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2-Ended Manifolds and Locally Splitting Actions of Abelian Groups

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The purpose of this paper is to prove the following result.

THEOREM. *Let (S^1, M) be a locally splitting action and suppose that the universal covering manifold $\tilde{M}$ is 2-ended. Then, every isometry of M preserves the foliated structure defined by the action.*

In the sequel, we apply some consequences of this result in order to study the structure of the identity component $I_0(M)$, of the isometry group $I(M)$ of the Riemannian manifold M.

1. Introduction.

The closed connected subgroup G of $I_0(M)$ acts *locally splitting* if and only if it has only one orbit type, the normal to the orbits distribution $\mathfrak{N}$ is integrable and every fundamental (Killing) vector field has constant length in directions normal to the orbits of the action [1].

For G abelian, it turns out that a locally splitting action of G is *free*, it has *parallel* fundamental vector fields and $\tilde{M}$ is a Riemannian product $\boldsymbol{R}^k \times \tilde{W}$, $k = \dim G$ [4, Theorem B]. In particular, if $G = S^1$ and $\tilde{M}$ has two *ends* then $\tilde{W}$ is compact and the orbits of the *lifted* in $\tilde{M}$ action (cf. [2] for the used terminology) admit a parametrization which makes them *lines* (see [6]). With this notation the motivation of the present work is twofold.

It is always interesting to decide when a Killing vector field of M preserves some foliated structure ([3], [5]). As every locally splitting action defines a local product structure consisting of orbits and integral manifolds, it seems natural to discuss similar problems in our case too. Our main result shows that, not only flows of Killing vector fields but, *every* isometry maps orbits of the given action to orbits and interchanges the maximal integral manifolds $N(x)$, $x \in M$, of $\mathfrak{N}$.

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On the other hand, locally splitting actions are originated on compatibility assumptions concerning the acting group and the underlying Riemannian structure [6]. Now, we are interested in the restrictions imposed on $I_0(M)$, provided that some of its one-parameter subgroups acts locally splitting and the action is controllable, in the topological sense. In this respect, we prove that under the hypothesis of the above stated Theorem, $I_0(M)$ is the *direct product* of S^1 and the compact subgroup K realized as the group of isometries which fix every $N(z)$. Note that M is not necessarily a direct product.

2. Proof of the theorem.

With the notation from 1, $N(gx)=gN(x)$ for all $g\in G$ and $N(x)\cap G(z)\neq\emptyset$ for all $z\in M$, where $G(z)$ is the orbit of $z\in M$ under G.

2.1 Lemma. *If (S^1, M) is locally splitting and $\tilde{M}$ is 2-ended then M is compact.*

Proof. Since $\tilde{W}$ is the common universal covering manifold of the various $N(z)$, all these are compact. As $M=S^1(N(x))$, it is compact too. In particular, $\tilde{M}$ and M are complete Riemannian manifolds.

Every isometry of M has a lifting to an isometry of $\tilde{M}$. The action (S^1, M) has a lifting to an action $(\boldsymbol{R}, \tilde{M})$ and $\tilde{M}\underset{\boldsymbol{R}}{=}\boldsymbol{R}(x)\times\tilde{W}$, as in 1, where $\underset{\boldsymbol{R}}{=}$ means equivariantly isometric. Furthermore, the covering projection $p\colon\tilde{M}\to M$ is a local isometry, equivariant with respect to the existing actions. These remarks reduce the proof of the following.

2.2. Proposition. *Suppose that the locally splitting action $(\boldsymbol{R}, P)$ on the 2-ended, complete, orientable Riemannian manifold P splits globally. Then every isometry of P preserves the decomposition defined by the splitting.*

Proof. Let $y\in P$, h be an isometry of P and $\{f_t : t\in\boldsymbol{R}\}$ the one-parameter subgroup of isometries defined by the action. Then, $P\underset{\boldsymbol{R}}{=}\boldsymbol{R}(y)\times S$ and we shall write ty for the point $f_t(y)$ of P. With this notation we set S_t for the compact manifold $\{ty\}\times S$. As $h(S_0)$ is compact and connected, there are unique t and s such that $h(S_0)\subset([t, s]y)\times S:=\Sigma$ and it is not contained in any other $([t_1, t_2]y)\times S$ with $t<t_1<t_2<s$.

If $x\in h(S_0)\cap S_t$, we shall show that the tangent spaces $T_xh(S_0)$ and T_xS_t are equal. Indeed, every geodesic γ_v with initial conditions $\gamma_v(0)=x$ and $\dot{\gamma}_v(0)=v\in T_xh(S_0)$ is the isometric image of a geodesic from S_0. Hence, it is contained in $h(S_0)$ and also in Σ. Suppose now that there exists some $w\in T_xh(S_0)$ not in T_xS_t. Then, there is some $t_0\in\boldsymbol{R}$ such that $\gamma_w(t_0)$ is not contained in S_t and the geodesic γ_w has points outside Σ. This contradicts $w\in T_xh(S_0)$.

As $h(S_0)$ and S_t are connected, totally geodesic submanifolds of the complete Riemannian manifold P and they have the same tangent space at some point, they are equal.

2.3 COROLLARY. *If the isometry ϕ of P does not commute with the isometries $\{f_t : t \in \boldsymbol{R}\}$ of the action, then there is exactly one $r \in \boldsymbol{R}$ such that $\phi(S_r) = S_r$ and $\phi(S_{r-t}) = S_{r+t}$ for all $t \in \boldsymbol{R}$.*

PROOF. The points $t\phi(y)$ and $\phi(ty)$ belong to $\boldsymbol{R}(\phi(y))$ and their distance from $\phi(y)$ is $|t|$. Hence either $\phi(ty) = t\phi(y)$, or $\phi(ty) = (-t)\phi(y)$. If there exists some $t_0 \in \boldsymbol{R}$ such that $\phi(t_0 y) = t_0 \phi(y)$, then it is easily seen that $\phi(sy) = s\phi(y)$ for all $s \in \boldsymbol{R}$. Suppose now that ϕ does not commute with $\{f_t : t \in \boldsymbol{R}\}$. Then, there is some $z \in P$ and $\phi(sz) = (-s)\phi(z)$, for all $s \in \boldsymbol{R}$. If $\tau z = \boldsymbol{R}(z) \cap N(\phi(z))$, we set $q = (\tau/2)z$ and we shall prove that $\phi(N(q)) = N(q)$. Because of the global equivariant splitting of the action $(\boldsymbol{R}, P)$ and 2.2, it is enough to show that $\phi(N(q)) \cap N(q)$ is not empty. Indeed, $(-\tau/2)(\tau z) = q$ and $(-\tau/2)\phi(z) = \phi((\tau/2)z) = \phi(q)$. Hence, $\phi(q)$ belongs to $(-\tau/2)N(\phi(z)) = N((-\tau/2)(\tau z)) = N((\tau/2)z) = N(q)$.

2.4 PROPOSITION. *Let F denote the fundamental vector field of the locally splitting action (S^1, M) and $\tilde{M}$ be 2-ended. Then $[F, Y] = 0$ for every Killing vector field Y of M, i.e., the flow of Y commutes with the action.*

PROOF. Consider the one-parameter subgroup of isometries $\{\Psi_s : s \in \boldsymbol{R}\}$ defined by the lifting $\tilde{Y}$ of Y in $\tilde{M}$. Every Ψ_s is homotopic to the identity, it preserves the splitting $\tilde{M} = \boldsymbol{R} \times \tilde{W}$ and induces a translation on the $\boldsymbol{R}$-factor. Because of 2.3, Ψ_s commutes with the one-parameter subgroup of isometries defined by the lifting of F in $\tilde{M}$. Projecting via p in M, we have the desired result.

3. Applications.

With the hypothesis of our Theorem still in force, let $\mathscr{X}_K(M)$ denote the Lie algebra of Killing vector fields of M. Since M is compact, every $f \in I_0(M)$ lies in the flow of some element of $\mathscr{X}_K(M)$. Every $X \in \mathscr{X}_K(M)$ has a decomposition $X = X(v) + X(h)$, where $X(v) = \langle F, X \rangle F$ is tangent to the orbits of the action and $X(h)$ belongs to the distribution $\mathfrak{N}$.

3.1 LEMMA. *Under the above notation, $X(v)$ and $X(h)$ are again Killing vector fields.*

PROOF. Because F is parallel and $[F, X] = 0$, an easy calculation shows that $X(v)$ satisfies the Killing equation

$$\langle \nabla_A X(v), B \rangle + \langle \nabla_B X(v), A \rangle = 0\,,$$

for all $A, B \in \mathscr{X}(M)$, i.e., $X(v)$ is a Killing vector field.

3.2 LEMMA. *Let $\mathscr{H} = \{Y \in \mathscr{X}_K(M) : Y(h) = 0\}$ and $\mathscr{K} = \{Y \in \mathscr{X}(M) : Y(v) = 0\}$. Then, $\mathscr{H}$ and $\mathscr{K}$ are ideals in $\mathscr{X}_K(M)$.*

PROOF. Since $\mathfrak{N}$ is integrable, $\mathscr{K}$ is a subalgebra of $\mathscr{X}_K(M)$. If $A \in \mathscr{K}$ and $Z \in \mathscr{X}_K(M)$, we claim that $\langle F, [A, Z] \rangle = 0$. Indeed, since $\langle F, A \rangle = 0$ and F is parallel, we have

$\langle F, \nabla_Z A\rangle = 0$. Because $Z \in \mathscr{X}_K(M)$ and $\langle F, Z\rangle = 0$,

$$\langle F, \nabla_A Z\rangle = -\langle A, \nabla_F Z\rangle = -\langle A, \nabla_Z F + [Z, F]\rangle = 0\,,$$

and our claim follows.

Using similar arguments we can prove that $\mathscr{H}$ is an ideal too.

3.3 PROPOSITION. *Let (S^1, M) be locally splitting and $\tilde{M}$ 2-ended. Then $I_0(M) = S^1 \times K$, where the elements of K fix every integral manifold of $\mathfrak{N}$.*

PROOF. Because of 3.2, $\mathscr{X}_K(M)$ has a direct sum decomposition $\mathscr{K} \oplus \mathscr{H}$ into ideals, where $\mathscr{H}$ consists of fundamental vector fields of (S^1, M) and the Killing vector fields of $\mathscr{K}$ are always tangent to the normal distribution $\mathfrak{N}$. The flows of fields of $\mathscr{K}$ form a connected Lie group of isometries K, which is a normal subgroup of $I_0(M)$ and preserves every $N(z)$. Since $\mathfrak{N}$ has compact leaves, K is itself compact. The corresponding normal subgroup for $\mathscr{H}$, may be identified with the acting group S^1. In view of 2.4, this implies that $S^1K = KS^1$ and $I_0(M) = S^1 \times K$.

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