

Classifying Hypersurfaces in the Lorentz-Minkowski Space with a Characteristic Eigenvector

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(Communicated by M. Sakai)

1. Introduction.

In a famous paper, [3], Cheng and Yau solved the Bernstein problem in the Lorentz-Minkowski space L^{n+1} showing that the only entire maximal hypersurfaces are hyperplanes. Maximal and constant mean curvature (CMC) hypersurfaces play a chief role in relativity theory as it is pointed out in a series of papers by Choquet, Fischer and Marsden, [4], Stumbles, [15], and Marsden and Tipler, [13]. CMC hypersurfaces are often closely related to either an eigenvalue problem or a differential equation stemming from the Laplacian. Perhaps the most remarkable case is that concerning to vanishing constant mean curvature. Let x denote an isometric immersion of a hypersurface M in the Lorentz-Minkowski space L^{n+1} and let H be the mean curvature vector field. In a recent paper, Markvorsen, [12], gives a pseudo-Riemannian version of the well-known Takahashi's theorem showing that the coordinate functions of the immersion x are eigenfunctions of the Laplacian Δ of M , associated to the same eigenvalue λ , if and only if M is a vanishing CMC hypersurface ($\lambda=0$), a de Sitter space $S_1^n(r)$ ($\lambda>0$) or a hyperbolic space $H^n(r)$ ($\lambda<0$). That means that vanishing mean curvature hypersurfaces in L^{n+1} are the only ones having harmonic coordinate functions.

More recently, Garay and Romero, [8], ask for hypersurfaces in L^{n+1} satisfying the condition $\Delta H = C$, C being a constant vector of L^{n+1} which is normal to M at every point, and show that C should vanish. As for surfaces in L^3 , we have shown in [7] that vanishing mean curvature surfaces are the only ones satisfying $\Delta H = 0$, so that it seems natural to ask for the following geometric question:

Does the equation $\Delta H = 0$ characterize the vanishing CMC hypersurfaces of L^{n+1} ?

That equation motivates ourselves to study a certain generalization of Takahashi's

condition in order to deal with hypersurfaces whose mean curvature vector field is an eigenvector for the Laplacian, in short, $\Delta H = \lambda H$, $\lambda \in \mathbf{R}$ (see [5], [7]). This equation jointly with constant mean curvature yields to M has zero mean curvature everywhere or M has constant scalar curvature. On the other hand, putting together the constancy of both mean and scalar curvatures the hypersurface must satisfy $\Delta H = \lambda H$, for a real constant λ . Therefore, the following problem also arises in a natural way:

Are the non-vanishing constant mean curvature and constant scalar curvature hypersurfaces of the Lorentz-Minkowski space characterized by the equation $\Delta H = \lambda H$?

An interesting class of hypersurfaces to which the above two problems can be considered is that of the Einstein ones. It is not difficult to see that the shape operator S of an Einstein hypersurface M satisfies the equation $S^2 - \text{tr}(S)S + \varepsilon\rho I = 0$, being ε the sign of M and ρ the constant involved when you write down the proportionality between the Ricci curvature of M and the metric. Then, in a more general situation, it is worth studying the family \mathcal{C}_λ of those hypersurfaces in L^{n+1} satisfying the condition $\Delta H = \lambda H$, for a real constant λ , and such that the minimal polynomial of the shape operator is at most of degree two. Throughout this paper we shall deal with hypersurfaces in \mathcal{C}_λ and get the size of this family.

2. Basic results.

Let M_s^n be a hypersurface in L^{n+1} with index $s=0, 1$. Denote by σ , A , H , ∇ and $\bar{\nabla}$ the second fundamental form, the shape operator, the mean curvature vector field, the Levi-Civita connection of M and the usual flat connection of L^{n+1} , respectively. Let N be a unit normal vector field of M and let α denote the mean curvature with respect to N , i.e., $H = \alpha N$.

Our first task will be to compute ΔH at a point p of M . To do that, let $\{E_1, \dots, E_n\}$ be a local orthonormal frame such that $\nabla_{E_i} E_j(p) = 0$. Then we have

$$\bar{\nabla}_{E_i} \bar{\nabla}_{E_i} H = E_i E_i(\alpha)N - 2E_i(\alpha)AE_i - \alpha(\nabla_{E_i} A)E_i - \alpha\sigma(AE_i, E_i),$$

from which we deduce

$$(2.1) \quad \Delta H = 2A(\nabla\alpha) + \alpha \text{tr} \nabla A + \{\Delta\alpha + \varepsilon\alpha \text{tr}(A^2)\}N,$$

where $\varepsilon = \langle N, N \rangle$, $\text{tr} \nabla A = \text{trace}\{(X, Y) \rightarrow (\nabla_X A)Y\}$ and $\nabla\alpha$ is the gradient of α .

In order to find a good expression of $\text{tr} \nabla A$, let h_{ij} be the components of the second fundamental form, i.e., $h_{ij} = \langle \sigma(E_i, E_j), N \rangle = \langle AE_i, E_j \rangle$. Then we have

$$AE_i = \sum_{j=1}^n \varepsilon_j h_{ij} E_j, \quad \alpha = \frac{\varepsilon}{n} \text{tr} A = \frac{\varepsilon}{n} \sum_{i=1}^n \varepsilon_i h_{ii}.$$

Now, from the Codazzi equation we get

$$E_i(h_{ij}) = E_j(h_{ii}),$$

and therefore we deduce

$$\begin{aligned} \text{tr} \nabla A &= \sum_i \varepsilon_i (\nabla_{E_i} A) E_i = \sum_i \varepsilon_i \nabla_{E_i} (A E_i) \\ &= \sum_{i,j} \varepsilon_i \varepsilon_j E_i(h_{ij}) E_j = \sum_{i,j} \varepsilon_i \varepsilon_j E_j(h_{ii}) E_j \\ &= \sum_j \varepsilon_j E_j(n\varepsilon\alpha) E_j = n\varepsilon \nabla \alpha. \end{aligned}$$

From here and equation (2.1) we obtain the following useful result (see [7]).

LEMMA 2.1. *Let M_s^n be a hypersurface in L^{n+1} with index $s=0, 1$. Then*

$$\Delta H = 2A(\nabla\alpha) + \frac{n\varepsilon}{2} \nabla\alpha^2 + \{\Delta\alpha + \varepsilon\alpha \text{tr}(A^2)\}N,$$

where $\nabla\alpha$ is the gradient of α and $\varepsilon = \langle N, N \rangle$.

From this lemma, we get the following easy consequence.

COROLLARY 2.2. *Let M^n be a hypersurface in L^{n+1} such that $\Delta H = \lambda H$ for a real constant λ . Then $\nabla\alpha^2$ is a principal direction with associated principal curvature $-(n\varepsilon/2)\alpha$ in the open set $\mathcal{U} = \{p \in M : \nabla\alpha^2(p) \neq 0\}$.*

Throughout this paper the method of moving frames will be used, so we are going to give the structure equations because they look slightly different in Lorentz space with regard to the Riemannian case. Let $\{E_1, \dots, E_{n+1}\}$ be a local orthonormal frame in L^{n+1} and let $\{\omega^1, \dots, \omega^{n+1}\}$ and $\{\omega_i^j\}_{i,j}$ be the dual frame and the connection forms, respectively, given by

$$\omega^i(X) = \langle X, E_i \rangle, \quad \omega_i^j(X) = \langle \nabla_X E_i, E_j \rangle.$$

Then we have the structure equations

$$d\omega^i = - \sum_{j=1}^{n+1} \varepsilon_j \omega_j^i \wedge \omega^j, \quad d\omega_i^j = - \sum_{k=1}^{n+1} \varepsilon_k \omega_k^j \wedge \omega_i^k,$$

where $\varepsilon_i = \langle E_i, E_i \rangle$.

3. Some examples.

In this section we are describing some examples of hypersurfaces in L^{n+1} satisfying the condition $\Delta H = \lambda H$, $\lambda \in \mathbf{R}$.

EXAMPLE 3.1. Take $k \in \{1, 2, \dots, n-1\}$ and let $f : L^{n+1} \rightarrow \mathbf{R}$ be a real function

defined by

$$f(x_1, \dots, x_{n+1}) = \delta_1(-x_1^2 + x_2^2 + \dots + x_k^2) + x_{k+1}^2 + \delta_2(x_{k+2}^2 + \dots + x_{n+1}^2),$$

where δ_1 and δ_2 belong to the set $\{0, 1\}$ and they do not vanish simultaneously. Taking $r > 0$ and $\varepsilon = \pm 1$, the set $M = f^{-1}(\varepsilon r^2)$ is a hypersurface of L^{n+1} provided $(\delta_1, \delta_2, \varepsilon) \neq (0, 1, -1)$.

A straightforward computation shows that the unit normal vector field is written as $N = (1/r)(\delta_1 x_1, \dots, \delta_1 x_k, x_{k+1}, \delta_2 x_{k+2}, \dots, \delta_2 x_{n+1})$, and the principal curvatures are $\mu_1 = -\delta_1/r$ and $\mu_2 = -\delta_2/r$ with multiplicities k and $n-k$, respectively. Thus M is an isoparametric hypersurface of L^{n+1} and therefore, by Lemma 2.1, we get $\Delta H = \varepsilon \operatorname{tr}(A^2)H$, with $\operatorname{tr}(A^2) = (k\delta_1 + (n-k)\delta_2)/r^2$. The adjoint table shows all possibilities (see [1]):

δ_1	δ_2	ε	Hypersurface	ΔH
0	1	1	$L^k \times S^{n-k}(r)$	$\frac{n-k}{r^2} H$
1	0	-1	$H^k(r) \times R^{n-k}$	$-\frac{k}{r^2} H$
1	0	1	$S_1^k(r) \times R^{n-k}$	$\frac{k}{r^2} H$
1	1	-1	$H^n(r)$	$-\frac{n}{r^2} H$
1	1	1	$S_1^n(r)$	$\frac{n}{r^2} H$

EXAMPLE 3.2. In [9], L.K.Graves constructs a new surface in L^3 as follows. Let $x(s)$ be a null curve in L^3 with Cartan frame $\{A, B, C\}$, i.e., $\{A, B, C\}$ is a pseudo-orthonormal frame of vector fields along $x(s)$ satisfying:

$$(3.1) \quad \begin{aligned} \dot{x} &= A, \\ \dot{A} &= k(s)C, \quad k(s) \neq 0, \\ \dot{B} &= aC, \quad a \text{ being a nonzero constant}, \\ \dot{C} &= aA + k(s)B. \end{aligned}$$

If we define $\Psi(s, u) = x(s) + uB(s)$, then Ψ determines a Lorentz surface which is called a *B-scroll*. An easy computation leads to $N(s, u) = -auB(s) - C(s)$ and $H = aN$. Then we have $\Delta H = 2a^2H$.

As a generalization of that surface we construct the following hypersurface (see [10]). Let $x(s)$ be a null curve in L^{n+1} with local pseudo-orthonormal frame $\{A, B, C, X_1, \dots, X_{n-2}\}$ along $x(s)$ satisfying (3.1). Let M be the hypersurface in L^{n+1} locally defined as follows:

$$\Psi(s, u, x_1, \dots, x_{n-2}) = x(s) + uB(s) + \sum_{j=1}^{n-2} x_j X_j(s) - \frac{1}{a} C(s) + \sqrt{\frac{1}{a^2} - \sum_{j=1}^{n-2} x_j^2} C(s).$$

It is not difficult to see that

$$N(s, u, x_1, \dots, x_{n-2}) = -auB(s) - \sqrt{1 - a^2 \sum_{j=1}^{n-2} x_j^2} C(s) - a \sum_{j=1}^{n-2} x_j X_j(s)$$

is the unit normal vector field to M and the shape operator can be put, in the usual frame, in the form

$$\begin{bmatrix} a & 0 & & 0 \\ k(s) & a & & \\ & & \ddots & \\ 0 & & & a \end{bmatrix}.$$

Thus the minimal polynomial of A is $p(t) = (t - a)^2$ and we get $\alpha = a$ and $\text{tr}(A^2) = na^2$. Therefore Lemma 2.1 allows us to write $\Delta H = na^2 H$. Then M is said to be a *generalized umbilical hypersurface*.

4. The characterization theorems.

In this section we are going to describe the set \mathcal{C}_λ . Then the shape operator of a hypersurface M in \mathcal{C}_λ takes one of the following forms (see [11]):

$$\text{I. } \begin{bmatrix} \mu_1 & & & & 0 \\ & \ddots & & & \\ & & \mu_1 & & \\ & & & \mu_2 & \\ 0 & & & & \mu_2 \end{bmatrix} \quad \text{II. } \begin{bmatrix} \beta & 0 & & & 0 \\ 1 & \beta & & & \\ & & \beta & & \\ & & & \ddots & \\ 0 & & & & \beta \end{bmatrix} \quad \text{III. } \begin{bmatrix} \beta & \gamma \\ -\gamma & \beta \end{bmatrix}$$

LEMMA 4.1. *Let M be a hypersurface of L^{n+1} in the set \mathcal{C}_λ . Then M has constant mean curvature or, at the points of the open set \mathcal{U} , the shape operator A is diagonalizable. Moreover, if this is the case, $-(n\epsilon/2)\alpha$ is a principal curvature with multiplicity one.*

PROOF. Let us suppose \mathcal{U} is not empty and let p be any point of \mathcal{U} . By Corollary 2.2 we know that $-(n\epsilon/2)\alpha$ is a principal curvature of M and therefore case III cannot hold. If A_p falls in case II, then it must be $\beta = \epsilon\alpha = -(n\epsilon/2)\alpha$ and so $\alpha(p) = 0$, which is

a contradiction with the choice of p . Thus A_p always falls in case I. As for last statement, if \mathcal{U} is not empty there are exactly two distinct principal curvatures $\mu_1 \neq \mu_2$, with $\mu_1 = -(n\varepsilon/2)\alpha$. Let D be the distribution associated with μ_1 and let $\{E_1, \dots, E_n\}$ be a local orthonormal frame of principal directions such that E_1 is in the direction of $\nabla\alpha^2$. If we assume $\dim D > 1$, we can work as in [14] and we deduce $X(\mu_1) = 0$, for any vector field X in D . In particular, $E_1(\alpha) = 0$ on \mathcal{U} , so that, E_1 and $\nabla\alpha^2$ being parallel, we get α is constant on \mathcal{U} , which is a contradiction. Therefore $-(n\varepsilon/2)\alpha$ is a principal curvature with multiplicity one.

Now, we are going to show the following major result.

THEOREM 4.2. *Let M be a hypersurface of L^{n+1} in the set \mathcal{C}_λ . Then M is a CMC hypersurface.*

PROOF. We aim to show \mathcal{U} is empty. Otherwise, from Lemma 4.1 we know that, at the points of \mathcal{U} , $-(n\varepsilon/2)\alpha$ is the principal curvature of multiplicity one with principal direction $\nabla\alpha^2$. Thus, on \mathcal{U} , we can choose a local orthonormal frame $\{E_1, \dots, E_{n+1}\}$ adapted to M , such that $\{E_1, \dots, E_n\}$ are eigenvectors of A with associated eigenvalues $\{\mu_1, \dots, \mu_n\}$, with E_1 in the direction of $\nabla\alpha^2$ and E_{n+1} normal to M . Therefore $\mu_1 = -(n\varepsilon/2)\alpha$ and $\mu_2 = \dots = \mu_n = (3n\varepsilon/2(n-1))\alpha$. Let $\{\omega^1, \dots, \omega^n\}$ and $\{\omega_i^j\}_{i,j}$ be the dual frame and the connection forms of the chosen frame, respectively. Then we have

$$(4.1) \quad \omega_{n+1}^1 = \frac{n\varepsilon}{2} \alpha \omega^1,$$

$$(4.2) \quad \omega_{n+1}^j = -\frac{3n\varepsilon}{2(n-1)} \alpha \omega^j, \quad j=2, \dots, n,$$

$$(4.3) \quad d\alpha = \varepsilon_1 E_1(\alpha) \omega^1.$$

If we take exterior differentiation in (4.1) and use the well-known structure equations we deduce $d\omega^1 = 0$. Thus one locally has $\omega^1 = du$, for a certain function u , which along with (4.3) implies $d\alpha \wedge du = 0$. Then α depends on u , $\alpha = \alpha(u)$, and we obtain $d\alpha = \alpha' du = \alpha'(u) \omega^1$ which implies $E_1(\alpha) = \varepsilon_1 \alpha'$.

Taking exterior differentiation in (4.2) and the structure equations we have

$$(4.4) \quad (n+2)\alpha \omega_j^1 = 3\varepsilon_1 \alpha' \omega^j, \quad j=2, \dots, n.$$

Now, taking once more exterior differentiation in (4.4) and using (4.1) and (4.2) we obtain the following second order differential equation:

$$(4.5) \quad 4\alpha\alpha'' - \frac{4(n+5)}{n+2} (\alpha')^2 + \frac{n^2(n+2)}{n-1} \varepsilon\varepsilon_1 \alpha^4 = 0.$$

If we put $y = (d\alpha/du)^2$, the above equation turns into

$$2\alpha \frac{dy}{d\alpha} - \frac{4(n+5)}{n+2} y = -\varepsilon\varepsilon_1 \frac{n^2(n+2)}{n-1} \alpha^4,$$

whose solution is given by

$$(4.6) \quad y(\alpha) = C\alpha^{2(n+5)/(n+2)} - \varepsilon\varepsilon_1 \left(\frac{n(n+2)}{2(n-1)} \right)^2 \alpha^4,$$

where C is a constant.

Now we use the definition of $\Delta\alpha$, the fact that E_1 is parallel to $\nabla\alpha^2$ and equation (4.4) to get

$$\alpha\Delta\alpha = -\varepsilon_1\alpha\alpha'' + \frac{3(n-1)\varepsilon_1}{n+2} (\alpha')^2.$$

On the other hand, since M is a hypersurface in \mathcal{C}_λ , one has $\alpha\Delta\alpha = (\lambda - \varepsilon\text{tr}(A^2))\alpha^2$ and therefore we obtain

$$\alpha\Delta\alpha = \lambda\alpha^2 - \frac{\varepsilon n^2(n+8)}{4(n-1)} \alpha^4.$$

Putting together the two last displayed equations we have

$$(4.7) \quad \alpha\alpha'' - \frac{3(n-1)}{n+2} (\alpha')^2 = -\lambda\varepsilon_1\alpha^2 + \frac{\varepsilon\varepsilon_1 n^2(n+8)}{4(n-1)} \alpha^4.$$

We deduce, by using (4.5), (4.6) and (4.7), that α is locally constant on \mathcal{U} , which is a contradiction with the definition of \mathcal{U} .

The following theorem gives the size of \mathcal{C}_λ .

THEOREM 4.3. *Let M be a hypersurface of L^{n+1} in the set \mathcal{C}_λ . Then one of the following statements holds:*

- 1) M has zero mean curvature everywhere.
- 2) M is an open piece of one of the following hypersurfaces: $H^n(r)$, $H^k(r) \times \mathbf{R}^{n-k}$, $L^k \times S^{n-k}(r)$, $S_1^k(r) \times \mathbf{R}^{n-k}$, $S_1^n(r)$.
- 3) M is a B -scroll.
- 4) M is a generalized umbilical hypersurface.

PROOF. Since the mean curvature α is constant, then either α vanishes everywhere on M or, from Lemma 2.1, $\text{tr}(A^2) = \varepsilon\lambda$. Then M is isoparametric because $\text{tr}(A^2)$ and the minimal polynomial of A are constant. If the shape operator of M is diagonalizable we get statement (2) from [1, Theorem 5.1]; otherwise, from [10, Theorem 4.5] we obtain (3) and (4).

5. Applications.

From Theorem 4.3 we completely characterize the set \mathcal{C}_λ for a real constant λ . In particular, it is worthwhile to analyze the set \mathcal{C}_0 . Concretely we obtain the following

COROLLARY 5.1. *Let M be a hypersurface in L^{n+1} . Then M is in \mathcal{C}_0 if and only if M has zero mean curvature everywhere.*

This solution is quite similar to that given in the Euclidean case because, according to [6], minimal hypersurfaces in R^{n+1} are the only ones in \mathcal{C}_0 .

A special and interesting subset of \mathcal{C}_λ is that of spacelike hypersurfaces. In this case, we have the following

COROLLARY 5.2. *Let M be a spacelike hypersurface of L^{n+1} in \mathcal{C}_λ . Then one of the following statements holds:*

- 1) M is a maximal hypersurface;
- 2) M is an open piece of the hyperbolic space $H^n(r)$;
- 3) M is an open piece of a hyperbolic cylinder $H^k(r) \times R^{n-k}$.

As a final application, our main result can also be considered under the viewpoint of the finite type submanifolds (see [2]). In fact, it can be shown that an immersion satisfying the equation $\Delta H = \lambda H$ is either of infinite type or has zero mean curvature everywhere when $\lambda = 0$ and either of 1-type or of null 2-type when $\lambda \neq 0$. In this context we give

COROLLARY 5.3. *Let M be a hypersurface of L^{n+1} in \mathcal{C}_λ . Then M is of null 2-type if and only if it is an open piece of one of the following hypersurfaces: a hyperbolic cylinder $H^k(r) \times R^{n-k}$, a Lorentzian cylinder $L^k \times S^{n-k}(r)$, a cylinder over a De Sitter space $S_1^k(r) \times R^{n-k}$.*

We finish by noticing that Corollary 5.3 gives us the following characterization of the hyperbolic cylinder.

COROLLARY 5.4. *Let M be a spacelike hypersurface of L^{n+1} in \mathcal{C}_λ . Then M is of null 2-type if and only if it is an open piece of a hyperbolic cylinder.*

References

- [1] N. ABE, N. KOIKE and S. YAMAGUCHI, Congruence theorems for proper semi-Riemannian hypersurfaces in a real space form, *Yokohama Math. J.*, **35** (1987), 123–136.
- [2] B. Y. CHEN, *Total Mean Curvature and Submanifolds of Finite Type*, Ser. in Pure Math., **1** (1984), World Scientific.
- [3] S. CHENG and S. T. YAU, Maximal spacelike hypersurfaces in the Lorentz-Minkowski space, *Ann. of Math.*, **104** (1976), 407–419.
- [4] Y. CHOQUET, A. FISCHER and J. MARSDEN, Maximal hypersurfaces and positivity of mass, *Proc. of*

the Enrico Fermi Summer School, J. Ehlers, ed., 396–456, Ital. Phys. Soc., 1979.

- [5] A. FERRÁNDEZ, O. J. GARAY and P. LUCAS, On a certain class of conformally flat Euclidean hypersurfaces, *Global Differential Geometry and Global Analysis, Berlin 1990*, Ferus, Pinkall, Simon and Wegner, ed., 48–54, Lecture Notes in Math., **1481** (1991), Springer.
- [6] A. FERRÁNDEZ and P. LUCAS, Null finite type hypersurfaces in space forms, *Kodai Math. J.*, **14** (1991), 406–419.
- [7] A. FERRÁNDEZ and P. LUCAS, On surfaces in the 3-dimensional Lorentz-Minkowski space, *Pacific J. Math.*, **152** (1992), 93–100.
- [8] O. J. GARAY and A. ROMERO, An isometric embedding of the complex hyperbolic space in a pseudo-Euclidean space and its applications to the study of real hypersurfaces, *Tsukuba J. Math.*, **14** (1990), 293–313.
- [9] L. GRAVES, Codimension one isometric immersions between Lorentz spaces, *Trans. Amer. Math. Soc.*, **252** (1979), 367–392.
- [10] M. A. MAGID, Lorentzian isoparametric hypersurfaces, *Pacific J. Math.*, **118** (1985), 165–197.
- [11] A. MÁLTSEV, *Fundamentos de Algebra Lineal*, Editorial Mir, Mosc (URSS), 1972.
- [12] S. MARKVORSEN, A characteristic eigenfunction for minimal hypersurfaces in space forms, *Math. Z.*, **202** (1989), 375–382.
- [13] J. MARSDEN and F. TIPLER, Maximal hypersurfaces and foliations of constant mean curvature in general relativity, *Bull. Amer. Phys. Soc.*, **23** (1978), p. 84.
- [14] P. J. RYAN, Homogeneity and some curvature conditions for hypersurfaces, *Tôhoku Math. J.*, **21** (1969), 363–388.
- [15] S. STUMBLES, Hypersurfaces of constant mean extrinsic curvature, *Ann. of Physics*, **133** (1981), 28–56.

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