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Exceptional Minimal Surfaces Whose Gauss Images Have Constant Curvature

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Dedicated to Professor Y. Hatakeyama on his 60th birthday

0. Introduction.

Let M be a minimal surface in the N-dimensional Euclidean space $\boldsymbol{R}^N$ with Gaussian curvature K $(\leqq 0)$ with respect to the induced metric ds^2. We consider the Gauss map from M to the Grassmann manifold $G_{2,N}$ of 2-planes in $\boldsymbol{R}^N$, where $G_{2,N}$ may be identified with the complex quadric Q_{N-2} in the $(N-1)$-dimensional complex projective space CP^{N-1} of constant holomorphic sectional curvature 2. Then the metric $d\hat{s}_0^2$ on M induced by the Gauss map is $-Kds^2$, which is degenerate at points where $K=0$ (see [7]). Let $\hat{K}_0$ denote the Gaussian curvature of M with respect to $d\hat{s}_0^2$, which is the Gaussian curvature of the Gauss image of M. Lawson [7], Hoffman and Osserman [4] discussed minimal surfaces in $\boldsymbol{R}^N$ with constant $\hat{K}_0$. In particular, they showed that if M is a minimal surface lying fully in $\boldsymbol{R}^N$ with constant $\hat{K}_0$, then $\hat{K}_0$ must be of the form $2/m$ for some positive integer m, and $m+1\leqq N\leqq 2m+2$. Some examples of minimal surfaces in $\boldsymbol{R}^N$ with constant $\hat{K}_0$ are given in [3].

In [5] Johnson studied a class of minimal surfaces in space forms, which are called exceptional minimal surfaces. First, in this paper, we discuss exceptional minimal surfaces in $\boldsymbol{R}^N$ with constant $\hat{K}_0$.

THEOREM 1. *Let M be an exceptional minimal surface lying fully in $\boldsymbol{R}^N$ with constant $\hat{K}_0$. Then $\hat{K}_0=1/n$ when $N=2n+1$, and $\hat{K}_0=2/n$ when $N=2n+2$.*

REMARK 1. (i) We will also show that for every positive integer n, there are exceptional minimal surfaces lying fully in $\boldsymbol{R}^{2n+1}$ with $\hat{K}_0=1/n$, and in $\boldsymbol{R}^{2n+2}$ with $\hat{K}_0=2/n$.

(ii) By Theorem 1 and [3], we can find that there are non-exceptional minimal surfaces in $\boldsymbol{R}^N$ with constant $\hat{K}_0$.

Next, we deal with the case where the ambient spaces are other space forms. Let M be a minimal surface in the N-dimensional simply connected space form $X^N(c)$ of constant curvature c. We denote by K $(\leqq c)$ the Gaussian curvature of M with respect

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to the induced metric ds^2. We consider Obata's Gauss map from M to the space of all totally geodesic 2-subspaces in $X^N(c)$ (see [8]). The metric $d\hat{s}_c^2$ on M induced by the Gauss map is $(c-K)ds^2$, which is degenerate at points where $K=c$ (see [8]). Let $\hat{K}_c$ denote the Gaussian curvature of M with respect to $d\hat{s}_c^2$, which is the Gaussian curvature of the Gauss image of M. We discuss exceptional minimal surfaces in $X^6(c)$ with constant $\hat{K}_c$, where $c\neq 0$.

THEOREM 2. *Let M be an exceptional minimal surface in $X^6(c)$ with constant $\hat{K}_c$, where $c>0$. Then M has constant curvature $c/3$, $c/6$ or 0.*

THEOREM 3. *There are no exceptional minimal surfaces in $X^6(c)$ with constant $\hat{K}_c$, where $c<0$.*

REMARK 2. Bryant [1] classified minimal surfaces with constant curvature in space forms. Minimal surfaces with positive constant curvature in $X^N(c)$, where $c>0$, are parts of minimal 2-spheres. So they are exceptional (see [5] and [2]). In [9] we noted that for every positive integer n, there are flat exceptional minimal surfaces lying fully in $X^{2n+1}(c)$, where $c>0$.

1. Exceptional minimal surfaces.

In this section, we follow [5] and recall the definition of exceptional minimal surfaces. Suppose M is a minimal surface in $X^N(c)$. Assume that M lies fully in $X^N(c)$, namely, does not lie in a totally geodesic submanifold of $X^N(c)$. Let the integer n be given by $N=2n+1$ or $2n+2$, and let indices have the following ranges:

$$1\leqq i,j\leqq 2\,,\quad 3\leqq \alpha\leqq N\,,\quad 1\leqq A,B\leqq N\,.$$

Let $\tilde{e}_A$ be a local orthonormal frame field on $X^N(c)$, and let $\tilde{\theta}_A$ be the coframe dual to $\tilde{e}_A$. Then $d\tilde{\theta}_A=\sum_B \tilde{\omega}_{AB}\wedge\tilde{\theta}_B$, where $\tilde{\omega}_{AB}$ are the connection forms on $X^N(c)$.

Suppose that e_i is a local orthonormal frame field on M and that the frame $\tilde{e}_A$ is chosen so that on M, $e_i=\tilde{e}_i$ and $\tilde{e}_\alpha$ are normal to M. When forms and vectors on $X^N(c)$ are restricted to M, let them be denoted by the same symbol without tilde: $\theta_A=\tilde{\theta}_A|_M$, $\omega_{AB}=\tilde{\omega}_{AB}|_M$ and $e_A=\tilde{e}_A|_M$. Then $\omega_{\alpha i}=\sum_j h_{\alpha ij}\theta_j$, where $h_{\alpha ij}$ are the coefficients of the second fundamental form of M.

Let T_xM and $T_xX^N(c)$ denote the tangent space of M and $X^N(c)$, respectively, at a point x. Curves on M through x have their first derivatives at x in T_xM, but higher order derivatives will have components normal to M. The space spanned by the derivatives of order up to r is called the r-th osculating space of M at x, denoted $T_x^{(r)}M$.

The r-th normal space of M at x, denoted $\mathrm{Nor}_x^{(r)}M$, is the orthogonal complement of $T_x^{(r)}M$ in $T_x^{(r+1)}M$. At generic points of M, the dimension of $\mathrm{Nor}_x^{(r)}M$ is 2 when $1\leqq r\leqq n-1$, and the dimension of $\mathrm{Nor}_x^{(n)}M$ is 1 or 2, depending on whether N is odd or even. Those normal spaces that have dimension 2 are called the normal planes of M.

Let β_N denote the number of normal planes possessed by M at generic points: $\beta_N=n-1$ if $N=2n+1$, and $\beta_N=n$ if $N=2n+2$.

Choose the normal vectors e_α so that $\mathrm{Nor}_x^{(r)}M$ is spanned by $\{e_{2r+1}, e_{2r+2}\}$, where $1\leqq r\leqq\beta_N$. When $N=2n+1$, $\mathrm{Nor}_x^{(n)}M$ is spanned by $\{e_{2n+1}\}$. Set $\varphi=\theta_1+\sqrt{-1}\theta_2$. Then there are H_α such that $H_\alpha=h_{\alpha 11}+\sqrt{-1}h_{\alpha 12}$ for $\alpha=3$ and 4, for each r such that $2\leqq r\leqq\beta_N$

$$H_{2r-1}\omega_{\alpha,2r-1}+H_{2r}\omega_{\alpha,2r}=H_\alpha\bar{\varphi}$$

where $\alpha=2r+1$ and $2r+2$, and when $N=2n+1$

$$H_{2n-1}\omega_{2n+1,2n-1}+H_{2n}\omega_{2n+1,2n}=H_{2n+1}\bar{\varphi}$$

(see [5]).

The r-th normal plane, $\mathrm{Nor}_x^{(r)}M$, of M is called exceptional if $H_{2r+2}=\pm\sqrt{-1}H_{2r+1}$. The minimal surface M is called exceptional if all of its normal planes are exceptional. Note that when $N=2n+1$, $\mathrm{Nor}_x^{(n)}M$ is a line, not a plane, and the notion of exceptionality does not apply. So, every minimal surface in $X^3(c)$ is exceptional.

2. A lemma.

Let (M, ds^2) be a 2-dimensional Riemannian manifold with Gaussian curvature $K<c$. We denote by Δ the Laplacian of (M, ds^2). Set

$$A_0^c=1/2\,,\qquad A_1^c=c-K\,,$$

$$A_{p+1}^c=\begin{cases}A_p^c[\Delta\log(A_p^c)+A_p^c/A_{p-1}^c-2(p+1)K]\,, & \text{if } A_p^c>0\,,\\ 0\,, & \text{otherwise}\,.\end{cases}\tag{1}$$

Let $\hat{K}_c$ be the Gaussian curvature of M with respect to the metric $d\hat{s}_c^2=(c-K)ds^2$. Then

$$\hat{K}_c=\frac{K}{c-K}-\frac{1}{2(c-K)}\Delta\log(c-K)\,.\tag{2}$$

Now suppose that $c=0$ and $\hat{K}_0=2/m$, where m is a positive integer. Then by (2), we have

$$\Delta\log(-K)=2\left(1+\frac{2}{m}\right)K\,.\tag{3}$$

LEMMA. *Under the hypothesis above,*

$$A_p^0=p((p-1)!)^2\left\{\prod_{k=1}^{p-1}\left(\frac{2}{k}-\frac{2}{m}\right)\right\}(-K)^p\qquad \textit{for } 2\leqq p\leqq m+1\,,$$

and

$$A_p^0=0 \qquad \textit{for } p\geqq m+1 .$$

PROOF. By (1) and (3), we have

$$\begin{aligned} A_2^0&=(-K)[\Delta\log(-K)-2K-4K] \\ &=2\left(2-\frac{2}{m}\right)(-K)^2 , \end{aligned}$$

and

$$\begin{aligned} A_3^0&=2\left(2-\frac{2}{m}\right)(-K)^2\left[2\Delta\log(-K)-2\left(2-\frac{2}{m}\right)K-6K\right] \\ &=12\left(2-\frac{2}{m}\right)\left(1-\frac{2}{m}\right)(-K)^3 . \end{aligned}$$

So the lemma is true for $p=2$ and 3. Assume that the lemma is true for p and $p+1$, where $2\leqq p\leqq m-1$. Then by (1), (3) and the assumption,

$$\begin{aligned} A_{p+2}^0&=(p+1)(p!)^2\left\{\prod_{k=1}^{p}\left(\frac{2}{k}-\frac{2}{m}\right)\right\}(-K)^{p+1} \\ &\quad\times\left[(p+1)\Delta\log(-K)-p(p+1)\left(\frac{2}{p}-\frac{2}{m}\right)K-2(p+2)K\right] \\ &=(p+2)((p+1)!)^2\left\{\prod_{k=1}^{p+1}\left(\frac{2}{k}-\frac{2}{m}\right)\right\}(-K)^{p+2} . \end{aligned}$$

So the lemma is true for $p+2$. Therefore, by induction, the lemma is true for $2\leqq p\leqq m+1$. Thus we have $A_{m+1}^0=0$, and by (1) we have $A_p^0=0$ for $p\geqq m+1$. Q.E.D.

3. Proof of Theorem 1.

PROOF OF THEOREM 1. Let ds^2 and K be as in Section 0. We assume that $K<0$ in the theorem because $\hat{K}_0$ cannot be defined at points where $K=0$. Let Δ and A_p^0 be as in Section 1. By [4], $\hat{K}_0=2/m$ for some positive integer m. So the equation (3) and Lemma are valid.

When $N=2n+1$, by Theorem A of [5], $A_p^0\geqq 0$ for $1\leqq p\leqq n$ with equality only at isolated points, and the metric $(A_n^0)^{1/(n+1)}ds^2$ is flat at points where $A_p^0>0$ for $1\leqq p\leqq n$. So by Lemma, we find that $m\geqq n$ and

$$A_n^0=n((n-1)!)^2\left\{\prod_{k=1}^{n-1}\left(\frac{2}{k}-\frac{2}{m}\right)\right\}(-K)^n .$$

Using the lemma in Section 3 of [5] and the equation (3), we have

$$0=\Delta\log(A_n^0)-2(n+1)K=\left(\frac{4n}{m}-2\right)K\,.$$

Thus we have $m=2n$, and $\hat{K}_0=1/n$.

When $N=2n+2$, by Theorem A of [5], $A_p^0\geqq 0$ for $1\leqq p\leqq n$ with equality only at isolated points, and $A_{n+1}^0=0$ identically. So by Lemma, we have $m=n$, and $\hat{K}_0=2/n$.

Q.E.D.

We shall show the fact in Remark 1 (i). Let (M, ds^2) be a 2-dimensional Riemannian manifold with Gaussian curvature $K<0$. Let Δ, A_p^0 and $\hat{K}_0$ be defined as in Section 1.

First suppose that $\hat{K}_0=1/n$, where n is a positive integer. We note that there are such 2-dimensional Riemannian manifolds. Then the equation (3) and Lemma are valid for $m=2n$. So $A_p^0>0$ for $p\leqq 2n$. Using Lemma and (3) with $m=2n$, we have

$$\Delta\log(A_n^0)-2(n+1)K=0\,.$$

By the lemma in Section 3 of [5], the metric $(A_n^0)^{1/(n+1)}ds^2$ is flat. By Theorem B of [5], (M, ds^2) can be realized locally as an exceptional minimal surface lying fully in $\boldsymbol{R}^{2n+1}$. Therefore, for every positive integer n, there are exceptional minimal surfaces lying fully in $\boldsymbol{R}^{2n+1}$ with $\hat{K}_0=1/n$.

Next suppose that $\hat{K}_0=2/n$, where n is a positive integer. Then the lemma is valid for $m=n$. So $A_p^0>0$ for $p\leqq n$ and $A_{n+1}^0=0$. By Theorem B of [5], (M, ds^2) can be realized locally as an exeptional minimal surface lying fully in $\boldsymbol{R}^{2n+2}$. Therefore, for every positive integer n, there are exceptional minimal surfaces lying fully in $\boldsymbol{R}^{2n+2}$ with $\hat{K}_0=2/n$.

4. Proof of Theorems 2 and 3.

In this section we prove the following proposition. Combining the proposition with [1], we have Theorems 2 and 3.

PROPOSITION. *Let M be an exceptional minimal surface in $X^6(c)$ with constant $\hat{K}_c$, where $c\neq 0$. Then M has constant curvature.*

PROOF. Let ds^2 and K be as in Section 0. We assume that $K<c$ in the proposition because $\hat{K}_c$ cannot be defined at points where $K=c$. Let Δ and A_p^c be as in Section 1. We assume that $\hat{K}_c=a$. Then by (2), we have

$$\Delta\log(c-K)=2\{(a+1)K-ca\}\,. \tag{4}$$

By (1) and (4),

$$\begin{aligned} A_2^c&=(c-K)[\Delta\log(c-K)+2(c-K)-4K]\\ &=2(c-K)\{(a-2)K-c(a-1)\}\,. \end{aligned} \tag{5}$$

Set $M_1=\{x\in M\,;\,A_2^c>0\}$. By (1), (4) and (5),

$$(6)\quad A_3^c=A_2^c[\Delta\log(c-K)+\Delta\log\{(a-2)K-c(a-1)\}+2\{(a-2)K-c(a-1)\}-6K]$$
$$=A_2^c[\Delta\log\{(a-2)K-c(a-1)\}+2\{2(a-2)K-c(2a-1)\}]$$

on M_1.

Now suppose that M lies fully in $X^N(c)$ where $3\leqq N\leqq 6$. When $N=3$, by Theorem A and the lemma in Section 3 of [5],

$$(7)\qquad \Delta\log(c-K)=4K\,.$$

By (4) and (7), we can see that K is constant. When $N=4$, $A_2^c=0$ identically by Theorem A of [5]. Then by (5), we can see that K is constant.

When $N=5$, by Theorem A of [5], M_1 is M minus isolated points and the metric $(A_2^c)^{1/3}ds^2$ is flat on M_1. Using the lemma in Section 3 of [5], the equations (4) and (5), we have

$$(8)\qquad 0=\Delta\log(A_2^c)-6K$$
$$=\Delta\log(c-K)+\Delta\log\{(a-2)K-c(a-1)\}-6K$$
$$=\Delta\log\{(a-2)K-c(a-1)\}+2\{(a-2)K-ca\}$$

on M_1. By (8) we can see that $a\neq 2$. By (4) and (8), we have

$$\Delta K=F(K)=b_0+b_1K+b_2K^2+b_3K^3$$

and

$$|\nabla K|^2=G(K)=b_4+b_5K+b_6K^2+b_7K^3+b_3K^4$$

on M_1 and, by continuity, on M, where

$$b_0=-\frac{2c^2a(2a^2-6a+5)}{a-2}\,,\qquad b_1=2c(6a^2-9a-1)\,,$$

$$b_2=-12a(a-2)\,,\qquad b_3=\frac{2(a-2)(2a-1)}{c}\,,$$

$$b_4=\frac{2c^3a(a-1)(2a-3)}{a-2}\,,\qquad b_5=-\frac{2c^2a(8a^2-24a+17)}{a-2}\,,$$

$$b_6=2c(12a^2-18a+1)\,,\qquad b_7=-2(8a^2-16a+3)\,.$$

If K is not constant, then

$$(9)\qquad GK+(F-G')\left(F-\frac{1}{2}G'\right)+G\left(F'-\frac{1}{2}G''\right)=0\,,$$

where the prime denotes the differentiation with respect to K (see for example [6, p. 136]). The left-hand side of (9) is a polynomial of K such that the coefficient of K^5 is $-16(a-2)(2a-1)/c$ and the constant term is $-8c^4a(a-1)(5a-3)/(a-2)$. So it is a nontrivial polynomial. Thus K must be constant, which is a contradiction. Therefore, K is constant.

When $N=6$, by Theorem A of [5], M_1 is M minus isolated points and $A_3^c=0$ identically. By (6) we have

$$\Delta \log\{(a-2)K-c(a-1)\}+2\{2(a-2)K-c(2a-1)\}=0 \tag{10}$$

on M_1. By (10) we can see that $a\neq 2$. By (4) and (10), we have

$$\Delta K=P(K)=d_0+d_1K+d_2K^2+d_3K^3$$

and

$$|\nabla K|^2=Q(K)=d_4+d_5K+d_6K^2+d_7K^3+d_3K^4$$

on M_1 and, by continuity, on M, where

$$d_0=-\frac{2c^2(3a^3-9a^2+8a-1)}{a-2}, \qquad d_1=2c(9a^2-15a+2),$$

$$d_2=-6(a-2)(3a-1), \qquad d_3=\frac{6(a-1)(a-2)}{c},$$

$$d_4=\frac{2c^3(a-1)(3a^2-5a+1)}{a-2}, \qquad d_5=-\frac{2c^2(12a^3-39a^2+35a-7)}{a-2},$$

$$d_6=2c(18a^2-33a+10), \qquad d_7=-2(12a^2-29a+13).$$

If K is not constant, then

$$QK+(P-Q')\left(P-\frac{1}{2}Q'\right)+Q\left(P'-\frac{1}{2}Q''\right)=0 \tag{11}$$

(similar to (9)). The left-hand side of (11) is a polynomial of K such that the coefficient of K^5 is $6(a-1)(a-2)(3a-14)/c$ and the constant term is $-2c^4(a-1)(9a^3+12a^2-14a+1)/(a-2)$. When $a=1$, the coefficient of K is $16c^3\neq 0$. So the left-hand side of (11) is a nontrivial polynomial. Thus K must be constant, which is a contradiction. Therefore, K is constant. Q.E.D.

Minimal 2-spheres in $X^N(c)$, where $c>0$, are always exceptional (see [5] and [2]). So, by Theorem 2, we have the following:

COROLLARY. *Let M be a minimal 2-sphere in $X^6(c)$ with constant $\hat{K}_c$, where $c>0$. Then M has constant curvature $c/3$ or $c/6$.*

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