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A Cartan Subalgebra for an Inclusion of Factors with Index 3

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Abstract. For an outer action α of the symmetric group S_3 on an AFD factor M, we will determine when the pair $M\rtimes_\alpha S_3 \supset M\rtimes_\alpha S_2$ contains a Cartan subalgebra.

§1. Introduction.

Motivated by a result of Jones-Popa [5], we showed in [14] that for an outer action α of a finite group G on an AFD factor M, $M\rtimes_\alpha G \supset M$ contains a common Cartan subalgebra if and only if $N(\alpha)=\{e\}$. On the other hand, it was shown that an inclusion of factors with index 3 is conjugate to $M\rtimes_\alpha Z_3 \supset M$ or $M\rtimes_\alpha S_3 \supset M\rtimes_\alpha S_2$ (see Kosaki [7], Ocneanu [9], Pimsner–Popa [10], [11], Popa [12] for the former, and Izumi [3], Ocneanu [9], Popa [12] for the latter).

Hence, the following question naturally arises: For an outer action α of the symmetric group S_3 on an AFD factor M, when does $M\rtimes_\alpha S_3 \supset M\rtimes_\alpha S_2$ contain a common Cartan subalgebra? We shall give a necessary and sufficient condition in terms of the invariant of the action α.

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§2. Preliminaries.

Here, we collect certain results needed later. A Cartan subalgebra in a von Neumann algebra M is by the definition (Feldman–Moore [1]), a regular MASA which is the range of a faithful normal conditional expectation from M.

THEOREM 2.1. *Let M be the AFD type II_1 factor R or the AFD type II_∞ factor $R_{0,1}$ and let $\alpha: G\to \mathrm{Aut}\,M$ be an outer action of a finite group G on M. Then $M\rtimes_\alpha G\supset M$ has a common Cartan subalgebra.*

THEOREM 2.2. *Let $\alpha: G\to \mathrm{Aut}\,M$ be an outer action of a finite group G on an AFD*

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type III *factor M. Then* $M \rtimes_\alpha G \supset M$ *contains a common Cartan subalgebra if and only if* $N(\alpha)=\{e\}$. *Here* $N(\alpha)=\alpha^{-1}(\mathrm{Cnt}M)$ *and* $e \in G$ *is the unit of G.*

We refer to Jones–Popa [5] and [14] for details, and to Jones [4], Ocneanu [8], Sutherland–Takesaki [15] and Kawahigashi–Sutherland–Takesaki [6] for the classification of actions.

§3. Results.

3.1. The case of type II factor.

PROPOSITION 3.1. *Let* $\alpha: G \to \mathrm{Aut}M$ *be an outer action of a finite group G on the AFD type* II_1 *or* II_∞ *factor M. Let H be a subgroup of G. Then* $M \rtimes_\alpha G \supset M \rtimes_\alpha H$ *possesses a common Cartan subalgebra.*

PROOF. By Theorem 2.1, $M \rtimes_\alpha G \supset M$ has a common Cartan subalgebra. It is automatically a Cartan subalgebra of $M \rtimes_\alpha H$ because of [5; Remark 2.4]. q.e.d.

3.2. The case of type III factor.

LEMMA 3.2. *Let* $M \supset N$ *be a pair of type* III *factors and* $E: M \to N$ *a faithful normal conditional expectation. Let* φ *be a faithful normal semi-finite weight on N. If* $M \supset N$ *has a common Cartan subalgebra, then so does the canonical inclusion* $\tilde{M}=M \rtimes_{\sigma^{\varphi \circ E}} \boldsymbol{R} \supset \tilde{N}=N \rtimes_{\sigma^\varphi} \boldsymbol{R}$. *In particular, we get* $\tilde{N}' \cap \tilde{M}=Z(\tilde{N})$.

PROOF. This follows from [14; Lemma 1] and the obvious fact:

$$\tilde{N}' \cap \tilde{M} \subset \tilde{A}' \cap \tilde{M}=\tilde{A} \subset \tilde{N},$$

where $\tilde{A}$ is a common Cartan subalgebra for $\tilde{M} \supset \tilde{N}$. q.e.d.

LEMMA 3.3. *Let* $\alpha: G \to \mathrm{Aut}P$ *be an action of a finite group G on a type* III *factor P. Let H be a subgroup of G and set* $M=P \rtimes_\alpha G \supset N=P \rtimes_\alpha H$. *Let* $E: M \to N$ *be the natural conditional expectation and let* $\tilde{M}=M \rtimes_{\sigma^{\varphi \circ E}} \boldsymbol{R} \supset \tilde{N}=N \rtimes_{\sigma^\varphi} \boldsymbol{R}$ *for some weight* φ *on N.*

(i) $\tilde{M} \supset \tilde{N}$ *is conjugate to* $\tilde{P} \rtimes_{\tilde{\alpha}} G \supset \tilde{P} \rtimes_{\tilde{\alpha}} H$,

(ii) $\tilde{N}' \cap \tilde{M} \supset Z(\tilde{N})$ *is (as a pair) anti-isomorphic to*

$$(Z(\tilde{P}) \rtimes_{\mathrm{id},\mu} N(\alpha))^\gamma \supset (Z(\tilde{P}) \rtimes_{\mathrm{id},\mu} N(\alpha|_H))^\gamma,$$

where $\tilde{P}$ *is the crossed product of P by the modular action,* $\tilde{\alpha}$ *means the canonical extension of* α *in the sense of Haagerup–Størmer* [2] *and* γ *is the action of H defined by*

$$\gamma_g\left(\sum_{h \in N(\alpha)} c_h z_h\right)=\sum_{h \in N(\alpha)} \lambda(g, h)\tilde{\alpha}_g(c_{g^{-1}hg})z_h, \qquad g \in H.$$

PROOF. It follows from the obvious modification of the proof of [13; Lemma

2.4]. In fact, since $\tilde{P}' \cap \tilde{M}$ is anti-isomorphic to $Z(\tilde{P}) \rtimes_{\mathrm{id},\mu} N(\alpha)$ and $\tilde{N}' \cap \tilde{M} = (\tilde{P}' \cap \tilde{M}) \cap \{\tilde{\lambda}_H\}'$, the result follows from the fact that $\mathrm{Ad}\tilde{\lambda}_h$ corresponds to γ_h. q.e.d.

THEOREM 3.4. *Let $\alpha: S_3 \to \mathrm{Aut} P$ be an outer action of the symmetric group S_3 on an AFD type* III *factor P and set $M = P \rtimes_\alpha S_3 \supset N = P \rtimes_\alpha S_2$. Then $M \supset N$ contains a common Cartan subalgebra if and only if $N(\alpha) = \{e\}$.*

PROOF. Let us assume that $M \supset N$ contains a common Cartan subalgebra. If we denote the canonical inclusion of type II_∞ von Neumann algebras by $\tilde{M} \supset \tilde{N}$, then it follows from Lemma 3.2 that $\tilde{N}' \cap \tilde{M} = Z(\tilde{N})$. We remark that S_3 is the group generated by a and b with the relations $a^2 = 1$, $b^3 = 1$ and $aba = b^2$. Since $N(\alpha)$ is a normal subgroup of S_3, the possibility of it is $\{e\}$, $\{e, b, b^2\}$ or S_3. However, for the cases of $\{e, b, b^2\}$ and S_3, we have $\tilde{N}' \cap \tilde{M} \neq Z(\tilde{N})$ because of Lemma 3.3. Thus we get the consequence.

The converse follows from Theorem 2.2 and [5; Remark 2.4]. q.e.d.

References

[1] J. FELDMAN and C. MOORE, Ergodic equivalence relations, cohomology and von Neumann algebras. I, II, Trans. Amer. Math. Soc., **234** (1977), 289–324, 325–359.

[2] U. HAAGERUP and E. STØRMER, Equivalence of normal states on von Neumann algebras and the flow of weights, Adv. in Math., **83** (1990), 180–262.

[3] M. IZUMI, Goldman's type theorem for index 3, Publ. RIMS, Kyoto Univ., **28** (1992), 833–842.

[4] V. JONES, *Actions of Finite Groups on the Hyperfinite Type II_1 Factor*, Mem. Amer. Math. Soc., **237** (1980).

[5] V. JONES and S. POPA, Some properties of MASA's in factors, Operator Theory, **6** (1982), 89–102, Birkhäuser.

[6] Y. KAWAHIGASHI, C. SUTHERLAND and M. TAKESAKI, The structure of the automorphism group of an injective factor and the cocycle conjugacy of discrete abelian group actions, Acta Math., **169** (1992), 105–130.

[7] H. KOSAKI, Characterization of crossed product (properly infinite case), Pacific J. Math., **137** (1989), 159–167

[8] A. OCNEANU, *Actions of Discrete Amenable Groups on von Neumann Algebras*, Lecture Notes in Math., **1138** (1985), Springer.

[9] A. OCNEANU, Quantized groups, string algebras and Galois theory for algebras, *Operator Algebras and Applications, Vol. 2*, London Math. Soc. Lecture Note Ser., **136** (1988), 119–172.

[10] M. PIMSNER and S. POPA, Entropy and index for subfactors, Ann. Sci. École Norm. Sup., **19** (1986), 57–106.

[11] M. PIMSNER and S. POPA, Finite dimensional approximation of pairs of algebras and obstructions for the index, J. Funct. Anal., **98** (1991), 270–291.

[12] S. POPA, Classification of subfactors: the reduction to commuting squares, Invent. Math., **101** (1990), 19–43.

[13] Y. SEKINE, Flows of weights of crossed products of type III factors by discrete groups, Publ. RIMS Kyoto Univ., **26** (1990), 655–666.

[14] Y. SEKINE, Cartan subalgebras in fixed point algebras of finite group actions, Math. Scand., **70** (1992), 281–292.

[15] C. SUTHERLAND and M. TAKESAKI, Actions of discrete amenable groups on injective factors of type

III_λ, $\lambda \neq 1$, Pacific J. Math., **137** (1989), 404–444.

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