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Q-polynomial of Pretzel Links

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Abstract. We find a formula for the Q-polynomial of a pretzel link $L(p_1, p_2, \cdots, p_n)$. The formula describes the Q-polynomial as a polynomial in $H_{p_1}(x), H_{p_2}(x), \cdots, H_{p_n}(x)$, where $H_{p_1}(x), H_{p_2}(x), \cdots, H_{p_n}(x)$ are Laurent polynomials in x and $H_{p_i}(x)$ depends on only p_i.

1. Introduction.

Brandt–Lickorish–Millett [BLM] and Ho [H] defined a polynomial invariant for unoriented links, and the invariant is called the Q-polynomial. Kauffman [K] defined a polynomial invariant, called the F-polynomial, for oriented links and he remarked that $Q_{\bar{L}}(x) = F_L(1, x)$ for any oriented link L, where $\bar{L}$ is the unoriented link obtained from L by forgetting about the orientation.

Lickorish calculated F-polynomial of two-bridge links. In this paper, we calculate Q-polynomial of pretzel links.

For an integer p we denote an integral tangle diagram depicted in Fig. 1 by $\tilde{I}_p$. Let $\tilde{L}(p_1, p_2, \cdots, p_n)$ be a link diagram that is a numerator of a tangle diagram obtained

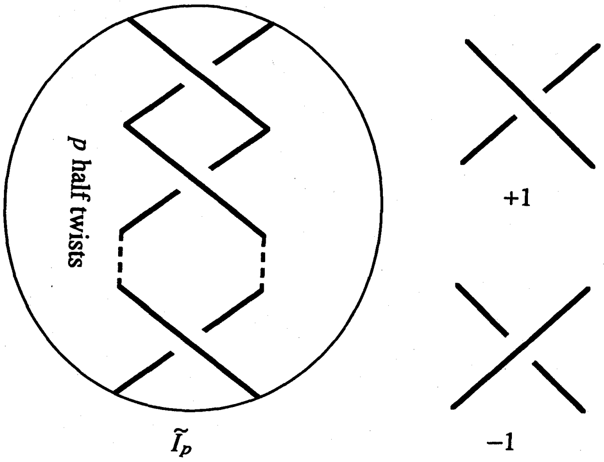

FIGURE 1

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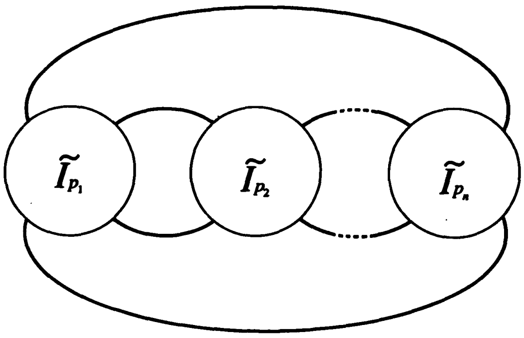


FIGURE 2

by summing integral tangle diagrams $\tilde{I}_{p_1}, \tilde{I}_{p_2}, \cdots, \tilde{I}_{p_n}$ (see Fig. 2). We call a link represented by a diagram $\tilde{L}(p_1, p_2, \cdots, p_n)$ a *pretzel link* $L(p_1, p_2, \cdots, p_n)$.

We set $\lambda_+ = (x+\sqrt{x^2-4})/2$ and $\lambda_- = (x-\sqrt{x^2-4})/2$. It is clear that

$$\begin{cases} \lambda_+ + \lambda_- = x\,, \\ \lambda_+\lambda_- \quad = 1\,. \end{cases} \tag{1}$$

For an integer p let

$$H_p(x) = \frac{\lambda_+^p - \lambda_-^p}{\lambda_+^2 - \lambda_-^2}\,.$$

We note that the two facts that λ_+ and λ_- satisfy (1) and that $H_p(x)$ is a symmetric polynomial in λ_+ and λ_- imply that $H_p(x)$ is a Laurent polynomial in x.

Let $\mathcal{M}_n$ be the set of mappings from $\{1, 2, \cdots, n\}$ to $\{-1, 0, 1\}$. For $\mu \in \mathcal{M}_n$ we set

$$S\mu = \sum_{i=1}^{n} \mu(i)\,,$$

and denote the number of elements of $\mu^{-1}(0)$ by $N\mu$.

THEOREM. *For nonzero integers $p_1, p_2, \cdots, p_n$, we have*

$$Q_{L(p_1, p_2, \cdots, p_n)}(x) = \sum_{\mu \in \mathcal{M}_n} G_{S\mu}(x)\left(\frac{x}{2-x}\right)^{N\mu} \pi[\mu; p_1, p_2, \cdots, p_n](x)\,,$$

where

$$G_p(x) = \left(1 - \frac{x}{2-x}\right)(H_{1+p}(x) + H_{1-p}(x)) + \frac{x}{2-x}\,,$$

$$\pi[\mu; p_1, p_2, \cdots, p_n](x) = \prod_{i \in \mu^{-1}(1)} H_{1+p_i}(x) \prod_{i \in \mu^{-1}(-1)} H_{1-p_i}(x) \prod_{i \in \mu^{-1}(0)} (1 - H_{1+p_i}(x) - H_{1-p_i}(x))\,,$$

and $\prod_{i \in \varnothing} = 1$. □

Since all $H_p(x)$'s are Laurent polynomials in x, we know that each $\pi[\mu; p_1, p_2, \cdots, p_n](x)$ is also a Laurent polynomial. We shall show that all $G_s(x)$'s are Laurent polynomials.

2. Proof.

In order to prove Theorem we need to show some properties of the polynomials $H_p(x)$ and $G_p(x)$.

LEMMA 1. *For an integer p we have*

$$H_{1+p}(x) + H_{-1+p}(x) = xH_p(x) .$$

PROOF. By (1) we have $\lambda_+ + \lambda_+^{-1} = x$ and $\lambda_- + \lambda_-^{-1} = x$. It follows that

$$\begin{aligned} H_{1+p}(x) + H_{-1+p}(x) &= \frac{\lambda_+^{1+p} - \lambda_-^{1+p}}{\lambda_+^2 - \lambda_-^2} + \frac{\lambda_+^{-1+p} - \lambda_-^{-1+p}}{\lambda_+^2 - \lambda_-^2} \\ &= \frac{\lambda_+^p(\lambda_+ + \lambda_+^{-1}) - \lambda_-^p(\lambda_- + \lambda_-^{-1})}{\lambda_+^2 - \lambda_-^2} \\ &= \frac{x(\lambda_+^p - \lambda_-^p)}{\lambda_+^2 - \lambda_-^2} \\ &= xH_p(x) . \end{aligned}$$

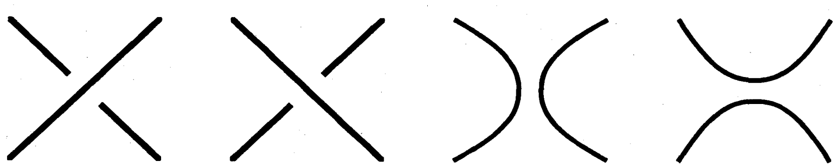
□

We now recall the definition of the Q-polynomial of links. That is the Laurent polynomial invariant for unoriented links defined inductively by

$$\begin{cases} Q_{\bigcirc}(x) = 1 , \\ Q_{L_+}(x) + Q_{L_-}(x) = xQ_{L_0}(x) + xQ_{L_\infty}(x) , \end{cases}$$

where $\bigcirc$ is the trivial knot and L_+, L_-, L_0 and L_∞ are idential except in the neighbourhood of one crossing point where they look as in Fig. 3.

We note that a link $L(\underbrace{1, 1, \cdots, 1}_{s})$ is a $(2, s)$-*torus link* and denote it by $T[s]$.



L_+ L_- L_0 L_∞

FIGURE 3

LEMMA 2. *Let s be an integer. Then*

$$Q_{T[s]}(x)=G_s(x)\,.$$

PROOF. We consider a sequence $\{X_s(x)\}_{s\in Z}$ in $Z[x^{\pm 1}]$, and a system of equations

$$\begin{cases} X_{1+s}(x)+X_{-1+s}(x)=xX_s(x)+x\,, \\ X_1(x)=X_{-1}(x)=1\,, \end{cases}$$

where Z is the set of integers. The system has a unique solution. Therefore to prove Lemma 2 we need to verify that both sequences $\{Q_{T[s]}(x)\}_{s\in Z}$ and $\{G_s(x)\}_{s\in Z}$ are solutions of the system.

By the definition of the Q-polynomial we have

$$\begin{aligned} Q_{T[1+s]}(x)+Q_{T[-1+s]}(x)&=xQ_{T[s]}(x)+xQ_{\bigcirc}(x) \\ &=xQ_{T[s]}(x)+x\,. \end{aligned}$$

Since both $T[1]$ and $T[-1]$ are the trivial knots, $Q_{T[\pm 1]}(x)=1$. Hence $\{Q_{T[s]}(x)\}_{s\in Z}$ is a solution of the system.

By Lemma 1 we have

$$\begin{aligned} &G_{1+s}(x)+G_{-1+s}(x) \\ &=\left(1-\frac{x}{2-x}\right)(H_{2+s}(x)+H_{-s}(x)+H_s(x)+H_{2-s}(x))+\frac{2x}{2-x} \\ &=\left(1-\frac{x}{2-x}\right)(xH_{1+s}(x)+xH_{1-s}(x))+\frac{2x}{2-x} \\ &=xG_s(x)-\frac{x^2}{2-x}+\frac{2x}{2-x} \\ &=xG_s(x)+x\,. \end{aligned}$$

By $H_0(x)=0$, $H_2(x)=1$ we have $G_{\pm 1}(x)=1$. Hence we conclude that $\{G_s(x)\}_{s\in Z}$ is a solution of the system. This completes the proof. □

By Lemma 2 we know that all $G_s(x)$'s are Laurent polynomials in x.

We denote the Q-polynomial of $L(p_1, p_2, \cdots, p_n)$ by $Q[p_1, p_2, \cdots, p_n](x)$.

LEMMA 3. *For integers* $p_1, p_2, \cdots, p_n$ $(n\geq 2)$,

$$\begin{aligned} &Q[p_1, p_2, \cdots, p_n](x) \\ &\quad=H_{1+p_n}(x)Q[p_1, \cdots, p_{n-1}, 1](x)+H_{1-p_n}(x)Q[p_1, \cdots, p_{n-1}, -1](x) \\ &\quad\quad+\frac{x}{2-x}(1-H_{1+p_n}(x)-H_{1-p_n}(x))Q[p_1, p_2, \cdots, p_{n-1}](x)\,. \end{aligned}$$

PROOF. Fix $p_1, p_2, \cdots, p_{n-1}$. We consider a sequence $\{X_p(x)\}_{p\in Z}$ in $Z[x^{\pm 1}]$, and the system of equations

$$\begin{cases} X_{1+p}(x)+X_{-1+p}(x)=xX_p(x)+xQ[p_1, p_2, \cdots, p_{n-1}](x), \\ X_1(x)=Q[p_1, \cdots, p_{n-1}, 1](x), \\ X_{-1}(x)=Q[p_1, \cdots, p_{n-1}, -1](x). \end{cases}$$

The system has a unique solution. By the definition of the Q-polynomial the sequence $\{Q[p_1, \cdots, p_{n-1}, p](x)\}_{p\in \mathbf{Z}}$ is a solution of the system. By Lemma 1 we obtain that the sequence

$$\Big\{H_{1+p}(x)Q[p_1, \cdots, p_{n-1}, 1](x)+H_{1-p}(x)Q[p_1, \cdots, p_{n-1}, -1](x)$$
$$+\frac{x}{2-x}(1-H_{1+p}(x)-H_{1-p}(x))Q[p_1, p_2, \cdots, p_{n-1}](x)\Big\}_{p\in \mathbf{Z}}$$

is a solution of the system. □

Now we restate and prove Theorem.

THEOREM. *For nonzero integers $p_1, p_2, \cdots, p_n$,*

$$Q[p_1, p_2, \cdots, p_n](x)=\sum_{\mu\in\mathcal{M}} G_{S\mu}(x)\left(\frac{x}{2-x}\right)^{N\mu}\pi[\mu; p_1, p_2, \cdots, p_n](x).$$

PROOF. We prove Theorem by induction on $|p_1|+|p_2|+\cdots+|p_n|-n$.

If $|p_1|+|p_2|+\cdots+|p_n|-n=0$, then each p_i $(i=1, 2, \cdots, n)$ is either 1 or -1. It follows that the pretzel link $L(p_1, p_2, \cdots, p_n)$ is the $(2, p_1+p_2+\cdots+p_n)$-torus link. By Lemma 2 we have

$$Q[p_1, p_2, \cdots, p_n](x)=Q_{T[p_1+p_2+\cdots+p_n]}(x)=G_{(p_1+p_2+\cdots+p_n)}(x). \tag{2}$$

On the other hand let ν be the mapping in $\mathcal{M}_n$ with $\nu(i)=p_i$ for $i=1, 2, \cdots, n$. If $\mu\neq\nu$ for $\mu\in\mathcal{M}_n$, then there exists an integer j $(j=1, 2, \cdots, n)$ with $\mu(j)\neq p_j$. We note that

$$\begin{cases} H_{1+p_j}(x)=0 & \text{if } \mu(j)=1, \\ H_{1-p_j}(x)=0 & \text{if } \mu(j)=-1, \\ 1-H_{1+p_j}(x)-H_{1-p_j}(x)=0 & \text{if } \mu(j)=0. \end{cases}$$

Therefore we have $\pi[p_1, p_2, \cdots, p_n](x)=0$ if $\mu\neq\nu$. It follows that

$$\sum_{\mu\in\mathcal{M}_n} G_{S\mu}(x)\left(\frac{x}{2-x}\right)^{N\mu}\pi[\mu; p_1, p_2, \cdots, p_n](x) \tag{3}$$
$$=G_{S\nu}(x)\left(\frac{x}{2-x}\right)^{N\nu}\pi[\nu; p_1, p_2, \cdots, p_n](x).$$

By the defintion of ν and by $H_2(x)=1$ we have

$$S\nu=p_1+p_2+\cdots+p_n\,,$$
$$N\nu=0\,,$$
$$\pi[\nu;\,p_1,p_2,\cdots,p_n](x)=1\,.$$

It follows that

$$G_{S\nu}(x)\left(\frac{x}{2-x}\right)^{N\nu}\pi[\nu;\,p_1,p_2,\cdots,p_n](x)=G_{(p_1+p_2+\cdots+p_n)}(x)\,. \tag{4}$$

By (2), (3) and (4) we conclude that the formula of Theorem holds.

Next we prove in the case that $|p_1|+|p_2|+\cdots+|p_n|-n>0$.

If $n=1$, then $L(p_1)$ is the trivial knot. We have $Q[p_1](x)=1$. Since $\mathscr{M}_1$ contains exactly three mappings, the right side of the formula in Theorem is the sum of three polynomials $H_{1+p_1}(x)$, $H_{1-p_i}(x)$ and $1-H_{1+p_1}(x)-H_{1-p_i}(x)$. Therefore we conclude that the fomula holds.

If $n\geq 2$, then there exists an integer $j\,(j=1,2,,\cdots,n)$ with $|p_j|\geq 2$. Since both sides of the formula in Theorem do not depend on the order of $p_1,p_2,\cdots,p_n$, we may assume that $|p_n|\geq 2$. By Lemma 3 we have

$$\begin{aligned}(5)\qquad & Q[p_1,p_2,\cdots,p_n](x)\\ &=H_{1+p_n}(x)Q[p_1,\cdots,p_{n-1},1](x)+H_{1-p_n}(x)Q[p_1,\cdots,p_{n-1},-1](x)\\ &\quad+\left(\frac{x}{2-x}\right)(1-H_{1+p_n}(x)-H_{1-p_n}(x))Q[p_1,p_2,\cdots,p_{n-1}](x)\,.\end{aligned}$$

Since

$$|p_1|+|p_2|+\cdots+|p_{n-1}|+1-n<|p_1|+|p_2|+\cdots+|p_n|-n\,,$$

by the induction hypothesis we have

$$\begin{aligned}(6)\qquad & Q[p_1,\cdots,p_{n-1},1](x)\\ &=\sum_{\mu\in\mathscr{M}_n}G_{S\mu}(x)\left(\frac{x}{2-x}\right)^{N\mu}\pi[\mu;\,p_1,\cdots,p_{n-1},1](x)\,.\end{aligned}$$

We set

$$\mathscr{M}_n^+=\{\mu\in\mathscr{M}_n\mid\mu(n)=1\}\,,$$
$$\mathscr{M}_n^0=\{\mu\in\mathscr{M}_n\mid\mu(n)=0\}\,,$$
$$\mathscr{M}_n^-=\{\mu\in\mathscr{M}_n\mid\mu(n)=-1\}\,.$$

Then $\mathscr{M}_n^+$, $\mathscr{M}_n^0$ and $\mathscr{M}_n^-$ are mutually disjoint and $\mathscr{M}_n=\mathscr{M}_n^+\cup\mathscr{M}_n^0\cup\mathscr{M}_n^-$. The two equations $H_0(x)=0$ and $H_2(x)=1$ imply that

$$(7)\quad \begin{cases} H_{1+p_n}(x)\pi[\mu; p_1, \cdots, p_{n-1}, 1](x) = \pi[\mu; p_1, p_2, \cdots, p_n](x) & \text{if } \mu \in \mathcal{M}_n^+ , \\ \pi[\mu; p_1, \cdots, p_{n-1}, 1](x) = 0 & \text{otherwise} . \end{cases}$$

By (6) and (7) we have

$$(8)\quad \begin{aligned} &H_{1+p_n}(x)Q[p_1, \cdots, p_{n-1}, 1](x) \\ &= H_{1+p_n}(x) \sum_{\mu \in \mathcal{M}_n} G_{S\mu}(x)\left(\frac{x}{2-x}\right)^{N\mu} \pi[\mu; p_1, \cdots, p_{n-1}, 1](x) \\ &= H_{1+p_n}(x) \sum_{\mu \in \mathcal{M}_n^+} G_{S\mu}(x)\left(\frac{x}{2-x}\right)^{N\mu} \pi[\mu; p_1, \cdots, p_{n-1}, 1](x) \\ &= \sum_{\mu \in \mathcal{M}_n^+} G_{S\mu}(x)\left(\frac{x}{2-x}\right)^{N\mu} \pi[\mu; p_1, p_2, \cdots, p_n](x) . \end{aligned}$$

By the argument similar to that in case $H_{1+p_n}(x)Q[p_1, \cdots, p_{n-1}, 1](x)$, we can prove that

$$(9)\quad \begin{aligned} &H_{1-p_n}(x)Q[p_1, \cdots, p_{n-1}, -1](x) \\ &= \sum_{\mu \in \mathcal{M}_n^-} G_{S\mu}(x)\left(\frac{x}{2-x}\right)^{N\mu} \pi[\mu; p_1, p_2, \cdots, p_n](x) . \end{aligned}$$

We define the mapping $f: \mathcal{M}_n^0 \to \mathcal{M}_{n-1}$ as $f(\mu) = \mu|_{\{1,2,\cdots,n-1\}}$, then f is a bijection. For $\mu \in \mathcal{M}_n^0$, we have

$$(10)\quad \begin{cases} \pi[\mu; p_1, p_2, \cdots, p_n](x) \\ \qquad = (1 - H_{1+p_n}(x) - H_{1-p_n}(x))\pi[f(\mu); p_1, p_2, \cdots, p_{n-1}](x) , \\ Nf(\mu) = N\mu + 1 , \\ Sf(\mu) = S\mu . \end{cases}$$

Since

$$|p_1| + |p_2| + \cdots + |p_{n-1}| - (n-1) < |p_1| + |p_2| + \cdots + |p_n| - n ,$$

by the induction hypothesis we have

$$\begin{aligned} &\left(\frac{x}{2-x}\right)(1 - H_{1+p_n}(x) - H_{1-p_n}(x))Q[p_1, p_2, \cdots, p_{n-1}](x) \\ &\quad = \left(\frac{x}{2-x}\right)(1 - H_{1+p_n}(x) - H_{1-p_n}(x)) \\ &\qquad \times \sum_{\rho \in \mathcal{M}_{n-1}} G_{S\rho}(x)\left(\frac{x}{2-x}\right)^{N\rho} \pi[\mu; p_1, p_2, \cdots, p_{n-1}](x) . \end{aligned}$$

By (10) we have

$$\left(\frac{x}{2-x}\right)(1-H_{1+p_n}(x)-H_{1-p_n}(x))\sum_{\rho\in\mathcal{M}_{n-1}}G_{S_\rho}(x)\left(\frac{x}{2-x}\right)^{N\rho}\pi[\mu;p_1,p_2,\cdots,p_{n-1}](x)$$
$$=\left(\frac{x}{2-x}\right)\sum_{\mu\in\mathcal{M}_n^0}G_{S_\mu}(x)\left(\frac{x}{2-x}\right)^{N\mu-1}\pi[\mu;p_1,p_2,\cdots,p_n](x)$$
$$=\sum_{\mu\in\mathcal{M}_n^0}G_{S_\mu}(x)\left(\frac{x}{2-x}\right)^{N\mu}\pi[\mu;p_1,p_2,\cdots,p_n](x)\,.$$

Therefore it follows that

$$\left(\frac{x}{2-x}\right)(1-H_{1+p_n}(x)-H_{1-p_n}(x))Q[p_1,p_2,\cdots,p_{n-1}](x)$$
$$=\sum_{\mu\in\mathcal{M}_n^0}G_{S_\mu}(x)\left(\frac{x}{2-x}\right)^{N\mu}\pi[\mu;p_1,p_2,\cdots,p_n](x)\,. \tag{11}$$

By (5), (8), (9) and (11) we conclude that the formula of Theorem holds. This completes the proof. □

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