TOKYO J. MATH.
VOL. 16, NO. 1, 1993

Minimal Hypersurfaces Foliated by Geodesics of 4-Dimensional Space Forms

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(Communicated by M. Sakai)

§0. Introduction.

Minimal surfaces of a 3-dimensional Euclidean space have been studied by many researchers. One of the most classic example of minimal surfaces is a helicoid. The helicoid is a ruled surface, i.e., a surface foliated by lines of $\boldsymbol{R}^3$. The following fact is well-known: minimal, ruled surface of $\boldsymbol{R}^3$ is either a part of a plane $\boldsymbol{R}^2$, or a part of the helicoid (cf. [1]). Barbosa-Dajczer-Jorge [2] generalize this theorem to the ruled minimal submanifolds of higher dimensional space forms.

In this paper, we determine minimal hypersurfaces M given by $M=\{\exp_p(t\xi)\,;\, p\in\Sigma, t\in\boldsymbol{R}\}$, where Σ is a minimal surface of constant curvature in a 4-dimensional space form $\tilde{M}$, and ξ is a (local) unit normal vector field on Σ. Such a minimal surface Σ is classified by Kenmotsu [5]. In §2, we find the equations for a surface Σ and a unit normal vector field ξ on Σ with respect to which $M=\{\exp_p(t\xi)\,;\, p\in\Sigma, t\in\boldsymbol{R}\}$ is minimal in $\tilde{M}$. In §3, §4, and §5, we solve the equations when Σ is totally geodesic in $\tilde{M}$, the minimal Clifford torus $S^1\times S^1\subset S^3\subset S^4$, and Σ is a Veronese surface of S^4, respectively. As a consequence, we find all minimal hypersurfaces M of S^4 satisfying the following conditions (theorem 5.1): (1) M contains a Veronese surface Σ of S^4, (2) M is foliated by great circles S^1 of S^4 which intersect Σ orthogonally, (3) the type number (i.e., the rank of the shape operator) of M is equal to 3 on some open set which intersects Σ. The proof is reduced to solving a differential equation of a holomorphic function.

Concerning this theorem, we note that minimal hypersurfaces with type number 2 of n-dimensional space forms ($n\geqq 4$) are investigated by Dajczer-Gromoll [4]. In fact, such a minimal hypersurface is obtained by the image of a minimal surface under the Gauss map. But it seems that little is known about minimal hypersurfaces of S^4 with type number 3, other than the generalized Clifford torus $S^2\times S^1$ (cf. [6], [7]).

Received April 10, 1992

This research was partially supported by Grant-in-Aid for Scientific Research (No. 01740014), Ministry of Education, Science and Culture.

The author would like to express his sincere gratitude to Professor K. Ogiue for his valuable suggestions, and would like to thank Professor K. Tsukada for his encouragements.

§1. Preliminaries.

Let $\tilde{M}^n(c)$ be a space form of constant sectional curvature c, namely, $\tilde{M}^n(c)$ is the Euclidean sphere $S^n(c)$, the Euclidean space $\boldsymbol{R}^n$ or the hyperbolic space $H^n(c)$ according as c being positive, zero or negative. We will consider $S^n(c)$ and $H^n(c)$ as hypersurfaces of $\boldsymbol{R}^{n+1}$ and L^{n+1}, respectively, where L^{n+1} denotes the $(n+1)$-dimensional Lorentzian space with the canonical flat metric

$$d\sigma^2 = -dx_0{}^2 + \sum_{j=1}^{n} dx_j{}^2 .$$

We assume that the constant curvature c of $\tilde{M}^n(c)$ is equal to 1, 0 or -1, according as $c>0$, $c=0$, or $c<0$, unless otherwise stated. The exponential mapping of $\tilde{M}^n(c)$ has the following expression:

$$\exp_p(tV) = f_1(t)p + f_2(t)V , \tag{1.1}$$

where $p \in \tilde{M}^n(c)$ and $V \in T_p\tilde{M}^n(c)$ $(\|V\|=1)$ are considered as vectors in the ambient space. The functions f_1 and f_2 are given by

$$\begin{aligned} &f_1(t)=1 , && f_2(t)=t , && \text{if } c=0 ,\\ &f_1(t)=\cos t , && f_2(t)=\sin t , && \text{if } c=1 ,\\ &f_1(t)=\cosh t , && f_2(t)=\sinh t , && \text{if } c=-1 . \end{aligned} \tag{1.2}$$

Let Σ be a surface of $\tilde{M}=\tilde{M}^4(c)$. We give fundamental equations for $\Sigma \subset \tilde{M}$. Let e_1, e_2 be a local orthonormal frame field on Σ, and let ξ, η be a local orthonormal frame field of the normal bundle of Σ in $\tilde{M}$. Then the Gauss formula and the Weingarten formula are written as

$$\begin{aligned} &\tilde{\nabla}_{e_i}e_j = \nabla_{e_i}e_j + h^\xi_{ij}\xi + h^\eta_{ij}\eta ,\\ &\tilde{\nabla}_{e_i}\xi = -A_\xi e_i + \nabla^\perp_{e_i}\xi , \qquad (1 \le i, j \le 2) ,\\ &\tilde{\nabla}_{e_i}\eta = -A_\eta e_i + \nabla^\perp_{e_i}\eta , \end{aligned} \tag{1.3}$$

where $\tilde{\nabla}$ (resp. ∇ and $\nabla^\perp$) denotes the Riemannian connection of $\tilde{M}$ (resp. the induced Riemannian connection of Σ and the normal connection of Σ in $\tilde{M}$), h^ξ_{ij} and h^η_{ij} are the components of the second fundamental tensor of Σ in $\tilde{M}$, and A_ξ (resp. A_η) describes the shape operator with respect to ξ (resp. η) of Σ in $\tilde{M}$. Then we have $h^\xi_{ij} = \langle A_\xi e_i, e_j\rangle$, $h^\eta_{ij} = \langle A_\eta e_i, e_j\rangle$, $h^\xi_{ij} = h^\xi_{ji}$ and $h^\eta_{ij} = h^\eta_{ji}$, where $\langle\, ,\rangle$ stands for the induced metric on Σ. Let ω and s be connection forms for ∇ and $\nabla^\perp$, defined by $\omega(e_i) = \langle \nabla_{e_i}e_1, e_2\rangle$ and $s(e_i) = \langle \nabla^\perp_{e_i}\xi, \eta\rangle$ $(i=1, 2)$. We donote ω_i and s_i the components of ω and s, respectively.

Then we have $\nabla_{e_i}e_1=\omega_i e_2$, $\nabla_{e_i}e_2=-\omega_i e_1$, $\nabla^{\perp}_{e_i}\xi=s_i\eta$, $\nabla^{\perp}_{e_i}\eta=-s_i\xi$, respectively.

We define the covariant derivatives of the shape operators A_ξ and A_η by

$$(1.4)\qquad \begin{aligned} h_{ijk}^{\xi}&=\langle\nabla_{e_k}(A_\xi e_i)-A_{\nabla^{\perp}_{e_k}\xi}e_i-A_\xi(\nabla_{e_k}e_i),\, e_j\rangle\,,\\ h_{ijk}^{\eta}&=\langle\nabla_{e_k}(A_\eta e_i)-A_{\nabla^{\perp}_{e_k}\eta}e_i-A_\eta(\nabla_{e_k}e_i),\, e_j\rangle\,. \end{aligned}$$

Then the Codazzi equation is described as;

(1.5) h_{ijk}^{ξ} and h_{ijk}^{η} are symmetric with respect to i, j and k, respectively.

We also define the covariant derivative of the 1-form s by

$$(1.6)\qquad s_{ij}=e_j s_i-s(\nabla_{e_j}e_i)\,.$$

Then we can see that

$$(1.7)\qquad s_{ij}=\langle\nabla^{\perp}_{e_j}\nabla^{\perp}_{e_i}\xi-\nabla^{\perp}_{\nabla_{e_j}e_i}\xi,\, \eta\rangle\,.$$

Now the Ricci equation is expressed as

$$(1.8)\qquad s_{12}-s_{21}=h_{12}^{\xi}(h_{22}^{\eta}-h_{11}^{\eta})+(h_{11}^{\xi}-h_{22}^{\xi})h_{12}^{\eta}\,.$$

§2. The normal exponential mapping of surfaces.

In this section, we consider the minimal hypersurface M which is the image of a subbundle of the normal bundle of some surface Σ in a 4-dimensional space from $\tilde{M}=\tilde{M}^4(c)$, under the normal exponential mapping of Σ in $\tilde{M}$. Let Σ be a surface in $\tilde{M}$. Then by (1.1) and (1.2), the position vector of the hypersurface $M=\{\exp_p(t\xi)\,;\, p\in\Sigma,\, t\in \boldsymbol{R}\}$ is parametrized as;

$$X(u, v, t)=f_1(t)p(u, v)+f_2(t)\xi(u, v)\,,$$

where $p=p(u, v)$ and $\xi=\xi(u, v)$ describe the position vector of Σ and a unit normal vector of Σ in $\tilde{M}$ at p, respectively, and (u, v) is a local coordinate of Σ. We denote by $\eta=\eta(u, v)$ a local field of unit normal vectors on Σ in $\tilde{M}$ orthogonal to ξ. The tangent vectors at the point $X=X(u, v, t)$ on M are expressed as;

$$(2.1)\qquad X_u=f_1p_u+f_2\xi_u\,,\quad X_v=f_1p_v+f_2\xi_v\,,\quad X_t=f'_1p+f'_2\xi\,,$$

where X_u, X_v, X_t, $\cdots$, etc. denote the derivatives of X, p and ξ with respect to u, v and t, respectively, which are considered as vectors in the ambient space, and f'_j means the differentiation of $f_j=f_j(t)$. Then we can see that the induced metric g on M is given by

$$\begin{pmatrix} \|X_u\|^2 & \langle X_u, X_v\rangle & 0\\ \langle X_u, X_v\rangle & \|X_v\|^2 & 0\\ 0 & 0 & 1 \end{pmatrix},$$

with respect to the local coordinate (u, v, t) of M. Since the integral curves of $\partial/\partial t$ are geodesics of $\tilde{M}$, the mean curvature vector H_M of M in $\tilde{M}$ is given by

$$H_M=\frac{1}{3(\det g)}(\|X_v\|^2(X_{uu})^{\perp}+\|X_u\|^2(X_{vv})^{\perp}-\langle X_u, X_v\rangle(X_{uv}+X_{vu})^{\perp}),$$

where $(\ )^{\perp}$ denotes the projection of the normal space of M in $\tilde{M}$, and $\det g=\|X_u\|^2\|X_v\|^2-\langle X_u, X_v\rangle^2$.

Assume that $c\neq 0$. Then M is minimal in $\tilde{M}$ if and only if

$$p\wedge\xi\wedge X_u\wedge X_v\wedge(\|X_v\|^2X_{uu}+\|X_u\|^2X_{vv}-\langle X_u, X_v\rangle(X_{uv}+X_{vu}))=0, \tag{2.2}$$

since $X\wedge X_t=p\wedge\xi$. If $c=0$, then the same argument shows that M is minimal in $\boldsymbol{R}^4$ if and only if

$$\xi\wedge X_u\wedge X_v\wedge(\|X_v\|^2X_{uu}+\|X_u\|^2X_{vv}-\langle X_u, X_v\rangle(X_{uv}+X_{vu}))=0. \tag{2.3}$$

Here we note that (2.2) and (2.3) are independent of the choice of local coordinate (u, v) of Σ.

Take a local isothermal coordinate (u, v) of Σ, such that $p_u=\varphi e_1, p_v=\varphi e_2$, where (e_1, e_2) forms an orthonormal frame, and φ is a positive function on some open set in Σ. Suppose $c\neq 0$. Then, by (1.3) and (2.1), we have

$$\begin{aligned} X_u&=\varphi(f_1e_1+f_2D_{e_1}\xi)=\varphi(f_1e_1+f_2(-A_\xi e_1+\nabla^{\perp}_{e_1}\xi))\\ &=\varphi(f_1e_1+f_2(-h_{11}^{\xi}e_1-h_{12}^{\xi}e_2+s_1\eta)),\\ X_v&=\varphi(f_1e_2+f_2D_{e_2}\xi)=\varphi(f_1e_2+f_2(-A_\xi e_2+\nabla^{\perp}_{e_2}\xi))\\ &=\varphi(f_1e_2+f_2(-h_{12}^{\xi}e_1-h_{22}^{\xi}e_2+s_2\eta)), \end{aligned} \tag{2.4}$$

where D denotes the covariant differentiation in $\boldsymbol{R}^5$ or L^5. Note that the component of D_VW tangent to $\tilde{M}$ is $\tilde{\nabla}_VW$ for V, W tangent to M. From this, we get

$$\begin{aligned} \|X_u\|^2&=\varphi^2(f_1{}^2-2f_1f_2h_{11}^{\xi}+f_2{}^2((h_{11}^{\xi})^2+(h_{12}^{\xi})^2+s_1{}^2)),\\ \|X_v\|^2&=\varphi^2(f_1{}^2-2f_1f_2h_{22}^{\xi}+f_2{}^2((h_{12}^{\xi})^2+(h_{22}^{\xi})^2+s_2{}^2)),\\ \langle X_u, X_v\rangle&=\varphi^2(-2f_1f_2h_{12}^{\xi}+f_2{}^2(h_{12}^{\xi}(h_{11}^{\xi}+h_{12}^{\xi})+s_1s_2)). \end{aligned} \tag{2.5}$$

Similarly, X_{uu}, X_{uv}, X_{vu} and X_{vv} are written as

$$\begin{aligned} X_{uu}&=(e_1\varphi)X_u+\varphi^2(f_1D_{e_1}e_1+f_2(-D_{e_1}(A_\xi e_1)+D_{e_1}\nabla^{\perp}_{e_1}\xi)),\\ X_{uv}&=(e_2\varphi)X_u+\varphi^2(f_1D_{e_2}e_1+f_2(-D_{e_2}(A_\xi e_1)+D_{e_2}\nabla^{\perp}_{e_1}\xi)),\\ X_{vu}&=(e_1\varphi)X_v+\varphi^2(f_1D_{e_1}e_2+f_2(-D_{e_1}(A_\xi e_2)+D_{e_1}\nabla^{\perp}_{e_2}\xi)),\\ X_{vv}&=(e_2\varphi)X_v+\varphi^2(f_1D_{e_2}e_2+f_2(-D_{e_2}(A_\xi e_2)+D_{e_2}\nabla^{\perp}_{e_2}\xi)). \end{aligned}$$

Using (1.3), (1.4), (1.5) and (1.7), we get

(2.6)
$$\begin{aligned}
X_{uu} &\equiv \varphi^2(f_1 h_{11}^{\eta}\eta - f_2((h_{111}^{\xi} + 2s_1 h_{11}^{\eta})e_1 + (h_{112}^{\xi} + 2s_1 h_{12}^{\eta})e_2 \\
&\quad + (-s_{11} + h_{11}^{\xi}h_{11}^{\eta} + h_{12}^{\xi}h_{12}^{\eta})\eta)) \qquad \operatorname{mod}(p, \xi, X_u, X_v), \\
X_{uv} &\equiv \varphi^2(f_1 h_{12}^{\eta}\eta - f_2((h_{112}^{\xi} + s_1 h_{12}^{\eta} + s_2 h_{11}^{\eta})e_1 + (h_{122}^{\xi} + s_1 h_{22}^{\eta} + s_2 h_{12}^{\eta})e_2 \\
&\quad + (-s_{12} + h_{11}^{\xi}h_{12}^{\eta} + h_{12}^{\xi}h_{22}^{\eta})\eta)) \qquad \operatorname{mod}(p, \xi, X_u, X_v), \\
X_{uv} &\equiv \varphi^2(f_1 h_{12}^{\eta}\eta - f_2((h_{112}^{\xi} + s_1 h_{12}^{\eta} + s_2 h_{11}^{\eta})e_1 + (h_{122}^{\xi} + s_1 h_{22}^{\eta} + s_2 h_{12}^{\eta})e_2 \\
&\quad + (-s_{21} + h_{12}^{\xi}h_{11}^{\eta} + h_{22}^{\xi}h_{12}^{\eta})\eta)) \qquad \operatorname{mod}(p, \xi, X_u, X_v), \\
X_{vv} &\equiv \varphi^2(f_1 h_{22}^{\eta}\eta - f_2((h_{122}^{\xi} + 2s_2 h_{12}^{\eta})e_1 + (h_{222}^{\xi} + 2s_2 h_{22}^{\eta})e_2 \\
&\quad + (-s_{22} + h_{12}^{\xi}h_{12}^{\eta} + h_{22}^{\xi}h_{22}^{\eta})\eta)) \qquad \operatorname{mod}(p, \xi, X_u, X_v).
\end{aligned}$$

Clearly, $X_{uv} = X_{vu}$ holds, and we remark that this condition is equivalent to the Ricci equation (1.8), $e_1\varphi = \varphi\omega_2$, and $e_2\varphi = -\varphi\omega_1$. By straight computation, using (2.5) and (2.6), we obtain

(2.7)
$$\begin{aligned}
&\|X_v\|^2 X_{uu} + \|X_u\|^2 X_{vv} - \langle X_u, X_v\rangle(X_{uv} + X_{vu}) \\
&\equiv \varphi^4(f_1^{\,3} a_0\eta - f_1^{\,2} f_2(a_{10}\eta + a_{11}e_1 + a_{12}e_2) + f_1 f_2^{\,2}(a_{20}\eta + a_{21}e_1 + a_{22}e_2) \\
&\quad - f_2^{\,3}(a_{30}\eta + a_{31}e_1 + a_{32}e_2)) \qquad \operatorname{mod}(p, \xi, X_u, X_v),
\end{aligned}$$

where,

(2.8)
$$\begin{aligned}
a_0 &= h_{11}^{\eta} + h_{22}^{\eta}, \\
a_{10} &= -(s_{11} + s_{22}) + (h_{11}^{\xi} + 2h_{22}^{\xi})h_{11}^{\eta} - 2h_{12}^{\xi}h_{12}^{\eta} + (2h_{11}^{\xi} + h_{22}^{\xi})h_{22}^{\eta}, \\
a_{11} &= h_{111}^{\xi} + h_{122}^{\xi} + 2s_1 h_{11}^{\eta} + 2s_2 h_{12}^{\eta}, \\
a_{12} &= h_{112}^{\xi} + h_{222}^{\xi} + 2s_1 h_{12}^{\eta} + 2s_2 h_{22}^{\eta}, \\
a_{20} &= -2h_{22}^{\xi}s_{11} + 2h_{12}^{\xi}(s_{12} + s_{21}) - 2h_{11}^{\xi}s_{22} + (h_{22}^{\xi}(2h_{11}^{\xi} + h_{22}^{\xi}) - (h_{12}^{\xi})^2 + s_2^{\,2})h_{11}^{\eta} \\
&\quad - 2(h_{12}^{\xi}(h_{11}^{\xi} + h_{22}^{\xi}) + s_1 s_2)h_{12}^{\eta} + (h_{11}^{\xi}(h_{11}^{\xi} + 2h_{22}^{\xi}) - (h_{12}^{\xi})^2 + s_1^{\,2})h_{22}^{\eta}, \\
a_{21} &= 2h_{22}^{\xi}h_{111}^{\xi} - 4h_{12}^{\xi}h_{112}^{\xi} + 2h_{11}^{\xi}h_{122}^{\xi} + 4(s_1 h_{22}^{\xi} - s_2 h_{12}^{\xi})h_{11}^{\eta} + 4(s_2 h_{11}^{\xi} - s_1 h_{12}^{\xi})h_{12}^{\eta}, \\
a_{22} &= 2h_{22}^{\xi}h_{112}^{\xi} - 4h_{12}^{\xi}h_{122}^{\xi} + 2h_{11}^{\xi}h_{222}^{\xi} + 4(s_1 h_{22}^{\xi} - s_2 h_{12}^{\xi})h_{12}^{\eta} + 4(s_2 h_{11}^{\xi} - s_1 h_{12}^{\xi})h_{22}^{\eta}, \\
a_{30} &= -((h_{12}^{\xi})^2 + (h_{22}^{\xi})^2 + s_2^{\,2})s_{11} + (h_{12}^{\xi}(h_{11}^{\xi} + h_{22}^{\xi}) + s_1 s_2)(s_{12} + s_{21}) \\
&\quad - ((h_{11}^{\xi})^2 + (h_{12}^{\xi})^2 + s_1^{\,2})s_{22} + (h_{22}^{\xi}(h_{11}^{\xi}h_{22}^{\xi} - (h_{12}^{\xi})^2) + s_2(s_2 h_{11}^{\xi} - s_1 h_{12}^{\xi}))h_{11}^{\eta} \\
&\quad + (2h_{12}^{\xi}((h_{12}^{\xi})^2 - h_{11}^{\xi}h_{22}^{\xi}) + (s_1^{\,2} + s_2^{\,2})h_{12}^{\xi} - s_1 s_2(h_{11}^{\xi} + h_{22}^{\xi}))h_{12}^{\eta} \\
&\quad + (h_{11}^{\xi}(h_{11}^{\xi}h_{22}^{\xi} - (h_{12}^{\xi})^2) + s_1(s_1 h_{22}^{\xi} - s_2 h_{12}^{\xi}))h_{22}^{\eta}, \\
a_{31} &= ((h_{12}^{\xi})^2 + (h_{22}^{\xi})^2 + s_2^{\,2})h_{111}^{\xi} \\
&\quad - 2(h_{12}^{\xi}(h_{11}^{\xi} + h_{22}^{\xi}) + s_1 s_2)h_{112}^{\xi} \\
&\quad + ((h_{11}^{\xi})^2 + (h_{12}^{\xi})^2 + s_1^{\,2})h_{122}^{\xi} \\
&\quad + (2s_1((h_{12}^{\xi})^2 + (h_{22}^{\xi})^2) - 2s_2 h_{12}^{\xi}(h_{11}^{\xi} + h_{22}^{\xi}))h_{11}^{\eta} \\
&\quad + (2s_2((h_{11}^{\xi})^2 + (h_{12}^{\xi})^2) - 2s_1 h_{12}^{\xi}(h_{11}^{\xi} + h_{22}^{\xi}))h_{12}^{\eta},
\end{aligned}$$

$$a_{32}=((h_{12}^{\xi})^2+(h_{22}^{\xi})^2+s_2{}^2)h_{112}^{\xi}-2(h_{12}^{\xi}(h_{11}^{\xi}+h_{22}^{\xi})+s_1s_2)h_{122}^{\xi}$$
$$+((h_{11}^{\xi})^2+(h_{12}^{\xi})^2+s_1{}^2)h_{222}^{\xi}+(2s_1((h_{12}^{\xi})^2+(h_{22}^{\xi})^2)-2s_2h_{12}^{\xi}(h_{11}^{\xi}+h_{22}^{\xi}))h_{12}^{\eta}$$
$$+(2s_2((h_{11}^{\xi})^2+(h_{12}^{\xi})^2)-2s_1h_{12}^{\xi}(h_{11}^{\xi}+h_{22}^{\xi}))h_{22}^{\eta}\,.$$

By (2.4), the exterior product of X_u and X_v is written as

$$(2.9)\qquad X_u\wedge X_v=\varphi^2(f_1{}^2e_1\wedge e_2-f_1f_2((h_{11}^{\xi}+h_{22}^{\xi})e_1\wedge e_2$$
$$-s_2e_1\wedge\eta+s_1e_2\wedge\eta)+f_2{}^2((h_{11}^{\xi}h_{22}^{\xi}-(h_{12}^{\xi})^2)e_1\wedge e_2$$
$$+(s_1h_{12}^{\xi}-s_2h_{11}^{\xi})e_1\wedge\eta+(s_1h_{22}^{\xi}-s_2h_{12}^{\xi})e_2\wedge\eta))\,.$$

Substituting (2.7) and (2.9) into (2.2), we have the following equation,

$$(f_1{}^5b_0-f_1{}^4f_2b_1+f_1{}^3f_2{}^2b_2-f_1{}^2f_2{}^3b_3+f_1f_2{}^4b_4-f_2{}^5b_5)p\wedge\xi\wedge\eta\wedge e_1\wedge e_2=0\,,$$

where

$$(2.10)\qquad b_0=a_0\,,\quad b_1=(h_{11}^{\xi}+h_{22}^{\xi})a_0+a_{10}\,,$$
$$b_2=(h_{11}^{\xi}h_{22}^{\xi}-(h_{12}^{\xi})^2)a_0+(h_{11}^{\xi}+h_{22}^{\xi})a_{10}+s_1a_{11}+s_2a_{12}+a_{20}\,,$$
$$b_3=(h_{11}^{\xi}h_{22}^{\xi}-(h_{12}^{\xi})^2)a_{10}+(s_1h_{22}^{\xi}-s_2h_{12}^{\xi})a_{11}$$
$$+(s_2h_{11}^{\xi}-s_1h_{12}^{\xi})a_{12}+(h_{11}^{\xi}+h_{22}^{\xi})a_{20}+s_1a_{21}+s_2a_{22}+a_{30}\,,$$
$$b_4=(h_{11}^{\xi}h_{22}^{\xi}-(h_{12}^{\xi})^2)a_{20}+(s_1h_{22}^{\xi}-s_2h_{12}^{\xi})a_{21}$$
$$+(s_2h_{11}^{\xi}-s_1h_{12}^{\xi})a_{22}+(h_{11}^{\xi}+h_{22}^{\xi})a_{30}+s_1a_{31}+s_2a_{32}\,,$$
$$b_5=(h_{11}^{\xi}h_{22}^{\xi}-(h_{12}^{\xi})^2)a_{30}+(s_1h_{22}^{\xi}-s_2h_{12}^{\xi})a_{31}+(s_2h_{11}^{\xi}-s_1h_{12}^{\xi})a_{32}\,.$$

Because $f_1{}^5, f_1{}^4f_2, \cdots, f_2{}^5$ are mutually independent functions, we have $b_j=0$ $(0\leq j\leq 5)$. Hence (2.10) implies

$$(2.11)\qquad a_0=0\,,$$

$$(2.12)\qquad a_{10}=0\,,$$

$$(2.13)\qquad s_1a_{11}+s_2a_{12}+a_{20}=0\,,$$

$$(2.14)\qquad (s_1h_{22}^{\xi}-s_2h_{12}^{\xi})a_{11}+(s_2h_{11}^{\xi}-s_1h_{12}^{\xi})a_{12}$$
$$+(h_{11}^{\xi}+h_{22}^{\xi})a_{20}+s_1a_{21}+s_2a_{22}+a_{30}=0\,,$$

$$(2.15)\qquad (h_{11}^{\xi}h_{22}^{\xi}-(h_{12}^{\xi})^2)a_{20}+(s_1h_{22}^{\xi}-s_2h_{12}^{\xi})a_{21}+(s_2h_{11}^{\xi}-s_1h_{12}^{\xi})a_{22}$$
$$+(h_{11}^{\xi}+h_{22}^{\xi})a_{30}+s_1a_{31}+s_2a_{32}=0\,,$$

$$(2.16)\qquad (h_{11}^{\xi}h_{22}^{\xi}-(h_{12}^{\xi})^2)a_{30}+(s_1h_{22}^{\xi}-s_2h_{12}^{\xi})a_{31}+(s_2h_{11}^{\xi}-s_1h_{12}^{\xi})a_{32}=0\,.$$

We can see that (2.11)–(2.16) hold too, when $c=0$. We note that the above equations are independent of the choice of orthonormal basis e_1, e_2 at each point of Σ. Hence we get

PROPOSITION 2.1. *Let Σ be a surface in a 4-dimensional space form $\tilde{M}^4(c)$, and let M be the hypersurface which is the image of the subbundle, spanned by unit vector field ξ, of the normal bundle under the normal exponential mapping of Σ in $\tilde{M}$. Then M is minimal in $\tilde{M}$ if and only if Σ and the orthonormal normal frame ξ, η satisfy* (2.8) *and* (2.11)–(2.16).

REMARK 2.2. By (2.8), (2.11) says that ξ is proportional to the mean curvature vector H of Σ in $\tilde{M}$ (if $H\neq 0$). Then (2.11) and (2.12) imply that $s_{11}+s_{22}=-\operatorname{trace} A_\xi A_\eta$. The equations (2.11)–(2.16) are viewed as linear equations for s_{11}, s_{12}, s_{21}, s_{22}, $h_{11\bar{1}}^{\xi}$, $h_{11\bar{2}}^{\xi}$, $h_{12\bar{2}}^{\xi}$, $h_{22\bar{2}}^{\xi}$.

REMARK 2.3. In Proposition 2.1, the assumption about M is equivalent to the following two conditions: (a) M is foliated by geodesics of $\tilde{M}^4(c)$. (b) 2-dimensional distribution on M orthogonal to the geodesics in (a) is integrable (locally). Then Proposition 2.1 claims that M is minimal if and only if a leaf of the foliation of (b) satisfies (2.11)–(2.16). We note that an isoparametric minimal hypersurface in $S^4(1)$ with 3 distinct constant principal curvatures $-\sqrt{3}$, 0, $\sqrt{3}$ is foliated by geodesics of $S^4(1)$, and the orthogonal 2-dimensional distribution is not integrable.

§3. Construction of examples I.

In this section, we determine the unit normal vector field ξ on the totally geodesic surface $\Sigma=\tilde{M}^2(c)$ in $\tilde{M}^4(c)$ such that $M=\{\exp_p(t\xi)\,;\,p\in\Sigma,\,t\in\boldsymbol{R}\}$ is a minimal hypersurface. Since Σ is totally geodesic, we have $h_{ij}^{\xi}=0$, $h_{ij}^{\eta}=0$ and $h_{ijk}^{\xi}=0$ for $i,j,k=1,2$. Thus Proposition 2.1 implies

(3.1) $$s_{11}+s_{22}=0\,,$$

(3.2) $$s_2{}^2s_{11}+s_1{}^2s_{22}-s_1s_2(s_{12}+s_{21})=0\,,$$

and the Ricci equation (1.8) yields

(3.3) $$s_{12}=s_{21}\,.$$

EXAMPLE 3.1. Σ is a totally geodesic $\boldsymbol{R}^2$ in $\boldsymbol{R}^4$ ($c=0$).

We may put $\boldsymbol{R}^2=\{(x,\,y,\,0,\,0)\in\boldsymbol{R}^4\,;\,x,\,y\in\boldsymbol{R}\}$. Let $e_1=\partial/\partial x$, and $e_2=\partial/\partial y$. Then they form an orthonormal frame on $\boldsymbol{R}^2$ and satisfy $\nabla e_1=\nabla e_2=0$. Note that an orthonormal normal frame ξ, η for $\boldsymbol{R}^2$ in $\boldsymbol{R}^4$ is given by

(3.4) $$\xi=(0,0,\cos\theta,\sin\theta)\,,\qquad \eta=(0,0,-\sin\theta,\cos\theta)\,,$$

where $\theta=\theta(x,y)$, and the normal connection of $\boldsymbol{R}^2$ in $\boldsymbol{R}^4$ satisfies

$$\nabla^{\perp}_{e_1}\xi=\theta_x\eta\,,\quad \nabla^{\perp}_{e_2}\xi=\theta_y\eta\,,\qquad \left(\theta_x=\frac{\partial\theta}{\partial x}\,,\quad \theta_y=\frac{\partial\theta}{\partial y}\right),$$

i.e., $s_1=\theta_x$ and $s_2=\theta_y$. Moreover we can see that $s_{11}=\theta_{xx}$, $s_{12}=\theta_{xy}$, $s_{21}=\theta_{yx}$ and $s_{22}=\theta_{yy}$. Then (3.1), (3.2) and (3.3) are equivalent to

$$\theta_{xx}+\theta_{yy}=0\,,\qquad \theta_{xy}=\theta_{yx}\,,$$
$$\theta_y{}^2\theta_{xx}+\theta_x{}^2\theta_{yy}-\theta_x\theta_y(\theta_{xy}+\theta_{yx})=0\,. \tag{3.5}$$

We put

$$z=x+\sqrt{-1}\,y\,,\qquad w=\theta_x-\sqrt{-1}\,\theta_y\,, \tag{3.6}$$
$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-\sqrt{-1}\frac{\partial}{\partial y}\right)\quad\text{and}\quad\frac{\partial}{\partial \bar z}=\frac{1}{2}\left(\frac{\partial}{\partial x}+\sqrt{-1}\frac{\partial}{\partial y}\right).$$

Then by the definition of θ, we have

$$\theta=\mathrm{Re}\int w\,dz\,, \tag{3.7}$$

where Re means the real part of the complex number.

If $w\equiv 0$, then θ is a constant. So we assume that $w\not\equiv 0$. Using (3.5), we can see that $w=w(z)$ is holomorphic, i.e., $\partial w/\partial\bar z=0$ and

$$w^2\left(\overline{\frac{\partial w}{\partial z}}\right)+\bar w^2\left(\frac{\partial w}{\partial z}\right)=0\,.$$

Hence we can write the above equation as;

$$\frac{\partial w}{\partial z}=\sqrt{-1}\,\rho w^2\,,\qquad \rho=\rho(z)\text{ is a real valued function}\,.$$

Since w is holomorphic, ρ is also holomorphic. These facts imply that $\rho\equiv$constant. By integrating this equation, we obtain $w=-(\sqrt{-1}\,\rho z+C_1)^{-1}$, where C_1 is a complex constant. Then we obtain

$$\int w\,dz=\frac{\sqrt{-1}}{\rho}\log(\sqrt{-1}\,\rho z+C_1),\qquad \text{if}\quad \rho\neq 0\,,$$
$$\int w\,dz=C_2z+C_3\,,\qquad \text{if}\quad \rho=0\,,$$

where C_2 and C_3 are complex constants. Hence, by using (3.7), we obtain

$$\theta(x,y)=A_1{}^{-1}\arctan\frac{A_1y+A_3}{A_1x+A_2}+A_4\,,\quad (A_1\neq 0)\qquad \text{if}\quad \rho\neq 0\,,$$
$$\theta(x,y)=A_1x+A_2y+A_3,\qquad \text{if}\quad \rho=0\,, \tag{3.8}$$

where A_j $(1\leq j\leq 4)$ is a real constant. The hypersurface $M=\{\exp_p(t\xi)\,;\,p\in\Sigma,\,t\in\boldsymbol{R}\}$ is parametrized by the coordinate (x, y, t) as

$$X(x, y, t)=(x, y, t\cos\theta, t\sin\theta).$$

When $\rho\neq 0$, we put $r=((A_1x+A_2)^2+(A_1y+A_3)^2)^{1/2}$. Then using (3.8), we see that the parametrization of M is given by

$$X(r, \theta, t)=\left(\frac{r\cos(A_1\theta-A_4)-A_2}{A_1},\ \frac{r\sin(A_1\theta-A_4)-A_3}{A_1},\ t\cos\theta,\ t\sin\theta\right).$$

When $\rho=0$, we put $r=-A_2x+A_1y$. Then the parametrization on M is

$$X(r, \theta, t)=\left(\frac{-A_2r+A_1\theta-A_1A_3}{(A_1{}^2+A_2{}^2)^2},\ \frac{A_1r+A_2\theta-A_2A_3}{A_1{}^2+A_2{}^2},\ t\cos\theta,\ t\sin\theta\right).$$

In both cases, the above expression of M shows that M is a ruled hypersurface in $\boldsymbol{R}^4$, because r and t are linear parameters which span totally geodesic planes $\boldsymbol{R}^2$.

EXAMPLE 3.2. Σ is a totally geodesic $S^2(1)$ in $S^4(1)$ $(c=1)$. We put

$$S^2(1)=\{(x^2+y^2+1)^{-1}(x^2+y^2-1, 2x, 2y, 0, 0)\in\boldsymbol{R}^5\,;\ x, y\in\boldsymbol{R}\}.$$

Then (x, y) is a local isothermal coordinate on $S^2(1)$. Let

$$e_1=\frac{x^2+y^2+1}{2}\frac{\partial}{\partial x},\qquad e_2=\frac{x^2+y^2+1}{2}\frac{\partial}{\partial y}.$$

Then they form an orthonormal local frame on $S^2(1)$ and satisfy

$$\nabla_{e_1}e_1=ye_2,\quad \nabla_{e_1}e_2=-ye_1,\quad \nabla_{e_2}e_1=-xe_2,\quad \nabla_{e_2}e_2=xe_1.$$

Any orthonormal normal frame ξ, η for $S^2(1)$ in $S^4(1)$ is given by

(3.9) $$\xi=(0, 0, 0, \cos\theta, \sin\theta),\qquad \eta=(0, 0, 0, -\sin\theta, \cos\theta),$$

where $\theta=\theta(x, y)$. Then the normal connection of $S^2(1)$ in $S^4(1)$ satisfies

$$s_1=\frac{x^2+y^2+1}{2}\theta_x,\qquad s_2=\frac{x^2+y^2+1}{2}\theta_y.$$

Moreover, we have

$$s_{11}=\frac{x^2+y^2+1}{2}\left(\frac{x^2+y^2+1}{2}\theta_{xx}+x\theta_x-y\theta_y\right),$$

$$s_{12}=\frac{x^2+y^2+1}{2}\left(\frac{x^2+y^2+1}{2}\theta_{xy}+x\theta_y+y\theta_x\right),$$

$$s_{21}=\frac{x^2+y^2+1}{2}\left(\frac{x^2+y^2+1}{2}\theta_{yx}+x\theta_y+y\theta_x\right),$$

$$s_{22}=\frac{x^2+y^2+1}{2}\left(\frac{x^2+y^2+1}{2}\theta_{yy}-x\theta_x+y\theta_y\right).$$

Then (3.1)–(3.3) are equivalent to

$$\begin{aligned}&\theta_{xx}+\theta_{yy}=0\,, \qquad \theta_{xy}=\theta_{yx}\,,\\ &(x^2+y^2+1)(\theta_y{}^2\theta_{xx}+\theta_x{}^2\theta_{yy}-\theta_x\theta_y(\theta_{xy}+\theta_{yx})\\ &\qquad -2(x\theta_x+y\theta_y)(\theta_x{}^2+\theta_y{}^2)=0\,.\end{aligned}$$

We define z, w, $\partial/\partial z$ and $\partial/\partial\bar{z}$ by (3.6). Then we get

$$(|z|^2+1)\left(w^2\left(\overline{\frac{\partial w}{\partial z}}\right)+\bar{w}^2\frac{\partial w}{\partial z}\right)+2(zw+\overline{zw})|w|^2=0\,,$$

$$w=w(z)\text{ is holomorphic}\,.$$

If $w\equiv 0$, θ is constant. So we assume that $w\not\equiv 0$. Hence we can write the above equation as

$$(|z|^2+1)\frac{\partial w}{\partial z}+2\bar{z}w=\sqrt{-1}\,\rho w^2\,, \tag{3.10}$$

$$\rho=\rho(z)\text{ is a real valued function}.$$

Differentiating this equation by $\partial/\partial\bar{z}$, we have

$$z\frac{\partial w}{\partial z}+2w=\sqrt{-1}\,\rho_{\bar{z}}w^2\,, \tag{3.11}$$

where $\rho_{\bar{z}}=\partial\rho/\partial\bar{z}$. Since $w\not\equiv 0$, $\rho_{\bar{z}}$ is holomorphic, i.e, $\rho_{\bar{z}\bar{z}}=0$. Hence we can see that ρ is written as

$$\rho=A_0|z|^2+C_1\bar{z}+\bar{C}_1z+A_1\,,$$

where A_0, A_1 are real constants, C_1 is a complex constant, and $\rho_{\bar{z}}=A_0z+C_1$. By (3.10) and (3.11), we get

$$w=\frac{\sqrt{-1}\,w^2}{2}\left(-\bar{C}_1z^2+(A_0-A_1)z+C_1\right),$$

$$\frac{\partial w}{\partial z}=\sqrt{-1}\,w^2(\bar{C}_1z+A_1)\,.$$

Since $w\not\equiv 0$, we have $w=2\sqrt{-1}\,(\bar{C}_1z^2+(A_1-A_0)z-C_1)^{-1}$. Differentiating w by z and using the above equations, we can easily see that $A_0=-A_1$. Hence the holomorphic function w is written as $w=2\sqrt{-1}\,(\bar{C}_1z^2+2A_1z-C_1)^{-1}$. Then we obtain

$$\int w\,dz=\sqrt{-1}\,(A_1{}^2+|C_1|^2)^{-1/2}(\log((A_1{}^2+|C_1|^2)^{1/2}-A_1-\bar{C}_1 z)$$
$$-\log((A_1{}^2+|C_1|^2)^{1/2}+A_1+\bar{C}_1 z))+C_2, \qquad \text{if} \quad C_1\neq 0\,,$$
$$\int w\,dz=\sqrt{-1}\,A_2\log z+C_2 \quad (C_2\in \boldsymbol{C})\,, \quad A_2=A_1{}^{-1}\,, \qquad \text{if} \quad C_1=0\,,$$

where $C_2\in \boldsymbol{C}$. By virtue of (3.6), we get

$$\begin{aligned}(3.12)\qquad \theta&=(A_1{}^2+A_2{}^2)^{-1/2}\\ &\times\arctan\left(2(A_1{}^2+A_2{}^2)^{1/2}\frac{x\sin A_3-y\cos A_3}{A_2(x^2+y^2-1)+2A_1(x\cos A_3+y\sin A_3)}\right)+A_4\,,\\ C_1&=A_2(\cos A_3+\sqrt{-1}\,\sin A_3)\,,\\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad C_1\neq 0\,,\\ \theta&=A_2\arctan\frac{y}{x}+A_3\,, \qquad\qquad\qquad \text{if} \quad C_1=0\,,\end{aligned}$$

where $A_j\in \boldsymbol{R}$ $(1\leq j\leq 4)$.

EXAMPLE 3.3. Σ is a totally geodesic $H^2(-1)$ in $H^4(-1)$ $(c=-1)$. We put

$$H^2(-1)=\{(1-x^2-y^2)^{-1}(1+x^2+y^2,\,2x,\,2y,\,0,\,0)\in L^5\,;$$
$$x,\,y\in \boldsymbol{R}\,, \text{ and } x^2+y^2<1\}\,.$$

Then (x, y) is an isothermal coordinate on $H^2(-1)$. Let

$$e_1=\frac{1-x^2-y^2}{2}\frac{\partial}{\partial x}\,, \qquad e_2=\frac{1-x^2-y^2}{2}\frac{\partial}{\partial y}\,.$$

Then they form an orthonormal frame on $H^2(-1)$ and satisfy

$$\nabla_{e_1}e_1=-ye_2\,, \quad \nabla_{e_1}e_2=ye_1\,, \quad \nabla_{e_2}e_1=xe_2\,, \quad \nabla_{e_2}e_2=-xe_1\,.$$

Any orthonormal normal frame ξ, η for $H^2(-1)$ in $H^4(-1)$ is given by (3.8). Then the normal connection of $H^2(-1)$ in $H^4(-1)$ satisfies

$$s_1=\frac{1-x^2-y^2}{2}\theta_x\,, \qquad s_2=\frac{1-x^2-y^2}{2}\theta_y\,,$$

and

$$s_{11}=\frac{1-x^2-y^2}{2}\left(\frac{1-x^2-y^2}{2}\theta_{xx}-x\theta_x+y\theta_y\right),$$

$$s_{12}=\frac{1-x^2-y^2}{2}\left(\frac{1-x^2-y^2}{2}\theta_{xy}-x\theta_y-y\theta_x\right),$$
$$s_{21}=\frac{1-x^2-y^2}{2}\left(\frac{1-x^2-y^2}{2}\theta_{yx}-x\theta_y-y\theta_x\right),$$
$$s_{22}=\frac{1-x^2-y^2}{2}\left(\frac{1-x^2-y^2}{2}\theta_{yy}+x\theta_x-y\theta_y\right).$$

Then (3.1)–(3.3) are equivalent to

$$\theta_{xx}+\theta_{yy}=0\,,\quad \theta_{xy}=\theta_{yx}\,,$$
$$(1-x^2-y^2)(\theta_y{}^2\theta_{xx}+\theta_x{}^2\theta_{yy}-\theta_x\theta_y(\theta_{xy}+\theta_{yx})$$
$$+2(x\theta_x+y\theta_y)(\theta_x{}^2+\theta_y{}^2)=0\,.$$

We define z, w, $\partial/\partial z$ and $\partial/\partial\bar{z}$ by (3.5). Then we see

$$(1-|z|^2)\left(w^2\left(\overline{\frac{\partial w}{\partial z}}\right)+\bar{w}^2\frac{\partial w}{\partial z}\right)-2(zw+\overline{zw})|w|^2=0\,,$$
$$w=w(z)\text{ is holomorphic}\,.$$

By the same argument as in Example 3.2, we have $w=2\sqrt{-1}(\bar{C}_1z^2+2A_1z+C_1)^{-1}$ ($A_1\in\boldsymbol{R}$, $C_1\in\boldsymbol{C}$), provided $w\not\equiv 0$. Then we get

$$\int w\,dz=\sqrt{-1}(A_1{}^2-|C_1|^2)^{-1/2}(\log((A_1{}^2-|C_1|^2)^{1/2}-A_1-\bar{C}_1z)$$
$$-\log((A_1{}^2-|C_1|^2)^{1/2}+A_1+\bar{C}_1z))+C_2\,,\qquad \text{if}\quad C_1\neq 0,\text{ and } A_1{}^2\neq|C_1|^2\,,$$
$$\int w\,dz=-2\sqrt{-1}\,\frac{e^{\sqrt{-1}A_3}}{A_1(z+e^{\sqrt{-1}A_3})}+C_2\,,\qquad \text{if}\quad C_1\neq 0\,,\text{ and } C_1=A_1e^{\sqrt{-1}A_3},$$
$$\int w\,dz=-\sqrt{-1}\,A_2\log z+C_2\,,\qquad \text{if}\quad C_1=0\,,$$

where $C_2\in\boldsymbol{C}$. By means of (3.6), we obtain

(3.13) $$\theta=(A_1{}^2-A_2{}^2)^{-1/2}$$
$$\times\arctan\left(2(A_1{}^2-A_2{}^2)^{1/2}\frac{x\sin A_3-y\cos A_3}{A_2(1+x^2+y^2)+2A_1(x\cos A_3+y\sin A_3)}\right)+A_4,$$
$$C_1=A_2(\cos A_3+\sqrt{-1}\,\sin A_3),$$
$$\text{if}\quad C_1\neq 0,\text{ and } A_1{}^2-A_2{}^2>0\,,$$

$$\theta=(A_2{}^2-A_1{}^2)^{-1/2}$$
$$\times\operatorname{arctanh}\left(2(A_2{}^2-A_1{}^2)^{1/2}\frac{x\sin A_3-y\cos A_3}{A_2(1+x^2+y^2)+2A_1(x\cos A_3+y\sin A_3)}\right)+A_4\,,$$
$$C_1=A_2(\cos A_3+\sqrt{-1}\,\sin A_3),$$
$$\text{if}\quad C_1\neq 0,\ \text{and}\ A_1{}^2-A_2{}^2<0\,,$$

$$\theta=2\frac{x\sin A_3-y\cos A_3}{A_1(x^2+y^2+1+2(x\cos A_3+y\sin A_3))}+A_4,\qquad \text{if}\quad C_1=A_1e^{\sqrt{-1}A_3}\neq 0\,,$$

$$\theta=A_2\arctan\frac{y}{x}+A_4\,,\qquad \text{if}\quad C_1=0\,,$$

where $A_j\in \boldsymbol{R}$ $(1\leq j\leq 4)$.

In consideration of these examples, we have

PROPOSITION 3.1. *Let Σ be a totally geodesic surface $\tilde{M}^2(c)$ in a 4-dimensional space form $\tilde{M}^4(c)$, and let ξ be a unit normal (local) vector field of Σ in $\tilde{M}$ given by* (3.4) *and* (3.9), *according as $c=0$ and $c\neq 0$. Then the hypersurface $M=\{\exp_p(t\xi)\,;\,p\in\Sigma,\,t\in\boldsymbol{R}\}$ is minimal if and only if the function θ which determines ξ is given by* (3.8), (3.12) *and* (3.13), *according as $c=0$, $c>0$ and $c<0$, respectively.*

§4. Construction of examples II.

In this section, we determine the unit normal vector field ξ on the minimal Clifford torus $S^1\times S^1\subset S^3(1)\subset S^4(1)$ such that $M=\{\exp_p(t\xi)\,;\,p\in\Sigma,\,t\in\boldsymbol{R}\}$ is a minimal hypersurface in $S^4(1)$. The position vector of Σ in $\boldsymbol{R}^5$ is given by

$$((1/\sqrt{2})\cos x,\ (1/\sqrt{2})\sin x,\ (1/\sqrt{2})\cos y,\ (1/\sqrt{2})\sin y,\ 0)\,,$$

for an isothermal coordinate (x, y) of Σ. Let $e_1=\sqrt{2}\,\partial/\partial x$ and $e_2=\sqrt{2}\,\partial/\partial y$. Then they form an orthonormal frame on Σ, and satisfy $\nabla e_1=\nabla e_2=0$. An orthonormal normal frame is given by

$$\text{(4.1)}\qquad \xi_1=(-(1/\sqrt{2})\cos x,\ -(1/\sqrt{2})\sin x,\ (1/\sqrt{2})\cos y,\ (1/\sqrt{2})\sin y,\ 0)\,,$$
$$\xi_2=(0,0,0,0,1)\,.$$

Then the shape operator and the normal connection of Σ in $S^4(1)$ are given by

$$A_{\xi_1}=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad A_{\xi_2}=\begin{pmatrix}0&0\\0&0\end{pmatrix},\quad \nabla^{\perp}\xi_1=\nabla^{\perp}\xi_2=0\,.$$

Moreover, we can see that the covariant derivative of the second fundamental form of Σ in $S^4(1)$ is identically zero. Any orthonormal normal frame ξ, η for Σ in $S^4(1)$ is given by

$$\xi=\cos\theta\,\xi_1+\sin\theta\,\xi_2\,,\quad \eta=-\sin\theta\,\xi_1+\cos\theta\,\xi_2\,,$$

where $\theta=\theta(x, y)$. So we have $h_{1}{}^{\xi}_{1}=-h_{2}{}^{\xi}_{2}=\cos\theta$, $h_{1}{}^{\xi}_{2}=0$, $h_{1}{}^{\eta}_{1}=-h_{2}{}^{\eta}_{2}=-\sin\theta$, $h_{1}{}^{\eta}_{2}=0$, and $h_{ij}{}^{\xi}_{k}=0$ $(i, j, k=1, 2)$. The normal connection satisfies $s_1=\sqrt{2}\,\theta_x$, $s_2=\sqrt{2}\,\theta_y$, $s_{11}=2\theta_{xx}$, $s_{12}=2\theta_{xy}$, $s_{21}=2\theta_{yx}$, and $s_{22}=2\theta_{yy}$.

Suppose $M=\{\exp_p(t\xi)\,;\,p\in\Sigma,\,t\in\boldsymbol{R}\}$ is a minimal hypersurface of $S^4(1)$. Since (2.12), (2.14) and (2.16) hold, we have

$$s_{11}+s_{22}=-\operatorname{trace}A_\xi A_\eta\,,\qquad \|\nabla^\perp\xi\|^2\|A_\xi\|^2\operatorname{trace}A_\xi A_\eta=0\,,$$

where $\|A_\xi\|^2=\sum_{i,j}(h_{ij}^\xi)^2=2\cos^2\theta$, $\operatorname{trace}A_\xi A_\eta=\sum_{i,j}h_{ij}^\xi h_{ij}^\eta=-2\cos\theta\sin\theta$, and $\|\nabla^\perp\xi\|^2=s_1{}^2+s_2{}^2=2\|\operatorname{grad}\theta\|^2$. If $\cos\theta\sin\theta\neq 0$, then θ is constant. Hence $s_{11}+s_{22}=2(\theta_{xx}+\theta_{yy})=0$, and this is a contradiction. Consequently, we have $\cos\theta\equiv 0$ or $\sin\theta\equiv 0$, i.e., $\xi=\xi_1$ or $\xi=\xi_2$. When $\xi=\xi_1$, M is a totally geodesic $S^3(1)$, and when $\xi=\xi_2$, M is a "cylinder". Thus, we obtain

PROPOSITION 4.1. *Let Σ be the minimal Clifford torus $S^1\times S^1\subset S^3(1)\subset S^4(1)$, and let ξ_1, ξ_2 be a natural orthonormal normal frame of Σ in $S^4(1)$ defined by* (4.1). *We define a unit normal vector field ξ of Σ in $S^4(1)$ by $\xi=\cos\theta\,\xi_1+\sin\theta\,\xi_2$, where θ is a function on Σ. Suppose $M=\{\exp_p(t\xi)\,;\,p\in\Sigma,\,t\in\boldsymbol{R}\}$ is a minimal hypersurface. Then $\theta\equiv 0 \bmod \pi/2$ holds. As a consequence, $M=S^3(1)$ (totally geodesic) or M is a "cylinder".*

More generally, we have the following examples:

EXAMPLE 4.2. Let Σ be a surface in $\tilde{M}^3(c)$, and let ξ be a unit normal vector field of Σ in $\tilde{M}^3(c)$. If we regard Σ as a surface in $\tilde{M}^4(c)$, then the hypersurface $M=\{\exp_p(t\xi)\,;\,p\in\Sigma,\,t\in\boldsymbol{R}\}$ is itself a totally geodesic $\tilde{M}^3(c)$ in $\tilde{M}^4(c)$.

EXAMPLE 4.3. Let Σ be a minimal surface in $\tilde{M}^3(c)$, and let ξ be a unit normal vector field of a totally geodesic $\tilde{M}^3(c)$ in $\tilde{M}^4(c)$. Then we can easily see that $M=\{\exp_p(t\xi)\,;\,p\in\Sigma,\,t\in\boldsymbol{R}\}$ is minimal in $\tilde{M}^4(c)$. In particular, when $c=0$, we have $M=\Sigma\times\boldsymbol{R}\subset\boldsymbol{R}^3\times\boldsymbol{R}=\boldsymbol{R}^4$, and M is a cylinder. In each case, a short calculation yields that the type number (that is, the rank of the shape operator) is equal to 0 or 2 at each point.

§5. Construction of examples III.

In this section, we construct minimal hypersurfaces M from the Veronese surface in $S^4(3)$ $(c=3)$. Let $H(3, \boldsymbol{R})=\{Y\in M(3, \boldsymbol{R})\,;\,{}^tY=Y\}$ be the set of (3×3)-symmetric matrices. $H(3, \boldsymbol{R})$ is a 6-dimensional linear subspace of $M(3, \boldsymbol{R})=\boldsymbol{R}^9$. We define a metric in $H(3, \boldsymbol{R})$ by

$$\langle Y, Z\rangle=\frac{1}{2}\operatorname{trace}(YZ)\,,\qquad Y, Z\in H(3, \boldsymbol{R})\,.$$

Define a map $\Psi: S^2(1) \to H(3, \boldsymbol{R})$ as

$$\Psi(\boldsymbol{y}) = \boldsymbol{y}{}^t\boldsymbol{y}\,, \qquad \boldsymbol{y} = {}^t(y_0, y_1, y_2) \in S^2(1)\,.$$

Then we have $\Sigma = \Psi(S^2(1)) = \{Y \in H(3, \boldsymbol{R})\,;\ Y^2 = Y, \text{trace}\, Y = 1\}$. Using this fact, we can see that $\langle Y - \frac{1}{3}I, Y - \frac{1}{3}I\rangle = 1/3$ holds for $Y \in \Sigma$, where I denotes the identity (3×3)-matrix. Hence Σ lies in the hypersphere $S^4(3)$ with center at $\frac{1}{3}I$ and of radius $1/\sqrt{3}$ in a 5-dimensional linear space $\{Y \in H(3, \boldsymbol{R})\,;\ \text{trace}\, Y = 1\}$ $(\cong \boldsymbol{R}^5)$. Moreover, it can be shown that Ψ is an isometric immersion (cf. [3]).

First, we write the position vector $\boldsymbol{y}$ of $S^2(1)$ by an isothermal coordinate (x, y) as follows:

$$\boldsymbol{y} = {}^t(x^2 + y^2 - 1, 2x, 2y)/(x^2 + y^2 + 1) \in S^2(1)\,.$$

We define $P = P(x, y)$ by

$$P(x, y) = (x^2 + y^2 + 1)^{-1}\begin{pmatrix} x^2 + y^2 - 1 & 2x & 2y \\ 2x & -x^2 + y^2 + 1 & -2xy \\ 2y & -2xy & x^2 - y^2 + 1 \end{pmatrix}.$$

Then P is an orthogonal (3×3)-matrix, and the position vector $Y = \Psi(\boldsymbol{y}) \in \boldsymbol{R}^5$ is written as

$$(5.1) \qquad Y = P\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}{}^tP\,.$$

Let

$$(5.2) \qquad e_1 = P\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}{}^tp\,, \qquad e_2 = P\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}{}^tP\,.$$

Then they form a local orthonormal basis of $T_Y(\Sigma)$ and satisfy

$$(5.3) \qquad e_1 = \frac{x^2 + y^2 + 1}{2}\Psi_*\left(\frac{\partial}{\partial x}\right), \qquad e_2 = \frac{x^2 + y^2 + 1}{2}\Psi_*\left(\frac{\partial}{\partial y}\right).$$

We put

$$(5.4) \qquad \xi_1 = P\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}{}^tP\,, \qquad \xi_2 = P\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}{}^tP\,,$$

then they form an orthonormal normal frame for Σ in $S^4(3)$ (locally). If we denote by D the canonical connection of $\boldsymbol{R}^5$, then direct calculations imply that

$$D_{e_1}e_1 = I - 3Y + ye_2 + \xi_1\,, \qquad D_{e_1}e_2 = -ye_1 + \xi_2\,,$$
$$D_{e_2}e_1 = -xe_2 + \xi_2\,, \qquad D_{e_2}e_2 = I - 3Y + xe_1 - \xi_1\,,$$
$$D_{e_1}\xi_1 = -e_1 + 2y\xi_2\,, \qquad D_{e_1}\xi_2 = -e_2 - 2y\xi_1\,,$$
$$D_{e_2}\xi_1 = e_2 - 2x\xi_2\,, \qquad D_{e_2}\xi_2 = -e_1 + 2x\xi_1\,.$$

Hence, using (1.4), we have

$$\begin{aligned} &\nabla_{e_1}e_1 = ye_2\,, && \nabla_{e_1}e_2 = -ye_1\,,\\ &\nabla_{e_2}e_1 = -xe_2\,, && \nabla_{e_2}e_2 = xe_1\,,\\ &A_{\xi_1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, && A_{\xi_2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\\ &\nabla^{\perp}_{e_1}\xi_1 = 2y\xi_2\,, && \nabla^{\perp}_{e_1}\xi_2 = -2y\xi_1\,,\\ &\nabla^{\perp}_{e_2}\xi_1 = -2x\xi_2\,, && \nabla^{\perp}_{e_2}\xi_2 = 2x\xi_1\,, \end{aligned} \tag{5.5}$$

and Σ is minimal in $S^4(3)$.

Any orthonormal normal frame of Σ in $S^4(3)$ is locally described as

$$\xi = \cos\theta\, \xi_1 + \sin\theta\, \xi_2\,, \qquad \eta = -\sin\theta\, \xi_1 + \cos\theta\, \xi_2\,, \tag{5.6}$$

where $\theta = \theta(x, y)$. Then we have

$$\begin{aligned} &h_{1\,1}^{\,\xi} = -h_{2\,2}^{\,\xi} = \cos\theta\,, && h_{1\,2}^{\,\xi} = \sin\theta\,,\\ &h_{1\,1}^{\,\eta} = -h_{2\,2}^{\,\eta} = -\sin\theta\,, && h_{1\,2}^{\,\eta} = \cos\theta\,. \end{aligned} \tag{5.7}$$

We note that $\operatorname{trace} A_\xi A_\eta = 0$ and $h_{ijk}^{\xi} = 0$ $(1 \le i, j, k \le 2)$ hold, because of (1.4), (5.5) and (5.7). The Ricci equation (1.8) is written as

$$s_{12} - s_{21} = 2\,, \tag{5.8}$$

by (5.7). By Proposition 2.1, the hypersurface $M = \{\exp_p(t\xi)\,;\, p \in \Sigma,\, t \in \boldsymbol{R}\}$ is minimal if and only if Σ satisfies

$$\begin{aligned} &s_{11} + s_{22} = 0\,,\\ &-2h_{2\,2}^{\,\xi} s_{11} + 2h_{1\,2}^{\,\xi}(s_{12} + s_{21}) - 2h_{1\,1}^{\,\xi} s_{22} = -\det A^M|_\Sigma\,,\\ &s_2{}^2 s_{11} + s_1{}^2 s_{22} - s_1 s_2(s_{12} + s_{21}) = 0\,. \end{aligned} \tag{5.9}$$

Here $\det A^M|_\Sigma$ denotes the determinant of the shape operator A^M of M in $\tilde{M}$ on a point of $\Sigma \subset M$ with respect to the orthonormal basis (e_1, e_2, ξ) and the normal vector η of M. Hence we have

$$\det A^M|_\Sigma = -h_{1\,1}^{\,\eta} s_2{}^2 + 2s_1 s_2 h_{1\,2}^{\,\eta} - h_{2\,2}^{\,\eta} s_1{}^2\,. \tag{5.10}$$

If we regard (5.8) and (5.9) as linear equations in s_{11}, s_{12}, s_{21} and s_{22}, then the determinant of the coefficient matrix is equal to $4\{h_{1\,2}^{\,\xi}(s_2{}^2 - s_1{}^2) + (h_{1\,1}^{\,\xi} - h_{2\,2}^{\,\xi})s_1 s_2\} = 4\det A^M|_\Sigma$ by (5.7).

First, we suppose $\det A^M|_\Sigma \neq 0$ (locally). Then, using (5.7) and (5.10), we see that

(5.8) and (5.9) are equivalent to

(5.11)
$$s_{11}=-\frac{1}{2}s_1s_2\,,\qquad s_{12}=\frac{1}{4}(s_1{}^2-s_2{}^2)+1\,,$$
$$s_{21}=\frac{1}{4}(s_1{}^2-s_2{}^2)-1\,,\qquad s_{22}=\frac{1}{2}s_1s_2\,.$$

By (5.3), (5.5) and (5.6), the components of the normal connection are given by

(5.12)
$$s_1=\frac{x^2+y^2+1}{2}\theta_x+2y\,,\qquad s_2=\frac{x^2+y^2+1}{2}\theta_y-2x\,,$$

and

$$s_{11}=\frac{x^2+y^2+1}{4}((x^2+y^2+1)\theta_{xx}+2x\theta_x-2y\theta_y)+2xy\,,$$
$$s_{12}=\frac{x^2+y^2+1}{4}((x^2+y^2+1)\theta_{xy}+2y\theta_x+2x\theta_y)-x^2+y^2+1\,,$$
$$s_{21}=\frac{x^2+y^2+1}{4}((x^2+y^2+1)\theta_{yx}+2y\theta_x+2x\theta_y)-x^2+y^2-1\,,$$
$$s_{22}=\frac{x^2+y^2+1}{4}((x^2+y^2+1)\theta_{yy}-2x\theta_x+2y\theta_y)-2xy\,.$$

Then we see that (5.11) is written as

(5.13)
$$\theta_{xx}=-\frac{1}{2}\theta_x\theta_y\,,\qquad \theta_{yy}=\frac{1}{2}\theta_x\theta_y\,,$$
$$\theta_{xy}=\theta_{yx}=\frac{1}{4}(\theta_x{}^2-\theta_y{}^2)\,.$$

If we define z, w, $\partial/\partial z$ and $\partial/\partial\bar{z}$ by (3.6), then (5.13) is equivalent to

$$\frac{\partial w}{\partial z}=-\frac{\sqrt{-1}}{4}w^2\,,\qquad w=w(z)\text{ is holomorphic.}$$

Therefore, we have $w=4(\sqrt{-1}\,z+C)^{-1}$ ($C\in\boldsymbol{C}$), provided $w\not\equiv 0$. By means of (3.7), we obtain

(5.14)
$$\theta(x,y)=4\arctan\frac{y+A_2}{x+A_1}+A_3\,,\qquad\text{or}\quad \theta(x,y)=A_1\,,$$

where $A_j\in\boldsymbol{R}$ $(1\le j\le 3)$. Then the hypersurface M is parametrized by a local coordinate (x, y, t) as

$$X(x, y, t)=\frac{1}{3}I+\cos\sqrt{3}\,t\left(Y-\frac{1}{3}I\right)+\frac{1}{\sqrt{3}}\sin\sqrt{3}\,t\,\xi\,.$$

By (5.7), (5.10), (5.12) and (5.14), we can see that $\det A^M|_\Sigma \not\equiv 0$, so M has type number 3 on some open set. Hence we obtain

THEOREM 5.1. *Let Σ be the Veronese surface in $S^4(3)$, and let ξ_1, ξ_2 be a natural orthonormal normal frame of Σ in $S^4(3)$ defined by* (5.4). *We define a unit normal vector field ξ of Σ in $S^4(3)$ by $\xi=\cos\theta\,\xi_1+\sin\theta\,\xi_2$, where θ is a function on Σ. Then $M=\{\exp_p(t\xi)\,;\,p\in\Sigma,\,t\in\boldsymbol{R}\}$ is minimal and $\det A^M|_\Sigma \not\equiv 0$ if and only if θ is given by* (5.14).

REMARK 5.2. By Theorem 5.1, we find all minimal hypersurfaces M of S_4 satisfying the following conditions: (1) M contains a Veronese surface Σ of S^4, (2) M is foliated by great circles S^1 of S^4 intersecting Σ orthogonally, (3) the type number of M is equal to 3 on some open set which intersects Σ.

REMARK 5.3. Examples of Theorem 5.1 are not complete because $\det A^M|_\Sigma$ diverges when (x, y) goes to point at infinity.

Finally we prove

PROPOSITION 5.4. *Let Σ be the Veronese surface in $S^4(3)$, and let ξ be a unit normal vector field on Σ in $S^4(3)$. We put $M=\{\exp_p(t\xi)\,;\,p\in\Sigma,\,t\in\boldsymbol{R}\}$. Suppose M is minimal in $S^4(3)$, and satisfies $\det A^M|_\Sigma\equiv 0$. Then M is a ruled hypersurface (i.e., M is foliated by the totally geodesic $S^2(3)$ in $S^4(3)$).*

PROOF. In order for M to be minimal in $S^4(3)$, Σ and ξ must satisfy (5.9). We note that the equations of (5.9) are independent of the choice of the orthonormal frame e_1, e_2 on Σ. So we can take the orthonormal frame on Σ such that $h_1{}^\xi_2=0$. Since Σ satisfies $\operatorname{trace} A_\xi A_\eta=0$, we have $h_1{}^\eta_1=h_2{}^\eta_2=0$. Using (5.10), we get $s_1s_2=0$, because $A_\eta\neq 0$. We note that the normal connection of Σ in $S^4(3)$ is not trivial. So we may assume that $s_1=0$ and $s_2\neq 0$. Then (5.9) yields

$$s_{11}+s_{22}=0\,, \qquad 2h_2{}^\xi_2 s_{11}+2h_1{}^\xi_1 s_{22}=0\,.$$

So we get $s_{11}=s_{22}=0$, because $A_\xi\neq 0$ and $\operatorname{trace} A_\xi=0$. By the definition (1.6) of s_{ij}, we obtain $\omega_1=0$ and $e_2s_2=0$. Hence an integral curve γ of e_1 satisfies

$$\bar\nabla_{e_1}e_1=h_1{}^\xi_1\xi\,, \qquad \bar\nabla_{e_1}\xi=-h_1{}^\xi_1 e_1\,.$$

Thus, γ lies on $S^2(3)$ which is totally geodesic in $S^4(3)$, and the tangent space of $S^2(3)$ on a point of γ is spanned by e_1 and ξ. Consequently, by the definition, M is a ruled hypersurface. Q.E.D.

An example of Proposition 5.4 is constructed as follows: We write the position vector $\boldsymbol{y}$ of $S^2(1)$ by a polar coordinate (u, v) as

$$y={}^t(\cos u\cos v,\ \cos u\sin v,\ \sin u)\in S^2(1)\,,$$

where $-\pi/2<u<\pi/2$ and $0\leq v<2\pi$. We define $P=P(u,v)$ by

$$P(u,v)=\begin{pmatrix}\cos u\cos v & -\sin u\cos v & -\sin v\\ \cos u\sin v & -\sin u\sin v & \cos v\\ \sin u & \cos u & 0\end{pmatrix}.$$

Then the position vector Y of Σ and the orthonormal local frame e_1, e_2 on Σ are given by (5.1) and (5.2), respectively. We define the orthonormal normal frame ξ_1, ξ_2 for Σ in $S^4(3)$, locally by (5.4). Then the similar computations as the first part of this section imply that

$$\begin{aligned}&\nabla_{e_1}e_1=\nabla_{e_1}e_2=0\,,\\ &\nabla_{e_2}e_1=-\tan u\, e_2\,, && \nabla_{e_2}e_2=\tan u\, e_1\,,\\ &A_{\xi_1}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, && A_{\xi_2}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\\ &\nabla^{\perp}_{e_1}\xi_1=\nabla^{\perp}_{e_1}\xi_2=0\,,\\ &\nabla^{\perp}_{e_2}\xi_1=-2\tan u\,\xi_2\,, && \nabla^{\perp}_{e_2}\xi_2=2\tan u\,\xi_1\,.\end{aligned}$$

We put $\xi=\xi_1$ and $\eta=\xi_2$. Then we have $s_1=0$, $s_2=-2\tan u$, and $\det A^M|_\Sigma=0$. Moreover, we get

$$s_{11}=s_{22}=0\,,\quad s_{12}=-2\tan^2 u\,,\quad s_{21}=-2(1+\tan^2 u)\,.$$

Therefore, Σ satisfies (5.8) and (5.9). By the proof of Proposition 5.4, we can see that the hypersurface $M=\{\exp_p(t\xi)\,;\,p\in\Sigma,\ t\in\boldsymbol{R}\}$ is a ruled minimal hypersurface in $S^4(3)$, so the type number of M is at most 2.

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