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A Characterization of the Poisson Kernel on the Classical Real Rank One Symmetric Spaces

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1. Introduction.

Let G be a connected real rank one semisimple Lie group and $G=KAN$ an Iwasawa decomposition for G. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}+\mathfrak{n}$ denote the corresponding Iwasawa decomposition for the Lie algebra $\mathfrak{g}$ of G. We define $H(x)$ $(x\in G)$ as the unique element in $\mathfrak{a}$ such that $x\in K\exp H(x)N$ and put $\rho(H)=\mathrm{tr}(\mathrm{ad}(H)|_{\mathfrak{n}})$ for $H\in\mathfrak{a}$. Let M be the centralizer of A in K and Δ the Laplace-Beltrami operator on G/K (cf. [H1], p. 386). Then the Poisson kernel $P: G/K\times K/M\to \boldsymbol{R}$ is given by $P(gK, kM)=\exp(-2\rho(H(g^{-1}k)))$ and for each $s\in\boldsymbol{C}$ P^s is an eigenfunction of Δ with eigenvalue, say, λ_s. Then our problem can be stated as follows.

PROBLEM. Let us suppose that a real valued, C^2 function F on G/K satisfies the following conditions:

(1) $\quad \Delta F=0\,,$

(2) $\quad \Delta F^2=\lambda_2 F^2\,,$

(3) $\quad F(eK)=1\,.$

Then is F determined by $F(gK)=P(gK, kM)$ for an element kM in K/M?

When $G=SO_0(n, 1)$ and $SU(n, 1)$, the affirmative answers are obtained by [CET], [T] and [KT]. However, their arguments are slightly dependent on each circumstance of the group. Therefore, it is worthy to give another proof which treats the problem simultaneously, and apply it to the case of $Sp(n, 1)$. Now let G be one of the classical real rank one semisimple Lie groups: $SO_0(n, 1)$, $SU(n, 1)$ and $Sp(n, 1)$. In §4 we shall prove that if a real valued, C^2 function F on G/K satisfies (1)–(3) and

(4) $\quad F$ is M-invariant ,

then F is determined by the Poisson kernels. In §5 we shall try to remove this additional

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condition. Although we can remove it in the cases $SO_0(n, 1)$ and $SU(n, 1)$, we need an assumption for the case of $Sp(n, 1)$, which can be stated as follows. We first identify G/K ($G=Sp(n, 1)$ and $K=Sp(n)\times Sp(1)$) with the unit ball D^n in $\boldsymbol{H}^n$ where $\boldsymbol{H}$ is the quaternions, and denote the standard coordinates on $\boldsymbol{H}^n$ by $w_1, w_2, \cdots, w_n$. Since F is real analytic from (1), F has a homogeneous expansion $F=\sum_{N=0}^{\infty} F_N$, where each F_N is a homogeneous polynomial with degree N. Then we easily see that there exists an element k_0 in K such that the rotation F_{k_0} of F defined by $F_{k_0}(w)=F(k_0\cdot w)$ ($w\in D^n$) satisfies

$$(F_{k_0})_1(w)=(2n+1)(w_1+\bar{w}_1)$$

(see (5.5)). Then the assumption which we need to obtain the affirmative answer of Problem is the following,

(5) $F_{k_0}(cw_1, w_2, \cdots, w_n)=F_{k_0}(w_1 c, w_2, \cdots, w_n)$ for all c in $\boldsymbol{H}$, i.e., F_{k_0} is a function of $w_1+\bar{w}_1, |w_1|^2$ and $w_s+\bar{w}_s$, $w_s i-i\bar{w}_s$, $w_s j-j\bar{w}_s$, $w_s k-k\bar{w}_s$ $(2\leq s\leq n)$.

Then if F satisfies (1)–(3) and (5), $F(gK)$ is determined by the Poisson kernel $P(gK, k_0M)$. Since $\boldsymbol{R}$ and $\boldsymbol{C}$ are abelian, the condition corresponding to (5) in the cases of $G=SO_0(n, 1)$ and $SU(n, 1)$ holds automatically, and thus, there is no need to assume (5).

2. The classical real rank one symmetric spaces.

Let F be one of $\boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{H}$ (the quaternions) and $x\to\bar{x}$ ($x\in \boldsymbol{F}$) the standard involution on $\boldsymbol{F}$. We put $|x|^2=x\bar{x}$. We consider $\boldsymbol{F}^{n+1}$ as a right vector space over $\boldsymbol{F}$ and define the quadratic form $Q(x)=|x_1|^2+|x_2|^2+\cdots+|x_n|^2-|x_{n+1}|^2$ for $x=(x_1, x_2, \cdots, x_{n+1})\in \boldsymbol{F}^{n+1}$. Then the connected component G of the group of all $\boldsymbol{F}$-linear transformations of $\boldsymbol{F}^{n+1}$ which are of determinant one, except for the case of $\boldsymbol{F}=\boldsymbol{H}$, and preserve Q is given as follows.

$$\begin{aligned}
&(1)\quad \text{If } \boldsymbol{F}=\boldsymbol{R},\ G=SO_0(n, 1)\,.\\
&(2)\quad \text{If } \boldsymbol{F}=\boldsymbol{C},\ G=SU(n, 1)\,.\\
&(3)\quad \text{If } \boldsymbol{F}=\boldsymbol{H},\ G=Sp(n, 1)\,.
\end{aligned}\tag{2.1}$$

Let $G=KAN$ be an Iwasawa decomposition for G where K is the maximal compact subgroup and A is the vector subgroup, respectively, consisting of all matrices in G of the form

$$k_{B,b}=\begin{pmatrix} B & 0\\ 0 & b\end{pmatrix}\tag{2.2a}$$

and

$$a_t=\begin{pmatrix}\cosh t & 0 & \sinh t\\ 0 & I_{n-1} & 0\\ \sinh t & 0 & \cosh t\end{pmatrix} \qquad (t\in \boldsymbol{R}), \tag{2.2b}$$

where B is an (n, n) matrix and $b\in \boldsymbol{F}$. We put $A^+=\{a_t\,;\, t>0\}$. Then the Cartan decomposition for G is given by $G=KCL(A^+)K$. We define $H: G\to \mathfrak{a}$ and $\rho: \mathfrak{a}\to \boldsymbol{R}$ as in §1. Let M be the centralizer of A in K. Then M consists of all matrices in G of the form

$$\begin{pmatrix}b & 0 & 0\\ 0 & B & 0\\ 0 & 0 & b\end{pmatrix}, \tag{2.3}$$

where B is an $(n-1, n-1)$ matrix, $b\in \boldsymbol{F}$ and $|b|^2=1$.

If $x=(x_1, x_2, \cdots, x_n)$ and $y=(y_1, y_2, \cdots, y_n)$ are in $\boldsymbol{F}^n$, we put $\langle x, y\rangle=\bar{x}_1y_1+\bar{x}_2y_2+\cdots+\bar{x}_ny_n$ and $\|x\|^2=\langle x, x\rangle$. Then G/K and K/M are identified with the unit ball $D^n=\{x\in \boldsymbol{F}^n\,;\, \|x\|^2<1\}$ and its boundary S^{nd-1}, the unit sphere in $\boldsymbol{F}^n$, respectively where $d=\dim_{\boldsymbol{R}}\boldsymbol{F}$. It is easy to see that G acts transitively on D^n and S^{nd-1} as follows.

$$g\cdot(x_1, x_2, \cdots, x_n)=(y_1y_{n+1}^{-1}, \cdots, y_ny_{n+1}^{-1}), \tag{2.4}$$

where $g(x_1, \cdots, x_n, 1)=(y_1, \cdots, y_n, y_{n+1})$.

The Poisson kernel is the function $P: G/K\times K/M\to \boldsymbol{R}$ given by

$$\begin{aligned}P(gK, kM)&=\exp(-2\rho(H(g^{-1}k)))\\ &=\left(\frac{1-\|z\|^2}{|1-\langle z, b\rangle|^2}\right)^l,\end{aligned} \tag{2.5}$$

where $z=gK$, $b=kM$ and $l=d(n+1)/2-1$. Then as a function on G/K, P^s $(s\in \boldsymbol{C})$ is an eigenfunction of the Laplace-Beltrami operator Δ on G/K with eigenvalue, say, λ_s. We easily see that $\lambda_1=0$ and $\lambda_2=2l^2$.

Now let $\zeta_1, \zeta_2, \cdots, \zeta_n$ denote the standard coordinates on $\boldsymbol{F}^n$. When we distinguish the coordinates according to the type of $\boldsymbol{F}$, we shall use the following notations.

$$\begin{aligned}&\text{If } \boldsymbol{F}=\boldsymbol{R}, \quad x_1, \cdots, x_n,\\ &\text{If } \boldsymbol{F}=\boldsymbol{C}, \quad z_1, \cdots, z_n \quad (z_s=x_s+ix_{n+s}),\\ &\text{If } \boldsymbol{F}=\boldsymbol{H}, \quad w_1, \cdots, w_n \quad (w_s=x_s+ix_{n+s}+jx_{2n+s}+kx_{3n+s}),\end{aligned} \tag{2.6}$$

where $x_s\in \boldsymbol{R}$ $(1\leq s\leq nd)$. Moreover, the polar coordinates, which will be useful to express spherical harmonics (see §3), are given as follows.

$$\text{If } \boldsymbol{F}=\boldsymbol{R}, \quad \begin{array}{l} x_1 = r\cos\xi \\ x_i = r\sigma_i \sin\xi \ (2\le i\le n) \end{array} ; \quad \sum_{i=2}^{n} \sigma_i^2 = 1,\ 0\le\xi\le\pi .$$

$$\text{If } \boldsymbol{F}=\boldsymbol{C}, \quad \begin{array}{l} z_1 = r\cos\xi\, e^{\sqrt{-1}\phi} \\ z_i = r\sigma_i \sin\xi \ (2\le i\le n) \end{array} ; \quad \sum_{i=2}^{n} |\sigma_i|^2 = 1,\ 0\le\xi\le\frac{\pi}{2},\ 0\le\phi\le 2\pi . \tag{2.7}$$

$$\text{If } \boldsymbol{F}=\boldsymbol{H}, \quad \begin{array}{l} w_1 = r\cos\xi\,(\cos\phi + y\sin\phi) \\ w_i = r\sigma_i \sin\xi \ (2\le i\le n) \end{array} ; \quad \begin{array}{l} \sum_{i=2}^{n} |\sigma_i|^2 = 1,\ 0\le\xi\le\frac{\pi}{2},\ 0\le\phi\le 2\pi , \\ y\in \boldsymbol{F},\ \Re(y)=0,\ |y|^2=1 . \end{array}$$

Then it is easy to see that if a function F on $\boldsymbol{F}^n$ is M-invariant, F is a function of only r, ξ when $\boldsymbol{F}=\boldsymbol{R}$ and only r, ξ, ϕ when $\boldsymbol{F}=\boldsymbol{C}$ and $\boldsymbol{H}$. Especially, if $\boldsymbol{F}=\boldsymbol{H}$, F is a function of r, $r_1=w_1+\bar{w}_1$ and $r_2^2=|w_1|^2$.

Let Δ_0 be the standard Laplacian on $\boldsymbol{F}^n$, that is,

$$\begin{aligned} \Delta_0 &= \sum_{i=1}^{nd} \frac{\partial^2}{\partial x_i^2} \\ &= \sum_{i=1}^{n} \frac{\partial^2}{\partial z_i \partial \bar{z}_i} \qquad \text{if} \quad \boldsymbol{F}=\boldsymbol{C} . \end{aligned} \tag{2.8}$$

Then we easily see that Δ can be written as

$$\Delta = \Delta_0 + \Delta_+ , \tag{2.9}$$

where Δ_+ does not contain a constant term and does not decrease the homogeneous order with respect to x_s $(1\le s\le nd)$ when it acts on a homogeneous polynomial (see [H1], p. 387).[1] Here we put

$$\nabla(f, g) = \Delta(fg) - \Delta f\cdot g - f\cdot \Delta g \tag{2.10a}$$

and

$$|\nabla|^2(f) = \nabla(f, f) \tag{2.10b}$$

for functions f and g on $\boldsymbol{F}^n$. Moreover, we shall define ∇_0 and $|\nabla_0|^2$ by replacing Δ with Δ_0.

3. Spherical harmonics.

Let $\hat{K}$ denote the equivalence classes of irreducible unitary representations of K and put $\hat{K}_M=\{(\tau, V_\tau)\in\hat{K};\ \dim V_\tau^M\neq 0\}$, where V_τ^M denotes the subspace of V_τ consisting of M-fixed vectors. Since G is of split rank one, $\dim V_\tau^M=1$ if $(\tau, V_\tau)\in\hat{K}_M$, and the Pater-Weyl's theorem insists that,

[1] The explicit form of Δ with respect to the coordinate x_s $(1\le s\le nd)$ can be found in [T], p. 3 and p. 21, respectively, for the cases of $G=SO_0(n, 1)$ and $SU(n, 1)$, and the one for the case of $Sp(n, 1)$ is pointed out to me by R. Takahashi.

$$L^2(S^{nd-1})=\sum_{\tau\in\hat{K}_M} V_\tau \tag{3.1}$$

as a representation space of K. Actually, the parametrization of $\tau\in\hat{K}_M$ and spherical harmonics which span V_τ^M are given as follows (see [JW]).

$$\begin{aligned}
&\text{If } \boldsymbol{F}=\boldsymbol{R}, \quad C\phi_{p0}=Cr^p\cos^p\xi\, F\left(-\frac{p}{2}, \frac{1-p}{2}; \frac{n-1}{2}; -\tan^2\xi\right)\\
&\qquad\qquad = Cx_1^p F(\cdots).\\
&\text{If } \boldsymbol{F}=\boldsymbol{C}, \quad C\phi_{pq}=Cr^{p+q}\cos^{p+q}\xi\, e^{\sqrt{-1}(p-q)\phi}F(-p, -q; n-1; -\tan^2\xi)\\
&\qquad\qquad = Cz_1^p\bar{z}_1^q F(\cdots).\\
&\text{If } \boldsymbol{F}=\boldsymbol{H}, \quad C\phi_{pq}=Cr^p\cos^p\xi\, \sin((q+1)\phi)\sin^{-1}\phi\\
&\qquad\qquad \cdot F\left(\frac{q-p}{2}, \frac{-p-q-2}{2}; 2(n-1); -\tan^2\xi\right)\\
&\qquad\qquad = Cr_1^q r_2^{p-q}\left(\sum_{i=0}^{[q/2]}(-1)^i 2^{q-2i}\binom{q-i}{i} r_1^{-2i} r_2^{2i}\right)F(\cdots),
\end{aligned} \tag{3.2}$$

where $r_1=w_1+\bar{w}_1=2x_1$ and $r_2^2=|w_1|^2=x_1^2+x_{n+1}^2+x_{2n+1}^2+x_{3n+1}^2$. Here the indices (p, q) move as

$$\Lambda=\begin{cases} (p, 0); \quad p=0, 1, 2, \cdots & (\boldsymbol{F}=\boldsymbol{R})\\ (p, q); \quad p, q=0, 1, 2, \cdots & (\boldsymbol{F}=\boldsymbol{C})\\ (p, q); \quad p, q=0, 1, 2, \cdots, \ p\geq q, \ p-q\in 2N & (\boldsymbol{F}=\boldsymbol{H}). \end{cases} \tag{3.3}$$

We put $\Lambda_0=\Lambda$ if $\boldsymbol{F}=\boldsymbol{R}$, $\boldsymbol{C}$, and $N\times N$ if $\boldsymbol{F}=\boldsymbol{H}$, and for $(p, q)\in\Lambda_0$ we put

$$e_{pq}(\zeta)=\begin{cases} x_1^p & (\boldsymbol{F}=\boldsymbol{R})\\ z_1^p\bar{z}_1^q & (\boldsymbol{F}=\boldsymbol{C})\\ r_1^p r_2^{2q} & (\boldsymbol{F}=\boldsymbol{H}). \end{cases} \tag{3.4}$$

Then it is easy to see that

$$e_{pq}\cos^{-2}\xi=r^2\begin{cases} e_{p-2,0} & (\boldsymbol{F}=\boldsymbol{R})\\ e_{p-1,q-1} & (\boldsymbol{F}=\boldsymbol{C})\\ e_{p,q-1} & (\boldsymbol{F}=\boldsymbol{H}). \end{cases} \tag{3.5}$$

Let v be a non-zero M-fixed vector in V_τ^M ($\tau\in\hat{K}_M$). Then the corresponding matrix coefficient $(\tau(k)v, v)$ ($k\in K$) is an M-invariant spherical function on K. We know that these matrix coefficients are nothing but the above spherical harmonics restricted on S^{nd-1}. For simplicity, we put

$$\dot{\zeta}=\frac{\zeta}{\|\zeta\|}=\left(\frac{\zeta_1}{\|\zeta\|},\cdots,\frac{\zeta_n}{\|\zeta\|}\right) \qquad (\zeta\in D^n)\,. \tag{3.6}$$

4. A property of M-invariant eigenfunctions of Δ.

Before we state a proposition, we shall recall some properties of M-invariant Poisson kernels. We easily see that the M-invariant Poisson kernels are of the form

$$\begin{aligned} P_c(\zeta)&=P_c(g)=P(gK,k_cM)\\ &=(1-r^2)^l|1-c\zeta_1|^{-2l}, \end{aligned} \tag{4.1}$$

where $\zeta=gK, r^2=\|\zeta\|^2$ and

$$k_c=\begin{pmatrix}\bar{c} & 0 & 0\\ 0 & c & 0\\ 0 & 0 & I_{n-1}\end{pmatrix}, \qquad c=\begin{cases}\pm 1 & (F=R, H)\\ c\in C, |c|=1 & (F=C)\,.\end{cases} \tag{4.2}$$

By using the Peter-Weyl's theorem, we can give an expansion of P_1 as follows.

$$P_1(\zeta)=\sum_{(p,q)\in\Lambda}P_{pq}(r)\phi_{pq}(\dot{\zeta})\,, \tag{4.3}$$

where $r=\tanh t$ and $P_{pq}(r)=\int_K P_1(ka_t)\bar{\phi}_{pq}(k)dk$. Moreover, since $P_c(\zeta)=P_1(k_c\cdot\zeta)$ for c in (4.2), it follows that

$$P_c(\zeta)=\sum_{(p,q)\in\Lambda}c_{pq}P_{pq}(r)\phi_{pq}(\dot{\zeta})\,, \tag{4.4}$$

where $c_{pq}=c^{p-q}$ when $F=R$, C and c^q when $F=H$. Here we recall the relations (3.2) and (3.5). They by substituting them for (4.4), we can find functions Q^c_{pq} of r such that

$$P_c(\zeta)=\sum_{(p,q)\in\Lambda_0}Q^c_{pq}(r)e_{pq}(\zeta)\,. \tag{4.5}$$

Obviously, since P_c is real valued, it follows that

$$\begin{aligned} &\text{if } F=R, H, \quad c_{pq}\in R \text{ and } Q^c_{pq} \text{ is real valued and}\\ &\text{if } F=C, \quad c_{pq}=\bar{c}_{qp} \text{ and } Q^c_{pq}=\bar{Q}^c_{qp}\,. \end{aligned} \tag{4.6}$$

Our first result can be stated as follows.

PROPOSITION 4.1. *Let us suppose that a real valued C^2 function F on G/K satisfies the following conditions*:

(1) $\Delta F=0$,

(2) $\Delta F^2=\lambda_2 F^2$,

(3) $F(eK)=1$,

(4) F *is* M-*invariant*.

Then F is an M-invariant Poisson kernel, that is, there exists a c in (4.2) *such that* $F=P_c$.

PROOF. Since F satisfies (1) and (4), it follows from (4.3) and Helgason's theorem (see [H2], p. 133 and [H3], p. 333) that there exist constants C_{pq} such that

$$F(\zeta)=\sum_{(p,q)\in\Lambda} C_{pq}P_{pq}(r)\phi_{pq}(\dot{\zeta}). \tag{4.7}$$

Then by the same argument which deduces (4.5) from (4.4) we can rewrite it as

$$F(\zeta)=\sum_{(p,q)\in\Lambda_0} \tilde{Q}_{pq}(r)e_{pq}(\zeta). \tag{4.8}$$

Obviously, since F is real valued, it follows that

$$\begin{aligned}&\text{if } \boldsymbol{F}=\boldsymbol{R}, \boldsymbol{H}, \quad C_{pq}\in\boldsymbol{R} \text{ and } \tilde{Q}_{pq} \text{ is real valued and}\\ &\text{if } \boldsymbol{F}=\boldsymbol{C}, \quad C_{pq}=\bar{C}_{qp} \text{ and } \tilde{Q}_{pq}=\bar{\tilde{Q}}_{qp}.\end{aligned} \tag{4.9}$$

Here we note that $\tilde{Q}_{00}=C_{00}P_{00}=C_{00}Q^c_{00}$, Q^c_{00} does not depend on c, and $1=F(eK)=C_{00}$ by (3). Therefore, we see that

$$\tilde{Q}_{00}=Q^c_{00}. \tag{4.10}$$

Next it follows from (1) and (2) that

$$\lambda_2F^2=|\nabla|^2(F). \tag{4.11}$$

Here we recall (2.9). Then if we put $\zeta=0$ in (4.1), we see from (3) and (4.8) that

$$\begin{aligned}\lambda_2&=|\nabla_0|^2(F)(0)\\ &=2\sum_{i=1}^{nd}\left(\frac{\partial F}{\partial x_i}\right)^2(0)\ \left(=2\sum_{i=1}^{n}\left|\frac{\partial F}{\partial z_i}\right|^2(0) \text{ if } \boldsymbol{F}=\boldsymbol{C}\right)=2|\tilde{Q}_{10}(0)|^2.\end{aligned} \tag{4.12}$$

Since P_1 also satisfies (1)–(4), we can obtain that $\lambda_2=2|Q^1_{10}(0)|^2$ by replacing F with P_1. Therefore, noting (4.6) and (4.9), we can choose a c in (4.2) such that

$$\tilde{Q}_{10}(0)=cQ^1_{10}(0)=Q^c_{10}(0). \tag{4.13}$$

Here the last equality follows from (4.4), (4.5) and the fact that $e_{10}(\zeta)=\phi_{10}(\dot{\zeta})$ when $\boldsymbol{F}=\boldsymbol{R}$, $\boldsymbol{C}$ and $\phi_{11}(\dot{\zeta})$ when $\boldsymbol{F}=\boldsymbol{H}$. Moreover, by using this fact and the orthogonality of ϕ_{pq} ($(p,q)\in\Lambda$) on K/M, we can deduce that

$$\tilde{Q}_{10}(r)r=\begin{cases}C_{10}P_{10}(r) & (\boldsymbol{F}=\boldsymbol{R}, \boldsymbol{C})\\ C_{11}P_{11}(r) & (\boldsymbol{F}=\boldsymbol{H})\end{cases} \tag{4.14a}$$

and

$$Q^c_{10}(r)r=\begin{cases} cP_{10}(r) & (F=\boldsymbol{R}, \boldsymbol{C}) \\ cP_{11}(r) & (F=\boldsymbol{H}). \end{cases} \tag{4.14b}$$

Therefore, (4.13) means that

$$\tilde{Q}_{10}=Q^c_{10}. \tag{4.15}$$

Now we shall show that $\tilde{Q}_{pq}=Q^c_{pq}$ for all $(p, q)\in\Lambda_0$ by induction. Obviously, this is nothing but our desired result: $F=P_c$. To accomplish the induction we shall compare with coefficients of e_{pq} in the equations (1) and (4.11). At that time we shall use (2.9) and the explicit forms of the actions of Δ_0 and $|\nabla_0|^2$:

$$\Delta_0\Big(\sum_{p,q}\tilde{Q}_{pq}(r)e_{pq}\Big)=\sum_{p,q}\Big(\frac{\partial^2\tilde{Q}_{pq}}{\partial r^2}+(nd-1)\frac{1}{r}\frac{\partial\tilde{Q}_{pq}}{\partial r}\Big)e_{pq} \\ +\sum_{p,q}\sum_{i=1}^{nd}\frac{x_i}{r}\frac{\partial\tilde{Q}_{pq}}{\partial r}\frac{\partial e_{pq}}{\partial x_i}+\sum_{p,q}\tilde{Q}_{pq}\Delta_0 e_{pq} \tag{4.16}$$

and

$$|\nabla_0|^2\Big(\sum_{p,q}\tilde{Q}_{pq}(r)e_{pq}\Big)=2\sum_{i=1}^{nd}\Big(\sum_{p,q}\frac{x_i}{r}\frac{\partial\tilde{Q}_{pq}}{\partial r}e_{pq}+\sum_{p,q}\tilde{Q}_{pq}\frac{\partial e_{pq}}{\partial x_i}\Big)^2 \tag{4.17}$$

respectively.

Case of $\boldsymbol{R}$. Let us suppose that $\tilde{Q}_{p0}=Q^c_{p0}$ $(0\le p\le N)$. We note that

$$\Delta_0 e_{p0}=p(p-1)e_{p-2,0} \quad\text{and}\quad x_i\frac{\partial e_{p0}}{\partial x_i}=\delta_{i1}pe_{p0} \ (1\le i\le n).$$

Then by applying (4.16) to (1) and comparing with the coefficients of $e_{N-1,0}$ in (1), we can deduce that

$$N(N+1)\tilde{Q}_{N+1,0}=\text{a function of } \tilde{Q}_{i0} \qquad (i\le N).$$

Since P_c also satisfies (1)–(4), it is easy to see that the same relation with Q^c is also valid. Therefore, it follows from the induction hypothesis that

$$N(N+1)(\tilde{Q}_{N+1,0}-Q^c_{N+1,0})=0,$$

that is, $\tilde{Q}_{N+1,0}=Q^c_{N+1,0}$. Then by (4.10), (4.15) and the induction we can obtain the desired result.

Case of $\boldsymbol{C}$. Let us suppose that $\tilde{Q}_{pq}=Q^c_{pq}$ $(0\le p+q\le N)$. We note that

$$\Delta_0 e_{pq}=pqe_{p-1,q-1} \quad\text{and}\quad z_i\frac{\partial e_{pq}}{\partial z_i}=\delta_{i1}pe_{pq}.$$

Then by applying (4.16) to (1) and comparing with the coefficients of e_{lm} $(l+m=N-1)$ in (1), we can deduce that

$$(l+1)(m+1)\tilde{Q}_{l+1,m+1}=\text{a function of } \tilde{Q}_{ij} \qquad (i+j\leq N)\,. \tag{4.18}$$

We next note that (4.17) is equal to

$$2\sum_{i=1}^{n}\left|\sum_{p,q}\frac{\bar{z}_i}{r}\frac{\partial\tilde{Q}_{pq}}{\partial r}e_{pq}+\sum_{p,q}\tilde{Q}_{pq}\frac{\partial e_{pq}}{\partial z_i}\right|^2$$
$$=2\sum_{p,q,p',q'}\left[\frac{\partial\tilde{Q}_{pq}}{\partial r}\frac{\partial\bar{\tilde{Q}}_{p'q'}}{\partial r}+\frac{1}{r}\left(p'\frac{\partial\tilde{Q}_{pq}}{\partial r}\bar{\tilde{Q}}_{p'q'}+p\tilde{Q}_{pq}\frac{\partial\bar{\tilde{Q}}_{p'q'}}{\partial r}\right)\right]e_{pq}\bar{e}_{p'q'}$$
$$+\sum_{p,q,p',q'}pp'\tilde{Q}_{pq}\bar{\tilde{Q}}_{p'q'}e_{p-1,q}\bar{e}_{p'-1,q'}\,.$$

Then by applying (4.17) to (4.11) and comparing with the coefficients of e_{lm} $(l+m=N)$ in (4.11), we can deduce that

$$\sum_{\substack{i+j'-1=l\\ j+i'-1=m}} ii'\tilde{Q}_{ij}\bar{\tilde{Q}}_{i'j'}=\text{a function of } \tilde{Q}_{ij} \qquad (i+j\leq N)\,. \tag{4.19}$$

As the process in the case of $\boldsymbol{R}$ (4.18) and (4.19) are also valid when we replace $\tilde{Q}$ by Q_c. Therefore, it follows from (4.6), (4.9) and the induction hypothesis that for $l+m=N-1$

$$(l+1)(m+1)(\tilde{Q}_{l+1,m+1}-Q^c_{l+1,m+1})=0$$

and for $l+m=N$

$$(m+1)(\tilde{Q}_{l,m+1}-Q^c_{l,m+1})\tilde{Q}_{10}+(l+1)(\tilde{Q}_{l+1,m}-Q^c_{l+1,m})Q_{10}=0\,.$$

Then it is easy to see that these relations imply that $\tilde{Q}_{lm}=Q^c_{lm}$ $(l+m=N+1)$. Therefore, by (4.10), (4.15) and the induction, we can obtain the desired result.

Case of $\boldsymbol{H}$. We suppose that $\tilde{Q}_{pq}=Q^c_{pq}$ $(0\leq p+2q\leq N)$. We note that

$$\Delta_0 e_{pq}=4p(p-1)e_{p-2,q}+4q(p+q)e_{p,q-1}$$

and

$$x_i\frac{\partial e_{pq}}{\partial x_i}=\begin{cases} pe_{pq}+2qe_{p,q-1}x_i^2 & \text{if } i=1\,,\\ 2qe_{p,q-1}x_i^2 & \text{if } i=n+1,\,2n+1,\,3n+1\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

Especially, $\sum_{i=1}^{nd}x_i(\partial e_{pq}/\partial x_i)=(p+2q)e_{pq}$. Then applying (4.16) to (1) and comparing with the coefficients of $e_{l-1,m}$ $(l+2m=N)$ in (1), we can deduce that

$$l(l+1)\tilde{Q}_{l+1,m}+(m+1)(l+m)\tilde{Q}_{l-1,m+1}$$
$$=\text{a function of } \tilde{Q}_{ij} \qquad (i+2j\leq N)\,. \tag{4.20}$$

We next note that (4.17) is equal to

$$2\sum_{p,q,p',q'}\left[\frac{\partial\tilde{Q}_{pq}}{\partial r}\frac{\tilde{Q}_{p'q'}}{\partial r}+\frac{2}{r}(p'+2q')\frac{\partial\tilde{Q}_{pq}}{\partial r}\tilde{Q}_{p'q'}\right]e_{pq}e_{p',q'}$$

$$+\sum_{p,q,p',q'}\tilde{Q}_{pq}\tilde{Q}_{p'q'}(pp'e_{p-1,q}e_{p'-1,q'}+q(p'+q')e_{p,q-1}e_{p',q'})\,.$$

Then by applying (4.17) to (4.11) and comparing with the coefficients of e_{lm} $(l+2m=N)$ in (4.11), we can deduce that

$$\sum_{\substack{i+i'=l+2\\ j+j'=m}} ii'\tilde{Q}_{ij}\tilde{Q}_{i'j'}+\sum_{\substack{i+i'=l\\ j+j'=m+1}}(i'+j')j\tilde{Q}_{ij}\tilde{Q}_{i'j'}$$
$$=\text{a function of } \tilde{Q}_{ij}\qquad (i+2j\leq N)\,. \tag{4.21}$$

As before, (4.20) and (4.21) are also valid when we replace $\tilde{Q}$ by Q^c. Therefore, it follows from the induction hypothesis that for $l+2m=N$

$$l(l+1)(\tilde{Q}_{l+1,m}-Q^c_{l+1,m})+(m+1)(l+m)(\tilde{Q}_{l-1,m+1}-Q^c_{l-1,m+1})=0$$

and

$$2(l+1)(\tilde{Q}_{l+1,m}-Q^c_{l+1,m})\tilde{Q}_{10}+(m+1)(\tilde{Q}_{l-1,m+1}-Q^c_{l-1,m+1})\tilde{Q}_{10}=0\,.$$

Then these relations imply that $\tilde{Q}_{l+1,m}=Q^c_{l+1,m}$ and $\tilde{Q}_{l-1,m+1}=Q^c_{l-1,m+1}$, and thus, $\tilde{Q}_{lm}=Q^c_{lm}$ $(l+2m=N+1)$. Therefore, by (4.10), (4.15) and the induction, we can obtain the desired result. □

5. Main theorem.

We shall try to remove the additional condition (4) in Proposition 4.1. Let us suppose that a real valued, C^2 function F on G/K satisfies (1)–(3). Since F is real analytic from (1), F has a homogeneous expansion $F=\sum_{N=0}^{\infty}F_N$ with respect to the real coordinates $x_s(1\leq s\leq nd)$. Especially, F_1 is of the form

$$\sum_{s=1}^{n}\sum_{l=1}^{d}a_s^l x_{s+n(l-1)}\qquad (a_s^l\in \boldsymbol{R})\,. \tag{5.1}$$

If we put

$$a_s=\begin{cases} a_s^1 & (\boldsymbol{F}=\boldsymbol{R})\\ a_s^1+ia_s^2 & (\boldsymbol{F}=\boldsymbol{C})\\ a_s^1+ia_s^2+ja_s^3+ka_s^4 & (\boldsymbol{F}=\boldsymbol{H})\,, \end{cases} \tag{5.2}$$

(5.1) can be written as $\sum_{s=1}^{n}2^{-1}(\bar{a}_s\zeta_s+\bar{\zeta}_s a_s)$. Then, applying the argument which deduces (4.13) from (1)–(3) in §4, we see that

$$\sum_{s=1}^{n} |a_s|^2 = \sum_{i=1}^{nd} \left(\frac{\partial F}{\partial x_i}\right)(0)^2 = \sum_{i=1}^{nd} \left(\frac{\partial P_1}{\partial x_i}\right)(0)^2 = l^2 , \tag{5.3}$$

and thus, we can find an element k_0 in K being of the form

$$\begin{pmatrix} a_1/l & \cdots\cdots & 0 \\ a_2/l & \cdots\cdots & 0 \\ & \cdots\cdots\cdots & \\ a_n/l & \cdots\cdots & 0 \\ 0 \quad 0 & \cdots & 0 \quad 1 \end{pmatrix} . \tag{5.4}$$

Then it is easy to see that $k_0 \cdot \sum_{s=1}^{n} (\bar{a}_s\zeta_s + \bar{\zeta}_s a_s) = l(\zeta_1 + \bar{\zeta}_1)$. Therefore, the rotation F_{k_0} of F defined by $F_{k_0}(\zeta) = F(k_0 \cdot \zeta)$ $(\zeta \in D^n)$ satisfies the following condition:

$$\frac{\partial F_{k_0}}{\partial x_1}(0) = l \quad \text{and} \quad \frac{\partial F_{k_0}}{\partial x_i}(0) = 0 \ (2 \le i \le nd) . \tag{5.5}$$

Our main theorem can be stated as follows.

THEOREM 5.1. (i) $G = SO_0(n, 1)$ *and* $G = SU(n, 1)$. *Let us suppose that a real valued, C^2 function F on G/K satisfies the following conditions*:

(1) $\Delta F = 0$,

(2) $\Delta F^2 = \lambda_2 F^2$,

(3) $F(eK) = 1$.

Then there exists an element k_0M in K/M for which F_{k_0} satisfies (5.5) *and F is determined by $F(gK) = P(gK, k_0M)$ $(g \in G)$* .

(ii) $G = Sp(n, 1)$. *Let us suppose that a real valued, C^2 function F on G/K satisfies* (1)–(3). *Then there exists an element k_0M in K/M for which F_{k_0} satisfies* (5.5). *Moreover, if*

(5) $F_{k_0}(cw_1, w_2, \cdots, w_n) = F_{k_0}(w_1c, w_2, \cdots, w_n)$ *for all c in H, i.e., F_{k_0} is a function of $w_1 + \bar{w}_1$, $|w_1|^2$ and $w_s + \bar{w}_s$, $w_si - i\bar{w}_s$, $w_sj - j\bar{w}_s$, $w_sk - k\bar{w}_s$ $(2 \le s \le n)$,*

then F is determined by $F(gK) = P(gK, k_0M)$ $(g \in G)$.

PROOF. We have already shown the existence of k_0M in K/M satisfying (5.5). By replacing F with the rotation F_{k_0} of F, without loss of generality, we may assume that $k_0 = e$. In what follows the conditions (5.5) and (5) mean the ones for $k_0 = e$. Then to prove the theorem it is enough to show that $F(gK) = P(gK, eM) = P_1(gK)$ $(g \in G)$ from (1)–(3), (5.5) and (5) where, as said in §1, the condition corresponding to (5) in the cases of $G = SO_0(n, 1)$ and $SU(n, 1)$ holds automatically. Then the desired result follows from Proposition 4.1 and the following,

PROPOSITION 5.2. *Let us suppose that a real valued, C^2 function F on G/K satisfies* (1)–(3), (5.5) *and* (5). *Then F is M-invariant.*

PROOF. First we put $[F](g)=\int_M F(mg)dm$ $(g\in G)$ and

$$F=[F]+R. \tag{5.6}$$

Then we easily see that $[F]$ and R satisfy the following conditions:

$$\Delta[F]=0, \tag{5.7}$$

$$[F](0)=1, \tag{5.8}$$

$$[F] \text{ is real valued}, \tag{5.9}$$

$$\frac{\partial[F]}{\partial x_1}(0)=l \quad \text{and} \quad \frac{\partial[F]}{\partial x_i}(0)=0 \ (2\le i\le nd), \tag{5.10}$$

$$\Delta R=0, \tag{5.11}$$

$$R(0)=0, \tag{5.12}$$

$$R \text{ is real valued}, \tag{5.13}$$

$$[R]=0, \tag{5.14}$$

$$\frac{\partial R}{\partial x_i}(0)=0 \ (1\le i\le nd). \tag{5.15}$$

Since $[F]$ is M-invariant and satisfies (5.7)–(5.10), it follows from the same argument in §4 that $[F]$ has an expansion being of the form (cf. (4.8), (4.10) and (4.14))

$$[F](\zeta)=\sum_{(p,q)\in\Lambda_0} S_{pq}(r)e_{pq}(\zeta) \qquad (\zeta\in D^n) \tag{5.16}$$

and

$$S_{00}=P_{00}, \tag{5.17}$$

$$S_{10}(r)r=\begin{cases} P_{10}(r) & (\boldsymbol{F}=\boldsymbol{R},\boldsymbol{C}) \\ P_{11}(r) & (\boldsymbol{F}=\boldsymbol{H}). \end{cases} \tag{5.18}$$

In particular, we see that

$$[F] \text{ and } R \text{ satisfy (5)}. \tag{5.19}$$

By substituting (5.6) for (2), we have

$$\lambda_2([F]^2+2[F]R+R^2)=|\nabla|^2([F])+2\nabla([F],R)+|\nabla|^2(R). \tag{5.20}$$

Here we note that $[\nabla([F],R)]=[\Delta([F]R)]=\Delta([F][R])=0$ by (5.14). Therefore, taking the average of (5.20) over M, we see that

$$\lambda_2([F]^2+[R^2])=|\nabla|^2([F])+[|\nabla|^2(R)]\,. \tag{5.21}$$

We now note that R is real analytic from (5.11) and then we can denote the homogeneous expansion of R as follows (see (5) and (5.19)).

$$\begin{aligned} R&=\sum_{N=0}^{\infty} R_N \\ &=\sum_{N=0}^{\infty} \sum_{\substack{(i,j)\in A_0 \\ m+i+j=N \text{ if } F=\boldsymbol{R},\boldsymbol{C}, \\ m+i+2j=N \text{ if } F=\boldsymbol{H}}} A_{ij}^m e_{ij}\,, \end{aligned} \tag{5.22}$$

where e_{ij} is defined by (3.4) and A_{ij}^m is a homogeneous polynomial of degree m with respect to x_s $(s\neq 1+n(k-1),\ 1\leq k\leq d)$. Obviously, (5.12), (5.13) and (5.15) mean that

$$\begin{aligned} &R_0=R_1=0 \qquad \text{and} \\ &\text{if } \boldsymbol{F}=\boldsymbol{R},\ \boldsymbol{H},\ A_{ij}^m \text{ is real valued and if } \boldsymbol{F}=\boldsymbol{C},\ A_{ij}^m=\bar{A}_{ji}^m\,. \end{aligned} \tag{5.23}$$

In what follows we shall prove that $R=0$ by induction. Let us suppose that $R_0=R_1=\cdots=R_N=0$. Then by comparing with the homogeneous polynomial with degree $2N$ in (5.21), we can obtain that

$$\begin{aligned} \left[\sum_{i=1}^{nd}\left(\frac{\partial R_{N+1}}{\partial x_i}\right)^2\right] &= \text{the homogeneous polynomial with degree } 2N \\ &\quad \text{in } \lambda_2[F]^2-|\nabla|^2([F])\,. \end{aligned} \tag{5.24}$$

Here let $\zeta_0=(0,\zeta_2,\zeta_3,\cdots,\zeta_n)\in D^n$. Then (5.16)–(5.18) mean that $[F](\zeta_0)=P_1(\zeta_0)$ and $(\partial[F]/\partial x_i)(\zeta_0)=(\partial P_1/\partial x_i)(x_0)$ $(1\leq i\leq nd)$. Therefore, if we put $\zeta=\zeta_0$ in (5.24), we can obtain that

$$\begin{aligned} \left[\sum_{i=1}^{nd}\left(\frac{\partial R_{N+1}}{\partial x_i}\right)^2(\zeta_0)\right] &= \text{the homogeneous polynomial with degree } 2N \\ &\quad \text{in } \lambda_2 P_1^2(\zeta_0)-|\nabla|^2(P_1)(\zeta_0) \\ &=0\,. \end{aligned} \tag{5.25}$$

Obviously, this means that $|(\partial A_{00}^{N+1}/\partial x_i)|^2=0$ $(i\neq 1+n(k-1),\ 1\leq k\leq d)$ and $A_{10}^N=0$. Therefore, A_{00}^{N+1} is a constant of degree $N+1$ and thus,

$$A_{00}^{N+1}=A_{10}^N \ (=A_{01}^N \text{ if } \boldsymbol{F}=\boldsymbol{C}) \ =0 \tag{5.26}$$

(see (5.23)).

We here note (2.9) and (5.22). Then comparing with the homogeneous polynomials with degree N in (5.20), we can obtain that

$$\frac{\partial R_{N+1}}{\partial x_1} \frac{\partial [F]}{\partial x_1}(0) = H \,, \tag{5.27}$$

where H is an M-invariant function on G/K.[2] Since $[R_{N+1}] = [R]_{N+1} = 0$ by (5.14) and $\partial/\partial x_1$ and [ ] commute each other, it follows that $0 = [H] = H$. Therefore, by (5.10)

$$\frac{\partial R_{N+1}}{\partial x_1} = 0 \,. \tag{5.28}$$

Moreover, comparing with the homogeneous polynomials with degree $N-1$ in (5.11), we can obtain that

$$\sum_{i=1}^{nd} \frac{\partial^2 R_{N+1}}{\partial x_i^2} \left(= \sum_{i=1}^{n} \frac{\partial^2 R_{N+1}}{\partial z_1 \partial \bar{z}_1} \text{ if } \boldsymbol{F} = \boldsymbol{C} \right) = 0 \,. \tag{5.29}$$

Then substituting (5.22) for (5.28) and (5.29), and comparing with coefficients of e_{ij} for R_{N+1}, we can deduce that

$$\text{if } \boldsymbol{F} = \boldsymbol{R}, \qquad (i+1) A^{N-i}_{i+1,0} = 0 \,, \tag{5.30a}$$

$$\text{if } \boldsymbol{F} = \boldsymbol{C}, \qquad \begin{cases} (i+1) A^{N-i-j}_{i+1,j} + (j+1) A^{N-i-j}_{i,j+1} = 0 \\ (i+1)(j+1) A^{N-i-j-1}_{i+1,j+1} + \sum_{s=2}^{n} \partial^2 A^{N+1-i-j}_{ij} / \partial z_s \partial \bar{z}_s = 0 \,, \end{cases} \tag{5.30b}$$

$$\text{if } \boldsymbol{F} = \boldsymbol{H}, \qquad \begin{cases} 2(i+1) A^{N-i-2j}_{i+1,j} + (j+1) A^{N-i-2j}_{i-1,j+1} = 0 \\ 4(i+1)(i+2) A^{N-1-i-2j}_{i+2,j} + 4(j+1)(i+j+1) A^{N-1-i-2j}_{i,j+1} \\ \quad + \sum_{s=2,\ s \neq 1+nk(k=1,2,3)}^{4n} \partial^2 A^{N+1-i-2j}_{ij} / \partial x_s^2 = 0 \,. \end{cases} \tag{5.30c}$$

Then, it follows from (5.26) and (5.30) that $A^m_{ij} = 0$ for $m+i+j = N+1$ if $\boldsymbol{F} = \boldsymbol{R}$, $\boldsymbol{C}$ and $m+i+2j = N+1$ if $\boldsymbol{F} = \boldsymbol{H}$, that is, $R_{N+1} = 0$. Therefore, by the induction we can conclude that $R = 0$. This means that $F = [F]$ is M-invariant. □

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