TOKYO J. MATH.
VOL. 15, NO. 2, 1992

Non-Existence of Homomorphisms between Quantum Groups

Takahiro HAYASHI

Nagoya University

(Communicated by T. Nagano)

§1. Introduction.

Let G be a connected complex semisimple Lie group. The (multi-parameter) quantum group $A_{\hbar,\varphi}(G)$ is a deformation of the function algebra $A(G)$ of G as a Hopf algebra (cf. [2, 5, 14, 8, 9, 10]). While the representation theory of $A_{\hbar,\varphi}(G)$ is similar to that of G, the "group theoretic" structure of $A_{\hbar,\varphi}(G)$ is rather different from that of G. For example, it seems that $A_{\hbar,\varphi}(G)$ does not have so many "subgroups" as G.

In this paper, we show that there exist no non-trivial Hopf algebra homomorphisms from $A_{\hbar,\varphi}(SL(N))$ into $A_{\hbar,\psi}(SO(N))$ $(N\geq 7)$ or $A_{\hbar,\psi}(Sp(N))$. In other words, there exists no quantum analogue of group inclusions $SO(N)\subset SL(N)$ and $Sp(N)\subset SL(N)$. The proof is done by considering the square of the antipode.

We refer the reader to Tanisaki [12] for the results on the representation theory of the quantized enveloping algebra, which we use below.

§2. Quantum groups.

Let G be a connected complex semisimple Lie group and let $\mathfrak{g}$ be its Lie algebra. Let $A=(a_{ij})_{1\leq i,j\leq l}$ be the Cartan matrix of $\mathfrak{g}$ and let $\boldsymbol{d}=(d_1,\cdots,d_l)$ be positive integers such that $d_ia_{ij}=d_ja_{ji}$. The quantized enveloping algebra $U_\hbar(\mathfrak{g})=U_{\hbar,\boldsymbol{d}}(\mathfrak{g})$ is the $\mathbf{C}[[\hbar]]$-algebra which is $\hbar$-adically generated by elements X_i, Y_i, H_i $(1\leq i\leq l)$ satisfying the following fundamental relations:

$$H_iH_j=H_jH_i\,,$$

$$H_iX_j-X_jH_i=a_{ij}X_j\,,\quad H_iY_j-Y_jH_i=-a_{ij}Y_j\,,$$

$$X_iY_j-Y_jX_i=\delta_{ij}\frac{K_i-K_i^{-1}}{q_i-q_i^{-1}}\,,$$

$$\sum_{0\leq n\leq 1-a_{ij}}(-1)^n\begin{bmatrix}1-a_{ij}\\ n\end{bmatrix}_{q_i}X_i^{1-a_{ij}-n}X_jX_i^n=0\qquad(i\neq j)\,,$$

Received December 10, 1991

$$\sum_{0\leq n\leq 1-a_{ij}} (-1)^n \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix}_{q_i} Y_i^{1-a_{ij}-n} Y_j Y_i^n = 0 \qquad (i\neq j),$$

where $q_i=\exp(\hbar d_i)$, $K_i=\exp(\hbar d_i H_i)$ and

$$\begin{bmatrix} n \\ m \end{bmatrix}_t = \frac{\prod_{1\leq r\leq n}(t^r-t^{-r})}{\prod_{1\leq r\leq m}(t^r-t^{-r})\prod_{1\leq r\leq n-m}(t^r-t^{-r})}.$$

Let $\varphi=(\varphi_{ij})_{1\leq i,j\leq l}$ be a skewsymmetric matrix. Then $U_\hbar(\mathfrak{g})$ becomes a topological Hopf algebra with the coproduct Δ_φ defined by

$$\Delta_\varphi(X)=\exp\Big(\hbar\sum_{i,j}\varphi_{ij}H_i\otimes H_j\Big)\Delta_0(X)\exp\Big(-\hbar\sum_{i,j}\varphi_{ij}H_i\otimes H_j\Big),$$

$$\Delta_0(H_i)=H_i\otimes 1+1\otimes H_i,$$

$$\Delta_0(X_i)=X_i\otimes 1+K_i\otimes X_i, \quad \Delta_0(Y_i)=Y_i\otimes K_i^{-1}+1\otimes Y_i.$$

We denote the topological Hopf algebra $(U_\hbar(\mathfrak{g}), \Delta_\varphi)$ by $U_{\hbar,\varphi}(\mathfrak{g})=U_{\hbar,d,\varphi}(\mathfrak{g})$ and call it the *multi-parameter quantized enveloping algebra* of $\mathfrak{g}$ (see [8, 9]).

Let P_G^{++} be the set of isomorphism classes of finite dimensional irreducible G-modules. For each $[L]\in P_G^{++}$, there exists an irreducible representation $(L_\hbar, \pi_L)$ of $U_{\hbar,\varphi}(\mathfrak{g})$ such that $L_\hbar/\hbar L_\hbar$ is isomorphic to L as $U_{\hbar,\varphi}(\mathfrak{g})/\hbar U_{\hbar,\varphi}(\mathfrak{g})\simeq U(\mathfrak{g})$-modules. Define a subspace $A_{\hbar,\varphi}(G)=A_{\hbar,d,\varphi}(G)$ of $U_{\hbar,\varphi}(\mathfrak{g})^*$ by

$$A_{\hbar,\varphi}(G)=\mathrm{span}\{\langle v, \pi_L(-)u\rangle \mid [L]\in P_G^{++}, v\in L^*, u\in L\}.$$

Then, the topological Hopf algebra structure on $U_{\hbar,\varphi}(\mathfrak{g})$ induces a Hopf algebra structure on $A_{\hbar,\varphi}(G)$. We call this Hopf algebra $A_{\hbar,\varphi}(G)$ the *(function algebra of) multi-parameter quantum group associated to G* (cf. [12]). The quantum group $A_{\hbar,0}(SL(N))$ has the following well-known expression:

$$\begin{aligned} A_{\hbar,0}(SL(N))=\langle x_{ij}\,(1\leq i,j\leq N) \mid\ & e^\hbar x_{im}x_{ik}=x_{ik}x_{im},\ e^\hbar x_{jk}x_{ik}=x_{ik}x_{jk},\\ & x_{jk}x_{im}=x_{im}x_{jk},\ x_{ik}x_{jm}-x_{jm}x_{ik}-(e^\hbar-e^{-\hbar})x_{jk}x_{im}=0\\ & (1\leq i<j\leq N,\ 1\leq k<m\leq N)\rangle, \end{aligned}$$

$$\Delta(x_{ij})=\sum x_{ik}\otimes x_{kj}.$$

§3. The result.

THEOREM. *Let G be a connected simple complex Lie group whose Lie algebra is either $\mathfrak{so}(N)$ $(N\geq 7)$ or $\mathfrak{sp}(N)$ $(N=2l,\ l\geq 2)$. Let φ and ψ be skewsymmetric matrices. Then, except for the map $f\mapsto\varepsilon(f)1$ $(f\in A_{\hbar,\varphi}(SL(N)))$, there exist no Hopf algebra homomorphisms from $A_{\hbar,\varphi}(SL(N))$ into $A_{\hbar,\psi}(G)$, where ε denotes the counit of*

$A_{\hbar,\varphi}(SL(N))$.

PROOF. Let $[a_{ij}]$ be the Cartan matrix of $\mathfrak{sl}(N)$, that is $a_{ij}=2$ $(i=j)$, -1 $(i=j\pm 1)$, 0 $(|i-j|\geq 2)$. Since $[a_{ij}]$ is symmetric, the corresponding positive integers $\boldsymbol{d}$ are of the form $\boldsymbol{d}=(d, \cdots, d)$ for some $d\in \boldsymbol{Z}_{>0}$. Let $V=(\bigoplus_{1\leq i\leq N}\boldsymbol{C}[[\hbar]]u_i, \pi)$ be the vector representation of $U_{\hbar,\varphi}(\mathfrak{sl}(N))$ and let $\omega\colon V\to V\otimes A_{\hbar,\varphi}(SL(N))$ be the corresponding coaction. Explicitly, the action is given by

$$\pi(H_i)=E_{ii}-E_{i+1\,i+1}\,, \quad \pi(X_i)=E_{i\,i+1}\,, \quad \pi(Y_i)=E_{i+1,i}\,,$$

where E_{ij} denotes a matrix defined by $E_{ij}u_k=\delta_{jk}u_i$. Let S be the antipode of $U_{\hbar,\varphi}(\mathfrak{sl}(N))$ and let $\mu\colon(V,\pi)\to(V,\pi\circ S^2)$ be an isomorphism of $U_{\hbar,\varphi}(\mathfrak{sl}(N))$-modules. Since $\mu(u_1)=\text{const.}\,u_1$, we may assume $\mu(u_1)=u_1$. Let f_i be an element of $U_{\hbar,\varphi}(\mathfrak{sl}(N))$ defined by $f_i=\exp(\hbar\sum_{jk}\varphi_{jk}a_{ji}H_k)Y_i$. Then we have $S^2(f_i)=e^{2\hbar d}f_i$ (cf. [8]). Using this and the relation $\mu(\pi(f_i)u_i)=\pi(S^2(f_i))\mu(u_i)$ $(1\leq i\leq N-1)$, we obtain $\mu=\operatorname{diag}(1, e^{2\hbar d}, e^{4\hbar d}, \cdots, e^{2(N-1)\hbar d})$.

Let $\mathfrak{g}$ be the Lie algebra of G. We choose a Cartan matrix of $\mathfrak{g}$ as in [4]. The corresponding positive integers $\boldsymbol{c}$ are of the form

$$\boldsymbol{c}=\begin{cases}(2c, \cdots, 2c, c) & (\mathfrak{g}=\mathfrak{so}(2l+1))\\ (c, \cdots, c, 2c) & (\mathfrak{g}=\mathfrak{sp}(2l))\\ (c, \cdots, c) & (\mathfrak{g}=\mathfrak{so}(2l))\end{cases}$$

for some $c\in\boldsymbol{Z}_{>0}$. Let $\gamma\colon A_{\hbar,\boldsymbol{d},\varphi}(SL(N))\to A_{\hbar,\boldsymbol{c},\psi}(G)$ be a Hopf algebra homomorphism. We define a representation $\pi_\gamma\colon U_{\hbar,\psi}(\mathfrak{g})\to\operatorname{End}(V)$ by $\pi_\gamma(x)u=\sum_{(u)}u_{(0)}\langle x, \gamma(u_{(1)})\rangle$ $(x\in U_{\hbar,\psi}(\mathfrak{g}),\ u\in V,\ \omega(u)=\sum_{(u)}u_{(0)}\otimes u_{(1)})$. We will show that $V_\gamma:=(V,\pi_\gamma)$ is isomorphic to a direct sum of trivial (one-dimensional) $U_{\hbar,\psi}(\mathfrak{g})$-modules. Suppose V_γ is isomorphic to the vector representation of $U_{\hbar,\psi}(\mathfrak{g})$. It is easy to see that μ is a $U_{\hbar,\psi}(\mathfrak{g})$-module isomorphism from (V,π_γ) onto $(V,\pi_\gamma\circ S^2)$. Hence, by similar arguments to the above, we see that μ is diagonalizable and that its eigenvalues are

$$\begin{cases}\{\alpha e^{4i\hbar c},\ \alpha e^{(4l+2)\hbar c},\ \alpha e^{4(i+l)\hbar c}\mid 1\leq i\leq l\} & \mathfrak{g}=\mathfrak{so}(2l+1)\\ \{\alpha e^{2i\hbar c},\ \alpha e^{2(i+l+1)\hbar c}\mid 1\leq i\leq l\} & \mathfrak{g}=\mathfrak{sp}(2l)\\ \{\alpha e^{2i\hbar c},\ \alpha e^{2(i+l-1)\hbar c}\mid 1\leq i\leq l\} & \mathfrak{g}=\mathfrak{so}(2l)\,,\end{cases}$$

where α denotes an invertible element of $\boldsymbol{C}[[\hbar]]$. This contradicts the previous formula for μ. Hence, V_γ is not isomorphic to the vector representation. On the other hand, the rank (dimension) of an irreducible representation W of $U_{\hbar,\psi}(\mathfrak{g})$ is greater than N if W is isomorphic to neither the trivial representation nor the vector representation. Hence V_γ is isomorphic to the direct sum of N copies of trivial representations. This implies $\gamma(x_{ij})=\delta_{ij}1$, where x_{ij} denotes the matrix element of $\omega\colon V\to V\otimes A_{\hbar,\varphi}(SL(N))$, that is, $\omega(u_j)=\sum_i u_i\otimes x_{ij}$. Since $A_{\hbar,\varphi}(SL(N))$ is generated by x_{ij}'s, we get $\gamma(f)=\varepsilon(f)1$ $(f\in A_{\hbar,\varphi}(SL(N))$. □

NOTE. (1) In the above, we discussed quantum groups under the Drinfeld's formal power series formulations. However, our proof is also applicable to quantum groups under the Jimbo's complex parameter formulations. Let q and q' be complex numbers which are transcendental over $\mathbf{Q}$. Let G be either $O(N)$, $SO(N)$ $(N \geq 7)$ or $Sp(N)$ $(N \geq 4,\ N\text{: even})$ and let G' be either $GL(N)$ or $SL(N)$. Let $A_{q'}(G')$ (respectively $A_q(G)$) be function algebras of quantum groups corresponding to G' and the parameter q' (respectively G and q). (See e.g. [11, 3] for a definition of these Hopf algebras.) Then, except for the map $f \mapsto \varepsilon(f)1$, there exist no Hopf algebra homomorphisms from $A_{q'}(G')$ into $A_q(G)$.

(2) In spite of our results, there still exists possibility of existence of interesting *algebra* homomorphisms from $U_\hbar(\mathfrak{so}(N))$ or $U_\hbar(\mathfrak{sp}(N))$ into $U_\hbar(\mathfrak{sl}(N))$. For example, there is an algebra homomorphism from $U_\hbar(\mathfrak{sl}(N))$ into $U_\hbar(\mathfrak{sl}(N) \oplus \mathfrak{sl}(N))$ defined by

$$H_i \mapsto H_i + H_{i+N-1}\,, \quad X_i \mapsto X_i + K_i X_{i+N-1}\,, \quad Y_i \mapsto Y_i K_{i+N-1}^{-1} + Y_{i+N-1}\,.$$

This type of homomorphisms are useful to construct representations of quantized affine algebras (cf. [4, 6]).

(3) A nice quantum deformation of the symmetric space $GL(N)/O(N)$ is constructed by [7] and [13], which is related to Macdonald symmetric polynomials.

ACKNOWLEDGEMENT. The author would like to express his thanks to Dr. S. Okada. He communicated to the author the characterization of the vector representations by their dimensions.

References

[1] E. ABE, *Hopf Algebras*, Cambridge Univ. Press, 1980.

[2] V. G. DRIN'FELD, Quantum groups, *Proc. ICM Berkeley 1986*, 798–820, Amer. Math. Soc., 1988.

[3] T. HAYASHI, Quantum deformation of classical groups, to appear in Publ. RIMS Kyoto Univ.

[4] T. HAYASHI, Q-analogues of Clifford and Weyl algebras—spinor and oscillator representations of quantum enveloping algebras, Commun. Math. Phys., **127** (1990), 129–144.

[5] M. JIMBO, A q-difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation, Lett. Math. Phys., **10** (1985), 63–69.

[6] S. KANG, K. MISRA and T. MIWA, Fock space representations of the quantized universal enveloping algebras $U_q(C_l^{(1)})$, $U_q(A_{2l}^{(2)})$ and $U_q(D_{l+1}^{(2)})$, preprint.

[7] M. NOUMI, in preparation.

[8] M. OKADO and H. YAMANE, R-matrices with gauge parameters and multi-parameter quantized enveloping algebras, preprint.

[9] N. RESHETIKHIN, Multiparameter quantum groups and twisted quasitriangular Hopf algebras, Lett. Math. Phys., **20** (1990), 331–335.

[10] A. SUDBERY, Consistent multiparameter quantisation of $GL(n)$, J. Phys. A: Math. Gen., **23** (1990), 697–704.

[11] M. TAKEUCHI, Matric bialgebras and quantum groups, Israel J. Math., **72** (1990), 232–251.

[12] T. TANISAKI, Finite dimensional representations of quantum groups, Osaka J. Math., **28** (1991), 37–53.

[13] K. UENO and T. TAKEBAYASHI, Zonal spherical functions on quantum symmetric spaces and

Macdonald's symmetric polynomials, preprint.
[14] S. L. WORONOWICZ, Tannaka-Klein duality for compact matrix pseudogroups. Twisted $SU(N)$ groups, Invent. Math., **93** (1988), 35–76.

Present Address:
DEPARTMENT OF MATHEMATICS, SCHOOL OF SCIENCE, NAGOYA UNIVERSITY
CHIKUSA-KU, NAGOYA 464, JAPAN