G_k; Free)$ (cf. [5; Prop. 1.7 (ii)]), while it is easy to see that $\partial(\bar{L}_*) = \{0\}$ in $W_*(G_k; Free)$ by the definition of $\bar{L}_*$. Thus the proposition 1.11 follows by Prop. 1.4 (ii). Next we consider the exact sequence;

$$(1.14) \qquad 0 \longrightarrow \mathfrak{N}_*(G_k; Af) \xrightarrow{j_*} \mathfrak{N}_*(G_k; Af, Free) \xrightarrow{\partial} \mathfrak{N}_{*-1}(G_k; Free) \longrightarrow 0,$$

which has a splitting homomorphism θ for ∂ defined by $\theta([M, A]) = [M \times_{Z_2} I, T \times \mathrm{id}]$ for each $[M, A] \in \mathfrak{N}_{*-1}(G_k; Free)$), ($I$; the unit interval) (cf. [2; Sect. 2]). Joining this and the exact sequence (i) in the above proposition by the forgetting map F, we have the corollary 1.12. q.e.d.

COROLLARY 1.15. *$[S^{4m+1}, T] = [CP(2)]^m [S^1, T] \neq 0$ and $[S^{4m+3}, T] = 0$ in $W_*(G_k; Af)$ for $m \geq 0$.*

PROOF. Using the relation (1.5), we see that $[S^{4m+3}, T] \in Im\partial$ in the above proposition, while $[S^{4m+1}, T] = [CP(2m)][S^1, T] = [CP(2)]^m [S^1, T] \neq 0$. Here we note that $[CP(2m)] = [CP(2)]^m$ (mod E_*) from the definition of E_* (cf. Lemma 1.10). q.e.d.

To study the module $W_*(G_k : Af)$, we define that

DEFINITION 1.16. $K_t = \{x \in W_*(G_k; Af) \mid j_*(x) \in Im(t)$ in $W_*(G_k; Af, Free)\}$ for each $t = e$ or s.

DEFINITION 1.17. For each $n \geq 0$, let $V(0, 2n+2)$ be an element in K_e such that $j_*(V(0, 2n+2)) = e(\xi_0^{2n+2})$ in $W_{2n+2}(G_k; Af, Free)$. Such $V(0, 2n+2)$ exists and non-zero since $\partial(e(\xi_0^{2n+2})) = e([S^{2n+1}, -1]) = 0$ in $W_{2n+1}(G_k; Free)$ and $[\xi_0^{2n+2}] \neq 0$ in H_*, the homology in the lemma 1.9 (ii). We define similarly an element $V(1, 2n+2)$ in K_s such that $j_*(V(1, 2n+2)) = s(\xi_0^{2n+2})$.

REMARK 1.18. For the above element $V(0, 2n+2)$, let A be an involution on

$\boldsymbol{RP}(2n+2)$ defined by $A([x_0:x_1:\cdots:x_{2n+2}])=[-x_0:x_1:\cdots:x_{2n+2}]$, and let $M=G_k\times \boldsymbol{RP}(2n+2)/-1\times A$ with the induced almost free G_k action $T\times\mathrm{id}$. Then $j_*([M, T\times\mathrm{id}])=e(\xi_0^{2n+2})+e(\lambda)$ in the exact sequence (1.14) where λ is the canonical line bundle over $\boldsymbol{RP}(2n+1)$. Since $e(\lambda)=0, j_*([M, T\times\mathrm{id}])=(j_*\circ F)(V(0, 2n+2))=e(\xi_0^{2n+2})$. Hence $F(V(0, 2n+2))=e_2([\boldsymbol{RP}(2n+2), A])$ in $\mathfrak{N}_*(G_k; Af)$.

Using these, we have

THEOREM 1.19. *For* $k\geq 2$, $\boldsymbol{W}_*(G_k; Af)\cong(K_e+K_s)\oplus L_*$ *as* $\boldsymbol{W}_*$*-modules, where*
(i) $K_e\cong(\boldsymbol{W}_*/E_*)\{\{[S^1, T], V(0, 2)\}\}\oplus(Q+Im(e))$,
(ii) $K_s\cong(\boldsymbol{W}_*/E_*)\{\{[G_k, T], V(1, 2)\}\}\oplus Im(s)$ *where e and s are two extensions from* $\boldsymbol{W}_*(\boldsymbol{Z}_2; All)$ *and Q is a* $\boldsymbol{Z}_2$ *vector space generated by the classes* $\{[CP(2)]^u w_n V(0, 2)\}$ *with* $u\geq 0$ *and* $n\equiv 3$ (*mod* 4).
Further $K_e\cap K_s\cong\mathscr{P}$ *by the definition of* K_t, *and*
(iii) L_* *is isomorphic to* $\bar{L}_*$ *and freely generated by the following* (iii-1) *and* (iii-2);
(iii-1) $V_{(k)}(2p+1, 2K)=D^{2p+2}\times S(\eta_{2K})\cup-(S^{2p+1}\times D(\eta_{2K}))$ *with an orientation-preserving action* $T_V=\bar{T}\times T\cup\bar{T}\times T$,
(iii-2) $V_{(2)}(2p, 2K)=S^1\times_R V_{(2)}(2p-1, 2K)$ *with the action* $\mathrm{id}\times T_V$ *where R is an orientation-reversing involution on* $V_{(2)}(2p-1, 2K)$ *obtained by the reflection in the first coordinate of* D^{2p} *if* $k=2$, *and* $V_{(k)}(2p, 2K)=\bar{e}(V_{(2)}(2p, 2K))$ *where* $\bar{e}$ *is the extension from* $G_2=\boldsymbol{Z}_4$ *actions if* $k\geq 3$.
In the above, $p\geq 1$ *and* $\{\eta_{2K}\}$ *are appeared in the proposition* 1.8.

We may omit the subscript number (k) of V after this, if no confusion can arise.

REMARK 1.20. When $p=0$, we note that $V(1, 2K)\in K_s$ since $j_*(V(1, 2K))=Q(1, 2K)=s(\eta_{2K})$, while we have an element $V(0, 2K)\in K_e$ such that $j_*(V(0, 2K))=Q(0, 2K)=e(\eta_{2K})$. In particular, we see that $V(\varepsilon, 2n+2)=V(\varepsilon, 2K)$ with $2K=(0, \cdots, 0)$; $(n+1)$-times of 0 for $\varepsilon=0$ or 1. Both $V(\varepsilon, 2K)$ are related by the map $[S^1, T]\otimes_{G_k}-$ as $V(1, 2K)=[S^1, T]\otimes_{G_k}V(0, 2K)$ (mod $\mathscr{P}$) in general by the proposition 1.11 (i). Hence $V(\varepsilon, 4m+2)$ $(m\geq 0)$ are uniquely determined in particular and $V(1, 4m+2)=[S^1, T]\otimes_{G_k}V(0, 4m+2)$. Further $V_{(k)}(0, 4m+2)=\bar{e}(V_{(k-1)}(0, 4m+2))$; the extension from G_{k-1} action.

PROOF OF THEOREM 1.19. The sequence (1.14) has a splitting map ρ for j_* defined by $\rho([M, T])=[\boldsymbol{RP}(v\oplus \boldsymbol{R}), T\times\mathrm{id}]$ for the normal bundle v in M over the fixed point set of $\boldsymbol{Z}_2$ in G_k (cf. [7; Lemma 4.2.4]). We note that $\theta\colon \mathfrak{N}_{*-1}(G_k; Free)\cong\mathfrak{N}_*(X_{1,k})$ and $\rho\colon \sum_{i\neq 1}\mathfrak{N}_*(X_{i,k})\cong\mathfrak{N}_*(G_k; Af)\cong Im(e_2)\oplus Im(s_2)\oplus L_{*,2}$ where e_2 and s_2 are the two extensions from $\mathfrak{N}_*(\boldsymbol{Z}_2; All)$ and $L_{*,2}=\rho(\bar{L}_{*,2})$ (cf. the proof of Proposition 1.8). Here we see that $j_*(V(q, 2K))=Q(q, 2K)$ by definition hence $\rho(Q(q, 2K))=V(q, 2K)$. Thus the above classes (iii-1) and (iii-2) generate $L_{*,2}$ freely over $\mathfrak{N}_*$ via the isomorphism $\rho|\bar{L}_{*,2}\colon \bar{L}_{*,2}\cong L_{*,2}$. Let L_* be a submodule in $\boldsymbol{W}_*(G_k; Af)$ generated by these classes. Then the map $\rho_0\colon \bar{L}_*\to L_*$, $\rho_0(Q(q, 2K))=\mathrm{V}(q, 2K)$, is an isomorphism $\rho_0\colon \bar{L}_*\cong L_*$ with

$\rho_0^{-1}=j_*|_{L_*}$. Now for each $x\in W_*(G_k; Af)$, put $j_*(x)=y_1+y_2$ ($y_1\in Im(e)\oplus Im(s)$, $y_2\in \bar{L}_*$) in the proposition 1.11 (i), then $j_*(x-\rho_0(y_2))=y_1\in Im(e)\oplus Im(s)$ since $j_*(\rho_0(y_2))=y_2$ as mentioned above, and $x-\rho_0(y_2)\in K_{e+s}=j_*^{-1}(Im(e)\oplus Im(s))$. For any element $z\in K_{e+s}$, put $j_*(z)=e(\xi)+s(\eta)$ for some ξ and $\eta\in W_*(Z_2; All, Free)$. Then $e(\partial(\xi))=s(\partial(\eta))$ in $W_*(G_k; Free)$ since $\partial(j_*(z))=0$. Note that $Im(e)\cap Im(s)=\{0\}$ here by the proposition 1.4 (i) (ii) and [5; Prop. 1.7 (ii)], so $e(\partial(\xi))=s(\partial(\eta))=0$ in $W_*(G_k; Free)$ and there is an element $z_t\in K_t$ for $t=e$ or s such that $j_*(z_e)=e(\xi)$ or $j_*(z_s)=s(\eta)$ by the definition of K_t. Therefore, $z=z_e+z_s$ (mod $\mathscr{P}$) and $z\in K_e+K_s$ since $\mathscr{P}\subset K_e+K_s$, so we have that $K_{e+s}=K_e+K_s$. Thus $W_*(G_k; Af)\cong(K_e+K_s)\oplus L_*$ in such a way that the sequence;

$$(1.21)\qquad 0\longrightarrow \mathscr{P}\xrightarrow{i_*} K_e+K_s\xrightarrow{j_*} Im(e)\oplus Im(s)\xrightarrow{\partial} Im\partial\longrightarrow 0$$

is exact and $j_*: L_*\cong\bar{L}_*$ in the proposition 1.11. To complete the theorem, we prove the following lemmas for the parts K_t. These imply the result. q.e.d.

LEMMA 1.22. *We have that*
$K_e\cong W_*\{\{V(0,2)\}\}+Im(e)+\mathscr{P}$, *and*
$K_s\cong W_*\{\{V(1,2)\}\}+Im(s)+\mathscr{P}$.

LEMMA 1.23. *For the classes* $\{V(0,2n+2)\mid n\geq 0\}$ *of* K_e, *we have that*
(i) $V(0,4m+4)\in Im(e)$,
(ii) $V(0,4m+2)-[CP(2m)]V(0,2)\in Im(e)$,
(iii) $[CP(2m)]V(0,2)\notin Im(e)$ *hence* $V(0,4m+2)\notin Im(e)$ *in general, and*
(iv) *If* $x\in E_*$ *with* $x\neq[CP(2)]^u w_n$ ($u\geq 0$, $n\equiv 3 (\text{mod } 4)$), *then* $x\cdot V(0,4m+2)\in Im(e)$. *In particular,* $v\cdot V(0,4m+2)\in Im(e)$ *for* $v=[CP(2u)]-[CP(2)]^u\in E_*$.

The same results hold for the classes $\{V(1,2n+2)\}$ *of* K_s. *In this case, the part* (iv) *holds for each* $x\in E_*$.

LEMMA 1.24. *We have that*
(i) $[CP(2)]^u[G_k, T]\notin Im(s)$, (ii) $[CP(2)]^u[S^1, T]\notin Im(e)$, *and*
(iii) $[CP(2)]^u V(\varepsilon, 2)\notin Im(t)$ *for* $\varepsilon=0$ *or* 1, *and* $t=e$ *or* s.

To prove the above lemmas, we use the following

LEMMA 1.25. *For the map* $\partial: W_*(Z_2; All, Free)\to\tilde{W}_*(BZ_2)\oplus E_*\{\{[Z_2, -1]\}\}$ *in* (1.13), *we have that* $\partial(Ker(e))\cong W_*\{\{dX(2n+2)\mid n\geq 0\}\}\oplus E_*\{\{[S^1, -1]\}\}$ *in* $Ker(e)\cong W_*\{\{dX(2n+2)\mid n\geq -1\}\}$ (cf. Prop. 1.4(ii)). *The same result holds for the map s.*

PROOF. Since $Ker(e)=Im(d)$ at $W_*(Z_2; All, Free)$ (cf. Lemma 1.9 (iii)), $\partial(Ker(e))=d(Im(\partial))=d(\tilde{W}_*(BZ_2)\oplus E_*\{\{[Z_2, -1]\}\})$. This implies the result since $d(\tilde{W}_*(BZ_2))$ is freely generated by $dX(2n+2)$ ($n\geq 0$). q.e.d.

PROOF OF LEMMA 1.22. Take each $x\in K_e$ and put $j_*(x)=e(\xi_x)$ for some $\xi_x\in W_*(Z_2; All, Free)$ (cf. Definition 1.16). Then $\partial(\xi_x)$ belongs to the kernel of $e: \tilde{W}_*(BZ_2)\oplus E_*\to W_*(G_k; Free)$, hence $\partial(\xi_x)=\sum_{n\geq -1} M_{2n+2}d(X(2n+2))$ ($M_{2n+2}\in W_*$)

and $\partial(\xi)=\sum_{n\geq 0} M_{2n+2}d(X(2n+2))$ for some $\xi \in Ker(e)$ in $W_*(Z_2; All, Free)$ by the above lemma. This implies that there is an element $y \in W_*(Z_2; All)$ such that $j_*(y)=\xi_x-(\xi+M_0\xi_0{}^2)$, and $j_*(x-M_0V(0, 2)-e(y))=0$ in $W_*(G_k; Af, Free)$. Hence the result for K_e follows by the proposition 1.11 (i). For an element $x \in K_s$, the proof is similar by using the lemma 1.9 (iii), so we omit it here. q.e.d.

Proof of Lemma 1.23. First we prove the part (ii). See the relation (1.5) in $W_*(Z_2; All)$, then we note that an element $[S^{4m+1}, -1]-[CP(2m)][S^1, -1]$ has a counter-image $\xi \in Ker(e)$ in $W_*(Z_2; All, Free)$ by the lemma 1.25. Let y be an element in $W_*(Z_2; All)$ such that $j_*(y)=\xi_0{}^{4m+2}-(\xi+[CP(2m)]\xi_0{}^2)$, then $(j_* \circ e)(y)=j_*(V(0, 4m+2)-[CP(2m)]V(0, 2))$ in $W_*(G_k; Af, Free)$ by definition. This implies that $e(y)-(V(0, 4m+2)-[CP(2m)]V(0, 2))=0$ in $\mathscr{P}$ by the dimensional condition, and the result holds. The proof of the part (i) is similar to this, so we omit it here. For part (iii), we suppose that $[CP(2m)]V(0, 2)=e(y)$ for some $y \in W_*(Z_2; All)$, then $(e \circ j_*)(y)=(j_* \circ e)(y)=[CP(2m)]e(\xi_0{}^2)$ in $W_*(G_k; Af, Free)$. This means that $j_*(y)-[CP(2m)]\xi_0{}^2 \in Ker(e)$ in $W_*(Z_2; All, Free)$ and $\partial(j_*(y)-[CP(2m)]\xi_0{}^2)=-[CP(2m)][S^1, -1] \in \partial(Ker(e))$ in $W_*(Z_2; Free)$. This is contrary to the lemma 1.25 since $[CP(2m)] \notin E_*$. Thus the part (iii) follows. Finally we prove the part (iv). From the lemma 1.25 and the relation (1.5) again, we see that $x\cdot[S^{4m+1}, -1]$ has a counter-image $\xi \in Ker(e)$ in $W_*(Z_2; All, Free)$ for each $x \in E_*$. Thus there is an element y in $W_*(Z_2; All)$ such that $e(y)-xV(0, 4m+2) \in \mathscr{P}$. When $\dim x \equiv 3 \pmod 4$, this difference may be $a=[CP(2)]^{\alpha}[S^1, T]$ in general. Then $xV(0, 4m+2) \notin Im(e)$ by the next lemma 1.24. If $x'=w_n$ for $n \not\equiv 3 \pmod 4$ and if $x'=w_{n_1}w_{n_2}$ for $n_i \equiv 3 \pmod 4$, then $x'V(0, 4m+2) \in Im(e)$. Thus if x belongs to the ideal in W_* generated by these elements x', we also have $xV(0, 4m+2) \in Im(e)$ in general. Hence we admit the case that $xV(0, 4m+2)=e(y)+a$ when $x=[CP(2)]^u w_n$ with $u \geq 0$ and $n \equiv 3 \pmod 4$. The corresponding relations among the classes $\{V(1, 2n+2)\}$ and $Im(s)$ are proved by using the exactness of (d, s) in the lemma 1.9 (iii). For the part (iv), we take an element y in $W_*(Z_2; All)$ such that $s(y)-xV(1, 4m+2) \in \mathscr{P}$ for each $x \in E_*$ as above, and this difference may be $b=\varepsilon[CP(2)]^{\alpha}[G_k, T]$ ($\varepsilon=0$ or 1) when $\dim x \equiv 1 \pmod 4$. However this implies that $0=[S^1, T]\otimes_{G_k} b=\varepsilon a$ in $W_*(G_k; Af)$ by the remark 1.20 (cf. [5: Theorem 2.22 (i)]) and $\varepsilon=0$. Hence $xV(1, 4m+2) \in Im(s)$ for each $x \in E_*$. q.e.d.

Proof of Lemma 1.24. The proof of (i) is easy. For part (ii), we suppose that $\varepsilon[CP(2)]^u[S^1, T]=e(y)$ for some $y \in W_*(Z_2; All)$. Since $(e \circ j_*)(y)=(j_* \circ e)(y)=0$ in $W_*(G_k; Af, Free)$, $j_*(y)=d(\xi)$ for some $\xi \in W_*(Z_2; All, Free)$ by the lemma 1.9 (iii). We have that $\partial(\xi)=\sum_{n\geq -1} M_{2n+2}d(X(2n+2))$ since $\partial(\xi) \in Ker(d)$ in $\tilde{W}_*(BZ_2) \oplus E_*\{\{[Z_2, -1]\}\}$. By the lemma 1.25, there is an element ξ' in $Ker(e)=Im(d)$ such that $\partial(\xi')=\sum_{n\geq 0} M_{2n+2}d(X(2n+2))$ and an element $z \in W_*(Z_2; All)$ such that $j_*(z)=\xi-(\xi'+M_0\xi_0{}^2)$ as usual. Then $(j_* \circ d)(z)=d(\xi)=j_*(y)$ implies that $y=d(z)$ by the dimensional condition (cf. (1.13)). Since $e \circ d=0$, we see that $e(y)=0$ and $\varepsilon=0$. This implies the result. The proof of part (iii) is proved by using the lemma 1.25. q.e.d.

REMARK 1.26. In the lemma 1.25, a conter-image $\xi \in Ker(e)$ of $dX(2n+2)$ under the map ∂ is constructed as follows. Since $X(2n+2)=[S^1 \times_R S^{2n+1}, \mathrm{id} \times -1]$ $(n \geq 0)$ (cf. Prop. 1.4), we see that $\partial[S^1 \times_R D^{2n+2}, \mathrm{id} \times -1]=\partial(\xi_1 \xi_0{}^{2n+1})=X(2n+2)$ by considering its fixed point data. Thus $\xi=d(\xi_1\xi_0{}^{2n+1})=\xi_1{}^2\xi_0{}^{2n} \in Ker(e)$ maps to $dX(2n+2)$ by ∂. We have another ξ as follows. Let $y_{n+1}=[S(\xi_n\xi_0)]$ $(n \geq 0)$ be a basis of $\tilde{W}_*(BZ_2)$ in [8; Lemma 7.1]. We note that $\Delta(y_{2n+2})=X(2n+1)=\Delta(X(2n+2))$ in $\mathfrak{N}_*(Z_2; Free)$ by the definition of ξ_{2n+1} and [4; Theorem 2.5] where Δ is the Smith homomorphism. Since $\varepsilon_* X(2n+2)=0$ too in $\mathfrak{N}_*$ for the augmentation map, this implies that $y_{2n+2}=X(2n+2)$ in $\mathfrak{N}_*(Z_2; Free)$ hence in $W_*(Z_2; Free)$. Thus we may take that $\xi=\xi_{2n+1}\xi_1 \in Ker(e)$ in $W_*(Z_2; All)$. Further, to find a counter-image ξ_x for each $x \in E_*\{\{[S^1, -1]\}\}$, it is sufficient to consider the case that $x=w_n[S^1, -1]$ $(n>4)$ where $w_n=[RP(\xi_{I_n})]$ for a suitable sequence I_n (cf. Definition 1.10). Put $\partial(\xi_{I_n})=\sum_{n \geq 0} M_{2n+2}X(2n+2)+\sum_{n \geq 0} M_{2n+1}[S^{2n+1}, -1]+w_n[Z_2, -1]$ in $W_*(Z_2; All)$ formally, then the element $d(\xi_{I_n}) \in Ker(e)$ maps to $\sum_{n \geq 0} M_{2n+2}d(X(2n+2))+x$ by ∂. Hence the element $\xi_x= d(\xi_{I_n})-\sum_{n \geq 0} M_{2n+2}\xi_1{}^2\xi_0{}^{2n} \in Ker(e)$ is a desired counter-image of x for example.

REMARK. In the theorem 1.19, the part Q may be really contained in $Im(e)$. However, as far as an application to the oriented theory in the next section is concerned, this is not at all serious.

2. Some applications.

Let $\Omega_*(G_k; Af)=\sum_{n \geq 0} \Omega_n(G_k; Af)$ be the oriented bordism group of all orientation-preserving almost free G_k actions. We note that a torsion element in $\Omega_*(G_k; Af)$ is of order 2^i for some $1 \leq i \leq k$, and a torsion free element comes from that of $\Omega_*(Z_2; All)$ essentially by the extension map $e: \Omega_*(Z_2; All) \otimes Z[1/2] \cong \Omega_*(G_k; Af) \otimes Z[1/2]$, where $Z[1/2]$ is the subring of the rationals, generated by Z and $1/2$ (cf. [10; Prop. 4.2 and 2.2]). Now let $\beta: W_*(G_k; Af) \to \Omega_{*-1}(G_k; Af)$ be the Bockstein homomorphism which sends $[M, T] \in W_n(G_k; Af)$ into $[N, T|_N] \in \Omega_{n-1}(G_k; Af)$, where N is the invariant submanifold of M dual to $\det \tau_M$ (cf. [6; Sect. 6] for example). For a typical type $x=[S^1 \times_R M, \mathrm{id} \times T]$ in the example 1.2, we see that $\beta(x)=[M, T]$ by definition. Then $\mathscr{T}=Im(\beta)$ is the subgroup of all elements of order 2 in $\Omega_*(G_k; Af)$, and there is a universal coefficient sequence:

$$(2.1) \qquad 0 \longrightarrow \Omega_*(G_k; Af) \otimes Z_2 \longrightarrow W_*(G_k; Af) \xrightarrow{\beta} \mathscr{T} \longrightarrow 0$$

induced from the Wall exact sequence.

DEFINITION 2.2. Let Δ be the set of all unordered sequence consisting of distinct even integers which are greater than 4 and not a power of 2. Then the classes $\{w_I=w_{2i(1)} \cdots w_{2i(t)} \mid I=(2i(1), \cdots, 2i(t)) \in \Delta\}$ is a base of the free $\Omega_* \otimes Z_2$ module of W_* (cf. [8; Sect. 11] for example).

THEOREM 2.3. *$\mathscr{T}$ is generated by the following* (i) *and* (ii):

(i) $\mathscr{T}_1=\beta(K_e+K_s)$ *is the sum of* $Z_2[2^{k-2}[CP(2)]^m[S^1, T] \mid m\geq 0]$ *as a* Z_2 *vector space,* $e(\mathrm{Tor}\,\Omega_*(Z_2; All))$ *and* $s(\mathrm{Tor}\,\Omega_*(Z_2; All))$ *(e, s: the two extensions from the torsion part of* $\Omega_*(Z_2; All)$*), and*

(ii) $\mathscr{L}_*=\beta(L_*)\cong(\Omega_*\otimes Z_2)\{\{\beta(w_I)V(2p+1, 2K), \beta(w_I V(2p, 2K)) \mid p\geq 1, 2K,$ *and* $I\in\Delta\}\}$ *as a free* $\Omega_*\otimes Z_2$ *module where the class* $\{p, 2K\}$ *is appeared in the proposition* 1.8. *Further,* $\mathscr{T}_1\cap\mathscr{L}_*=\{0\}$ *except that* $k=2$.

REMARK 2.4. In the above, an oriented manifold representing the class $\beta(w_I V(2p, 2K))$ is constructed by the method as mentioned in [8; Lemma 15.2].

First we prove the following lemmas.

LEMMA 2.5. *In general,* $\beta(V(0, 2n+2))=2^{k-2}[S^{2n+1}, T]$ *and* $\beta(V(1, 2n+2))=0$ *in* $\Omega_*(G_k; Af)$.

PROOF. We have that $j_*(V(0, 2n+2))=e(\xi_0{}^{2n+2})=[G_k\times_{Z_2}D^{2n+2}, T\times\mathrm{id}]$ in $W_{2n+2}(G_k; Af, Free)$ (cf. Definition 1.17). Since the composition $e\circ r=2\times\mathrm{id}: \Omega_*(G_k; Free)\to\Omega_*(G_{k-1}; Free)\to\Omega_*(G_k; Free)$ (r: the restriction) (cf. [10: Prop. 4.2]), we see that $\partial[G_k\times_{Z_2}D^{2n+2}, T\times\mathrm{id}]=2^{k-1}[S^{2n+1}, T]$ in $\Omega_*(G_k; Free)$ by induction. Hence we have an oriented manifold V (with an orientation-preserving almost free action T_V) whose boundary ∂V has 2^{k-1} copies of (S^{2n+1}, T) and Z_2 fixed point data is $e(\xi_0{}^{2n+2})$. By pasting these boundaries two by two, we obtain $V(0, 2n+2)$ with the induced action T and $\beta(V(0, 2n+2))=2^{k-2}[S^{2n+1}, T]$. Further $\beta(V(1, 2n+2))=2^{k-2}[S^1, T]\otimes_{G_k}[S^1, T]=0$ since $V(1, 2n+2)=[S^1, T]\otimes_{G_k}V(0, 2n+2)$ (mod $\mathscr{P}$) and $\beta(\mathscr{P})=\{0\}$ (cf. Remark 1.20). q.e.d.

LEMMA 2.6. *On the basis* $\{V(q, 2K)\}$ *of* L_*, *we have that*

$$\beta(V(2p, 2K))=2^{k-2}V(2p-1, 2K) \quad \text{and} \quad \beta(V(2p+1, 2K))=0 \quad (p\geq 1).$$

PROOF. From the theorem 1.19 (iii), we note that $\beta(V_{(k)}(2p, 2K))=\bar{e}(V_{(2)}(2p-1, 2K))=(\bar{e}\circ r)(V_{(k)}(2p-1, 2K))=2^{k-2}V_{(k)}(2p-1, 2K)$ by using the composition $e\circ r=2\times\mathrm{id}$: $\Omega_*(G_k; Af)\to\Omega_*(G_{k-1}: Af)\to\Omega_*(G_k; Af)$ as above. q.e.d.

PROOF OF THEOREM 2.3. Note that β maps the part $Im(t)$ (in K_t) onto $t(\mathrm{Tor}\,\Omega_*(Z_2; All))$ for $t=e$ or s since all torsion of $\Omega_*(Z_2; All)$ has order 2 (cf. [9; Theorem 3.4]). We note that $w_n\in\mathrm{Tor}\,\Omega_n$ for $n\equiv 3 \pmod 4$ hence $\beta(w_n V(0, 2))=2^{k-2}w_n[S^1, T]$ by the above lemma 2.5, which is zero if $k\geq 3$. When $k=2$, the fact that $w_n[Z_2, -1]=0$ in $\Omega_n(Z_2; All)$ implies that $w_n[S^1, T]=0$ in $\Omega_n(G_k; Af)$ (cf. [9; Theorem 3.1]). Thus we have that $\beta(Q)=\{0\}$ after all, and $\mathscr{T}_1$ consists of three parts as mentioned in (i). Note that

$$\beta(w_I V(2p+1, 2K))=\beta(w_I)V(2p+1, 2K),$$
$$\beta(w_I V(2p, 2K))=\beta(w_I)V(2p, 2K)+w_I\cdot 2^{k-2}V(2p-1, 2K) \qquad (2.7)$$

in $W_*(G_k; Af)$ from the lemma 2.6. Thus the linearly independence of the classes $\{V(q, 2K) \mid q \geq 2\}$ over W_* implies the result (ii) in particular by the definition 2.2. Next we study the intersection part $\mathscr{T}_1 \cap \mathscr{L}_*$. In case of $k \geq 3$, we note that $\mathscr{L}_* \subsetneq L_*$ via the embedding of the coefficient ring $\Omega_* \otimes Z_2 \subsetneq W_*$ since $\beta(w_I V(q, 2K)) = \beta(w_I)V(q, 2K)$ for all $q \geq 2$ by the formula (2.7), while $\mathscr{T}_1$ lies in $Im(e) + Im(s)$ in $W_*(G_k; Af)$. Thus each $x \in \mathscr{T}_1 \cap \mathscr{L}_*$ lies in $(Im(e) + Im(s)) \cap L_* = \{0\}$ (cf. Theorem 1.19), hence $x = 0$ in $\Omega_*(G_k; Af)$ by the embedding $\mathscr{L}_* \subsetneq L_*$. This implies that $\mathscr{T}_1 \cap \mathscr{L}_* = \{0\}$ and the result follows. q.e.d.

REMARK 2.8. In case of $k = 2$, define by L'_*, the submodule of $W_*(Z_4; Af)$ generated by all element of L_* and $\{V(1, 2K) \mid 2K\}$ for convenience. Including the latter classes, we see that $\{V(q, 2K) \mid q \geq 1, 2K\}$ are also linearly independent over W_* by using the fact that $j_*(V(1, 2K)) = s(\eta_{2K}) = s(\xi_{2K}{}^2)$ and the lemma 1.9 (ii) and (iii). Thus we have that $K_e \cap L'_* = \{0\}$ and $K_s \cap L'_* \cong W_*\{\{V(1, 2K) \mid 2K\}\}$. Then we note that $\mathscr{T}_1 \cap \mathscr{L}_*$ lies in $(K_e + K_s) \cap L'_* \cong W_*\{\{V(1, 2K) \mid 2K\}\}$ by the formula (2.7). Now, for each torsion element $x = \beta(\bar{x})$ in Ω_*, $xV(1, 2K) = \beta(\bar{x}V(1, 2K)) = \beta(xV(2, 2K))$ which does not vanish in $\mathscr{T}_1 \cap \mathscr{L}_*$. However we see that $[CP(2u)]V(1, 4m+2) \in \mathscr{L}_*$ does not belong to $\mathscr{T}_1$ for example by using the lemma 1.25. Hence $\mathscr{T}_1 \cap \mathscr{L}_*$ is properly contained in $(\Omega_* \otimes Z_2)\{\{V(1, 2K) \mid 2K\}\}$.

As an application, we study the image $Im(I_*)$ of the forgetful homomorphism $I_*: \Omega_*(Z_4 : Free) \to \Omega_*(Z_4; Af)$.

THEOREM 2.9. *$Im(I_*) \cong (\Omega_*/\mathrm{Tor}\,\Omega_*)\{\{[Z_4, i]\}\} \oplus C_*\{[S^1, i]\}$, where $C_*\{[S^1, i]\}$ is a $C_* \cong Z_2[CP(2m) \mid m \geq 1]$ module generated by $[S^1, i]$.*

First we prove the following

PROPOSITION 2.10. *As the elements in $\mathscr{T}$,*

(i) $[S^{4m+1}, i] = [CP(2m)][S^1, i] \neq 0$ *and* (ii) $[S^{4m+3}, i] = 0$

in $\Omega_*(Z_4; Af)$ for $m \geq 0$.

PROOF. We note that $[S^{4m+1}, i] \neq 0$ in $\Omega_*(Z_4; Af)$ by the corollary 1.15. Now, see the proof of the lemma 1.23 (ii). There an element $y \in W_*(Z_2; All)$ is chosen such that $j_*(y) = \xi_0{}^{4m+2} - (\sum_{\alpha=0}^{m-1} [CP(2\alpha)]\xi_1{}^2\xi_0{}^{4(m-\alpha)-2} + [CP(2m)]\xi_0{}^2)$ by using a counter-image ξ of $d(X(2n+2))$ mentioned in the remark 1.26. Thus $j_*(\beta(y)) = 0$ in $W_*(Z_2; All, Free)$ in particular (cf. [8; Theorem 4.2]). Since $\beta(y) \in Q_*^{(2)}$ which is the submodule of $W_*(Z_2; All)$ embedded in $W_*(Z_2; All, Free)$ (cf. [8; Sec. 8]), we have that $\beta(y) = 0$ in $W_*(Z_2; All)$ hence in $\Omega_*(Z_2; All)$. Thus $\beta(V(0, 4m+2) - [CP(2m)]V(0, 2)) = [S^{4m+1}, i] - [CP(2m)][S^1, i] = e(\beta(y)) = 0$ in $\Omega_{4m+1}(Z_4; Af)$ by the lemma 2.5. This implies the result (i). We see that $[S^{4m+3}, i] = 0$ in $\Omega_*(Z_4; Af)$ as follows. Consider an orientation-preserving Z_4 action T on $CP(2m+2)$ given by

$$T([z_0 : z_1 : z_2 : \cdots : z_{2m+1} : z_{2m+2}]) = [\bar{z}_0 : -\bar{z}_2 : \bar{z}_1 : \cdots : -\bar{z}_{2m+2} : \bar{z}_{2m+1}]$$

(cf. [11; Sec. 3]).

We note that the only stationary point of T is $[1:0:\cdots:0]$. Thus, deleting neighborhood of this point gives a manifold V^{4m+4} (with suitable orientation) whose boundary is equivariantly diffeomorphic to $[S^{4m+3}, i]$. q.e.d.

PROOF OF THEOREM 2.9. Consider that $\Omega_*(Z_4; Free) \cong \Omega_*\{\{[Z_4, i]\}\} \oplus \tilde{\Omega}_*(Z_4; Free)$ as usual. Then $\tilde{\Omega}_*(Z_4; Free) \cong \mathfrak{H}_2 \oplus \mathfrak{G}_2$ where $\mathfrak{H}_2$ is generated by the classes $\{[S^{2n+1}, i] \mid n \geq 0\}$ of order 4 and $\mathfrak{G}_2$ is generated by the classes $\{e(E^{2n+1}W(\omega))\}$, the extension of suitable elements $E^{2n+1}W(\omega) \in \tilde{\Omega}_*(Z_2; Free)$ of order 2 (cf. [5; Theorem 2.18]). Since $e\colon \Omega_*(Z_2; All) \otimes Z[1/2] \cong \Omega_*(Z_4; Af) \otimes Z[1/2]$, we see that the kernel of I_* from $\Omega_*\{\{[Z_4, i]\}\}$ is $(\mathrm{Tor}\,\Omega_*)\{\{[Z_4, i]\}\}$ in particular. On the other hand $I_*(\mathfrak{G}_2) = \{0\}$ since $\tilde{\Omega}_*(Z_2; Free)$ vanishes in $\Omega_*(Z_2; All)$ (cf. [9; Theorem 3.1]). The image $I(\mathfrak{H}_2)$ is shown by the above proposition and the fact that $C_* = \Omega_*/(\mathrm{Tor}\,\Omega_* + 2\Omega_*)$ is isomorphic to the Z_2 polynomial algebra on the classes $\{[CP(2m)]\}$ (cf. [12; p. 183]). q.e.d.

REMARK 2.11. We need to see the element $x_k(m) = 2^{k-2}[CP(2)]^m[S^1, T]$ in $\mathscr{T}_1$ is really non-zero. The fact $x_2(m) \neq 0$ is mentioned in the corollary 1.15. Next we prove that $x_3(m) \neq 0$ by using the theory of Wall manifolds only. Suppose that $x_3(m) = 0$ in $\Omega_{4m+1}(Z_8; Af)$ and let v be an element in $W_{4m+2}(Z_8; Af)$ such that $\beta(v) = [CP(2)]^m[S^1, T]$. We write that $v = \varepsilon[CP(2)]^m V_{(3)}(0, 2) + e(x) + s(y) + l$ for some $x, y \in W_*(Z_2; All)$ and l; a sum of elements $V_{(3)}(q, 2K)$ $(q \geq 2)$ in L_* (cf. Theorem 1.19). We consider the natural restriction $r\colon W_*(Z_8; Af) \to W_*(Z_4; Af)$. Then $r(v) = s(y) + r(l)$ where $r(l)$ is a sum of elements $V_{(2)}(2p+1, 2K)$ $(p \geq 1)$ since $r(V_{(3)}(0, 2)) = (r \circ \bar{e})(V_{(2)}(0, 2)) = 2V_{(2)}(0, 2) = 0$, $r(V_{(3)}(2p, 2K)) = 0$ for $p \geq 1$ by the same reason (cf. Remark 1.20 and Theorem 1.19 (iii-2)) and $(r \circ e)(x) = 2x = 0$. Hence we have that $(\beta \circ r)(v) = x_2(m) = s(y') + l'$ in $\Omega_{4m+1}(Z_4; Af)$ where $y' = \beta(y) \in \mathrm{Tor}\,\Omega_*(Z_2; All)$ and $l' = (\beta \circ r)(l)$. Consider again this in $W_*(Z_4; Af)$, then l' is a sum of elements $V_{(2)}(2p+1, 2K)$ $(p \geq 1)$ over $\Omega_* \otimes Z_2$. In particular, we see that $x_2(m) - s(y') = l' \in (K_e + K_s) \cap L_* = \{0\}$ and $x_2(m) = s(y')$ in $W_{4m+1}(Z_4; Af)$. Let $j_*(y') = \xi_{y'}$ in $W_{4m}(Z_2; All, Free)$, then $s(\xi_{y'}) = (s \circ j_*)(y') = j_*(x_2(m)) = 0$ in $W_{4m+1}(Z_4; Af, Free)$. Hence $(j_* \circ e)(y') = e(\xi_{y'}) = 0$ by the lemma 1.9 (iii), and $e(y') = \varepsilon[CP(2)]^m[Z_4, i]$ $(\varepsilon = 0$ or $1)$ by the proposition 1.11 (i). Using the following lemma 2.12 and Lemma 8.2 in [8], $0 \equiv \chi(\overline{y'}) = \varepsilon$ where $\overline{y'}$ is the orbit space of y' under Z_2. Hence $x_2(m) = s(y') = 0$ in $W_{4m+1}(Z_4; Af)$ which is contrary to $x_2(m) \neq 0$, and we have that $x_3(m) \neq 0$ in $\Omega_{4m+1}(Z_8; Af)$. In other words, we see that $[S^{4m+1}, T]$ is of order 4 in $\Omega_{4m+1}(Z_8; Af)$ by the same way as the proof of the proposition 2.10. However, this method does not apply to the case $k \geq 4$ since $(\beta \circ r)(v)$ vanishes in $W_*(G_{k-1}; Af)$ from the beginning, while it is easy to see that $x_k(0) \neq 0$ in general.

LEMMA 2.12. *If X is a non-oriented manifold with almost free G_k action T such that the Z_2 fixed point set F of T has even codimension, then the Euler characteristic modulo 2, χ of the orbit space $\partial X/T$ is zero.*

The proof is entirely analogous to that in [8; Lemma 8.1], so we omit it here.

References

[1] R. P. Beem, The action of free G-bordism on G-bordism, Duke Math. J., **42** (1975), 297–305.

[2] ———, On the bordism of almost free Z_{2^k} actions, Trans. Amer. Math. Soc., **225** (1977), 83–105.

[3] P. E. Conner, *Lectures on the Action of a Finite Group*, Lecture Notes in Math., **73** (1968), Springer.

[4] T. Hara, On Wall manifolds with (ε)-free involutions, Publ. R. I. M. S. Kyoto Univ., **22** (1986), 571–582.

[5] Y. Katsube, Principal oriented bordism algebra $\Omega_*(Z_{2^k})$, Hiroshima Math. J., **4** (1974), 265–277.

[6] K. Komiya, Oriented bordism and involutions, Osaka J. Math., **9** (1972), 165–181.

[7] C. Kosniowski, *Actions of Finite Abelian Groups*, Pitman (1978).

[8] C. Kosniowski and E. Ossa, The structure of the bordism module of oriented involutions, Proc. London Math. Soc., **44** (1982), 267–290.

[9] H. L. Rosenzweig, Bordism of involutions on manifolds, Illinois J. Math., **16** (1972), 1–10.

[10] R. J. Rowlett, The fixed-point construction in equivariant bordism, Trans. Amer. Math. Soc., **246** (1978), 473–481.

[11] R. E. Stong, Stationary point free group actions, Proc. Amer. Math. Soc., **18** (1967), 1089–1092.

[12] ———, *Notes on Cobordism Theory*, Princeton Univ. Press (1968).

[13] ———, Wall manifolds, Trans. Amer. Math. Soc., **251** (1979), 287–298.

[14] J. C. Su, A note on the bordism algebra of involutions, Michigan Math. J., **12** (1965), 25–31.

[15] F. Uchida, Bordism algebra of involutions, Proc. Japan Acad., **46** (1970), 615–619.

[16] C. T. C. Wall, Determination of the cobordism ring, Ann. of Math., **72** (1960), 292–311.

[17] E. R. Wheeler, The oriented bordism of Z_{2^k} actions, Trans. Amer. Math. Soc., **199** (1974), 113–121.

Present Address:
Department of Mathematics, Faculty of Science and Technology
Science University of Tokyo
Noda, Chiba 278, Japan