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On a Theorem of Shintani

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Dedicated to Professor Yukiyoshi Kawada on his 77$^{\text{th}}$ birthday

0. Introduction.

In his paper [4], T. Shintani gave the decompositions of the Dedekind η-function $\eta(z)$ and $\vartheta_1(w, z)$, $w, z \in \boldsymbol{C}$, $\mathrm{Im}\, z > 0$, (see (1.1), (1.3)) in terms of the double gamma-function and Stirling's modular form in the sense of Barnes [1]. The decomposition of ϑ_1 is also given in Barnes [2].

Shintani's starting point is the double Riemann zeta function

$$\zeta_2(s\,;\, w, \omega_1, \omega_2) = \sum_{m,n=0}^{\infty} (w + m\omega_1 + n\omega_2)^{-s}$$

$$\mathrm{Re}\, s > 0\,, \quad w + m\omega_1 + n\omega_2 \neq 0\,, \quad \text{for } m, n = 0, 1, 2, \cdots$$

and the definition of double gamma-function and Stirling's modular form by means of $\zeta_2'(0\,;\, w_1, \omega_1, \omega_2)$.

In the present paper, we shall go on the converse way to Shintani's; namely, we start with the decomposition of ϑ_1 and η by certain functions P and ρ (see (1.8), (1.10)), employing Stirling's formula for $\log \Gamma(z)$. Then we define two functions $\Gamma_2(w, \omega_1, \omega_2)$ and $\rho_2(\omega_1, \omega_2)$ (see (4.4), (4.5)) using P and ρ. It is shown that the function Γ_2 satisfies a certain limit condition (4.8) and the difference equations connecting $\Gamma_2(w + \omega_i\,;\, \omega_1, \omega_2)$ and $\Gamma_2(w;\, \omega_1, \omega_2)$, $i = 1, 2$ ((4.9), (4.10)).

Then Shintani's Proposition 2 in [3] shows that there exists a function $\lambda_2(\omega_1, \omega_2)$, depending only on ω_1, ω_2, such that

$$\log \frac{\Gamma_2(w\,;\, \omega_1, \omega_2)}{\lambda_2(\omega_1, \omega_2)} = \zeta_2'(0\,;\, w, \omega_1, \omega_2)\,.$$

Using the limit condition of Γ_2 and Binet's first formula for $\log \Gamma(z)$, we prove that

$$\lambda_2(\omega_1, \omega_2) = \rho_2(\omega_1, \omega_2)\,.$$

This shows the connection of our Γ_2, ρ_2 to $\zeta_2(s\,;\, w, \omega_1, \omega_2)$ and that our ρ_2 is nothing

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but the Stirling's modular form in the sense of Barnes [1]. Thus we arrive at Shintani's starting point.

It will be an interesting problem to ask what is the 'Dedekind sum' for ρ_2.

1. Decomposition of theta and eta.

We consider the theta function

$$\vartheta_1(w, z)=2e^{\pi iz/6}\sin\pi z\,\eta(z)\prod_{n=1}^{\infty}(1-e^{2\pi i(w+nz)})(1-e^{2\pi i(-w+nz)}) \tag{1.1}$$

$$w, z\in \boldsymbol{C}, \quad \operatorname{Im} z>0, \quad \pm w+nz\neq 0, \quad n=1, 2, \cdots,$$

where $\eta(z)$ is the Dedekind eta-function

$$\eta(z)=e^{\pi iz/12}\prod_{n=1}^{\infty}(1-e^{2\pi inz}). \tag{1.2}$$

To decompose

$$\prod_{n=1}^{\infty}(1-e^{2\pi i(w+nz)}) \tag{1.3}$$

appearing in (1.1), into the product of two functions, we note that

$$\Gamma(1+z)\Gamma(1-z)=\frac{\pi z}{\sin\pi z} \tag{1.4}$$

and for $|\arg z|\leq\pi-\delta$ $(\delta>0)$

$$\log\Gamma(z+a)=\left(z+a-\frac{1}{2}\right)\log z-z+\frac{1}{2}\log 2\pi \tag{1.5}$$

$$+\sum_{m=1}^{M}\frac{(-1)^{m-1}B'_{m+2}(a)}{m(m+1)(m+2)z^m}+O(z^{-M-1/2})$$

(Whittaker-Watson [5], p. 278). Here

$$\log z=\log|z|+i\arg z \qquad |\arg z|<\pi$$

and

$$\log(-z)=\log z-\pi i. \tag{1.6}$$

Further $B_m(u)$ is the m-th Bernoulli polynomial defined by

$$\frac{te^{ut}}{e^t-1}=\sum_{k=0}^{\infty}\frac{B_k(u)}{k!}t^k.$$

By (1.4), we have

$$1-e^{2\pi i(w+nz)}=e^{\pi i(w+nz)}(e^{-\pi i(w+nz)}-e^{\pi i(w+nz)})$$
$$=-2ie^{\pi i(w+nz)}\sin\pi(w+nz)$$
$$=-2\pi ie^{\pi i(w+nz)}\Gamma(w+nz)^{-1}\Gamma(1-(w+nz))^{-1}\,.$$

Then, observing that, by (1.5),

$$\log\Gamma(w+nz)=\left(w+nz-\frac{1}{2}\right)\log nz-nz+\frac{1}{2}\log(2\pi)+\frac{1/2-3w+3w^2}{6nz}+O(n^{-3/2})$$

holds for $z\in \boldsymbol{C}-(-\infty, 0]$, $w+nz\neq 0$, we see that

$$\sum_{n=1}^{\infty}\left\{\log\Gamma(w+nz)-\left(w+nz-\frac{1}{2}\right)\log(nz)+nz-\frac{1}{2}\log(2\pi)-\frac{1}{12nz}-\frac{w^2-w}{2nz}\right\}$$

is convergent. Therefore we define

$$P(w, z)=\prod_{n=1}^{\infty}\Gamma(w+nz)^{-1}\exp\left\{\left(w+nz-\frac{1}{2}\right)\log(nz)-nz+\frac{1}{2}\log(2\pi)+\frac{1}{12nz}+\frac{w^2-w}{2nz}\right\} \tag{1.8}$$

and we have

$$P(w, z)P(1-w, -z)=\prod_{n=1}^{\infty}(1-e^{2\pi i(w+nz)})\,. \tag{1.9}$$

For $z\in \boldsymbol{C}-(-\infty, 0]$, we define

(1.10) $\rho(z)=P(1, z)=P(0, z)$

$$=\prod_{n=1}^{\infty}\Gamma(1+nz)^{-1}\exp\left\{\left(nz+\frac{1}{2}\right)\log(nz)-nz+\frac{1}{2}\log(2\pi)+\frac{1}{12nz}\right\}.$$

THEOREM 1. *For* $\mathrm{Im}\,z>0$, $w+nz\neq 0$, $n=0, 1, 2, \cdots$, *we have*

(1) $$P(w, z)P(1-w, -z)P(-w, z)P(1+w, -z)=\frac{e^{-\pi iz/6}}{2\sin\pi w\;\eta(z)}\vartheta_1(w, z)\,,$$

(2) $$\rho(z)\rho(-z)=e^{-\pi iz/12}\eta(z)\,.$$

PROOF. (1) is obtained by making the product of (1.9) and (1.9) with $-w$ instead of w. (2) is nothing but the formula (1.9) with $w=1$. ■

2. Difference equations for $P(w, z)$.

Let γ be the Euler constant. Hence

$$1+\frac{1}{2}+\cdots+\frac{1}{N-1}-\log N=\gamma+O(N^{-1}) \tag{2.1}$$

holds. We shall prove the following.

THEOREM 2.

(1) $$P(w+z,z)=P(w,z)(2\pi)^{(z-1)/2}\Gamma(w+z)\exp\left\{-\left(z+w-\frac{1}{2}\right)\log z+\gamma\left(w+\frac{z}{2}-\frac{1}{2}\right)\right\},$$

(2) $$P(w+1,z)=P(w,z)\Gamma\left(\frac{w}{z}\right)\exp\left\{\log\frac{w}{z}+\frac{\gamma w}{z}\right\}.$$

PROOF. From (1.8), we have

$$\log P(w+z,z)^{-1}-\log P(w,z)^{-1}$$
$$=\sum_{n=1}^{\infty}\left\{\log\Gamma(w+(n+1)z)-\log\Gamma(w+nz)-z\log(nz)-\frac{2wz+z^2-z}{2nz}\right\}.$$

We consider the partial sum

$$\sum_{n=1}^{N}\left\{\log\Gamma(w+(n+1)z)-\log\Gamma(w+nz)-z\log(nz)-\frac{w+z/2-1/2}{n}\right\} \tag{2.2}$$
$$=\log\Gamma(w+(N+1)z)-\log\Gamma(w+z)-\sum_{n=1}^{N}\left\{z\log(nz)+\frac{1}{n}\left(w+\frac{1}{2}z-\frac{1}{2}\right)\right\}.$$

Using (1.5) again, we have

$$=-\log\Gamma(w+z)+\frac{1}{2}\log(2\pi)+\left((N+1)z+w-\frac{1}{2}\right)\log((N+1)z)$$
$$-(N+1)z-\sum_{n=1}^{N}\left\{z\log(nz)+\frac{1}{n}\left(w+\frac{1}{2}z-\frac{1}{2}\right)\right\}+O(N^{-1})$$
$$=-\log\Gamma(w+z)+\frac{1}{2}\log(2\pi)+\left(\left(N+\frac{1}{2}\right)z+\frac{1}{2}z+w-\frac{1}{2}\right)\log(N+1)$$
$$+\left((N+1)z+w-\frac{1}{2}\right)\log z-(N+1)z-Nz\log z-z\sum_{n=1}^{N}\log n$$
$$-\left(w+\frac{1}{2}z-\frac{1}{2}\right)\sum_{n=1}^{N}\frac{1}{n}+O(N^{-1})$$
$$=-\log\Gamma(w+z)+\frac{1}{2}\log(2\pi)+\left(z+w-\frac{1}{2}\right)\log z$$

$$+\left(w+\frac{1}{2}z-\frac{1}{2}\right)\left(\log(N+1)-\sum_{n=1}^{N}\frac{1}{n}\right)$$

$$+z\left\{\left(N+\frac{1}{2}\right)\log(N+1)-(N+1)-\sum_{n=1}^{N}\log n\right\}+O(N^{-1}).$$

Since

$$\log(N+1)-\sum_{n=1}^{N}\frac{1}{n}=-\gamma-O(N^{-1}) \qquad \text{by (2.1)},$$

$$\log\Gamma(N+1)=\left(N+\frac{1}{2}\right)\log(N+1)-(N+1)+\frac{1}{2}\log(2\pi)+O(N^{-1}) \qquad \text{by (1.5)},$$

and

$$\sum_{n=1}^{N}\log n=\log\Gamma(N+1),$$

we see, letting N tend to ∞, that (2.2) converges to

$$-\log\Gamma(w+z)+\frac{1}{2}\log(2\pi)+\left(z+w-\frac{1}{2}\right)\log z-\gamma\left(w+\frac{1}{2}z-\frac{1}{2}\right)-\frac{z}{2}\log(2\pi)$$

and get (1).

To prove (2), we need

$$\log\Gamma(z)=-\log z-\gamma z-\sum_{n=1}^{\infty}\left\{\log(n+z)-\log n-\frac{z}{n}\right\}. \tag{2.3}$$

Using

$$\log\Gamma(w+1+nz)=\log(w+nz)+\log\Gamma(w+nz)$$

in the formula

$$\log P(w+1,z)^{-1}=\sum_{n=1}^{\infty}\left\{\log\Gamma(w+1+nz)-\left(w+\frac{1}{2}+nz\right)\log(nz)\right.$$

$$\left.+nz-\frac{1}{2}\log(2\pi)-\frac{1}{12nz}-\frac{(w+1)^2-(w+1)}{2nz}\right\},$$

we get

$$\log P(w+1,z)^{-1}-\log P(w,z)^{-1}$$

$$=\sum_{n=1}^{\infty}\left\{\log(w+nz)-\log(nz)-\frac{w}{nz}\right\}$$

$$=\sum_{n=1}^{\infty}\left\{\log\left(n+\frac{w}{z}\right)-\log n-\frac{w}{nz}\right\}$$

$$=-\log\frac{w}{z}-\frac{\gamma w}{z}-\log\Gamma\left(\frac{w}{z}\right).$$

The last equality follows from (2.3). Thus (2) is obtained. (End of the proof of Theorem 2.)

3. Double Bernoulli polynomial and double Riemann zeta-function.

The double Riemann zeta-function is defined for $w, \omega_1, \omega_2 \in \boldsymbol{C}$, $\omega_2/\omega_1 \notin (-\infty, 0]$ by

$$\zeta_2(s\,;\,w,\omega_1,\omega_2)=\sum_{m,n=0}^{\infty}(w+m\omega_1+n\omega_2)^{-s}$$

$$\operatorname{Re} s>2\,,\quad w+m\omega_1+n\omega_2\neq 0\,,\quad m,n=0,1,2,\cdots$$

(see Barnes [1], §36). Here

$$w^s=e^{s\log w}$$

and

$$\log w=\log|w|+i\arg w\,,\qquad -\pi\leq\arg w<\pi\,.$$

Then it is known that for $\operatorname{Re} w>0$, $\operatorname{Re}\omega_1>0$, $\operatorname{Re}\omega_2>0$, ζ_2 can be represented by the following contour integral:

$$\zeta_2(s\,;\,w,\omega_1,\omega_2)=\frac{\Gamma(1-s)e^{-s\pi i}}{2\pi i}\int_{I(\varepsilon,\infty)}\frac{e^{-wt}t^{s-1}}{(1-e^{-\omega_1 t})(1-e^{-\omega_2 t})}dt\,. \tag{3.1}$$

Here $\varepsilon>0$ and $I(\varepsilon,\infty)$ means the path consisting of the infinite line from ∞ to ε, the circle of radius ε around the center 0 in the positive sense and the infinite line from ε to ∞ as shown in Fig. 1.

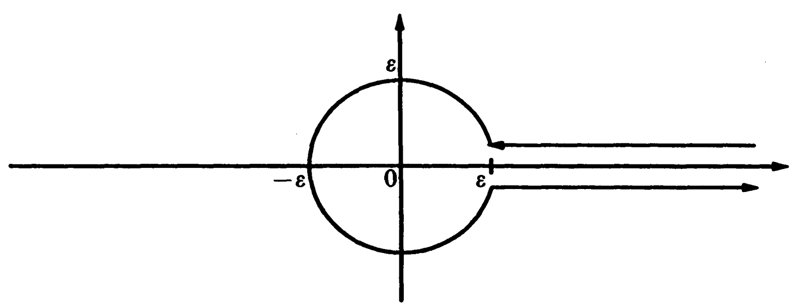

FIGURE 1

$I(\varepsilon, \infty)$ is taken so that it does not contain non-zero poles of the integrand. Since $I(\varepsilon, \infty)$ does not go through 0, the right hand side of (3.1) converges for any s. It is known that ζ_2 can be continued analytically to the whole s-plane except for simple poles at $s=1$, and $s=2$.

The double Bernoulli polynomials ${}_2S_n(w\,;\,\omega_1,\omega_2)$ are defined by

$$\frac{1-e^{-wt}}{(1-e^{-\omega_1 t})(1-e^{-\omega_2 t})}=\frac{w}{\omega_1\omega_2 t}-{}_2S_0(w\,;\,\omega_1,\omega_2)+\cdots$$
$$+(-1)^{n-1}\frac{{}_2S_n(w\,;\,\omega_1,\omega_2)}{n!}t^n+\cdots,\qquad |t|<\left|\frac{2\pi}{\omega_1}\right|,\left|\frac{2\pi}{\omega_2}\right|.$$

Differentiating this with respect to w, we have

$$\frac{-te^{-wt}}{(1-e^{-\omega_1 t})(1-e^{-\omega_2 t})}=\frac{1}{\omega_1\omega_2 t}-{}_2S_0'(w\,;\,\omega_1,\omega_2)+{}_2S_1'(w\,;\,\omega_1,\omega_2)t+\cdots \tag{3.2}$$
$$+(-1)^{n-1}\frac{{}_2S_n'(w\,;\,\omega_1,\omega_2)}{n!}t^n+\cdots.$$

Thus, writing symbolically as

$$\frac{t}{e^t-1}=e^{Bt},\qquad \frac{te^{wt}}{e^t-1}=e^{(B^*+w)t},$$

we have

$$\frac{-te^{-wt}}{(1-e^{-\omega_1 t})(1-e^{-\omega_2 t})}=\frac{1}{\omega_1\omega_2 t}e^{-(B\omega_1+B^*\omega_2+w)t}$$

and comparing this to (3.2),

$$ {}_2S_n'(w\,;\,\omega_1,\omega_2)=\frac{(B\omega_1+B^*\omega_2+w)^{n+1}}{(n+1)\omega_1\omega_2}. \tag{3.3}$$

Here $B^j=B_j$, $B^{*j}=B_j$ but $B^jB^{*k}\neq B_{j+k}$. In particular, we have, by (3.3),

$$ {}_2S_1'(w\,;\,\omega_1,\omega_2)=\frac{\omega_1^2+\omega_2^2+3\omega_1\omega_2}{12\omega_1\omega_2}+\frac{w^2}{2\omega_1\omega_2}-\frac{(\omega_1+\omega_2)w}{2\omega_1\omega_2} \tag{3.4}$$

and

$$ {}_2S_0'(w\,;\,\omega_1,\omega_2)=\frac{w}{\omega_1\omega_2}-\frac{\omega_1+\omega_2}{2\omega_1\omega_2}. \tag{3.5}$$

PROPOSITION 1. *For* $\mathrm{Re}\,w>0$, $\mathrm{Re}\,\omega_1>0$, $\mathrm{Re}\,\omega_2>0$,

$$\zeta_2(0\,;\,w,\omega_1,\omega_2)={}_2S_1'(w\,;\,\omega_1,\omega_2).$$

PROOF. We have

$$\zeta_2(0\,;\,w,\omega_1,\omega_2)=\mathrm{Res}_{t=0}\frac{e^{-wt}t^{-1}}{(1-e^{-\omega_1 t})(1-e^{-\omega_2 t})}$$
$$={}_2S_1'(w\,;\,\omega_1,\omega_2)$$

by the contour integral representation of ζ_2 and the residue principle. ∎

4. Definition and properties of Γ_2 and ρ_2.

We put, for $z\in \boldsymbol{C}-(-\infty,0]$, $w+nz\neq 0$, $n=1,2,\cdots$,

(4.1) $$\hat{\Gamma}_2(w\,;\,z)=P(w,z)^{-1}\rho(z)(2\pi)^{w/2}\Gamma(w)\exp\left[\left(\frac{w-w^2}{2z}-\frac{w}{2}\right)\log z+\frac{\gamma(w^2-w)}{2z}\right],$$

and

(4.2) $$\hat{\rho}_2(z)=\rho(z)(2\pi)^{3/4}\exp\left\{-\frac{\gamma}{12z}-\frac{z}{12}+z\zeta'(-1)+\left(\frac{z}{12}-\frac{1}{4}+\frac{1}{12z}\right)\log z\right\}.$$

Here ζ' is the derivative of the Riemann zeta-function $\zeta(s)$. Then, for $\mathrm{Im}\,z>0$, we have

(4.3) $$\hat{\rho}_2(z)\hat{\rho}_2(-z)=(2\pi)^{3/2}z^{-1/2}\eta(z)\exp\left\{\pi i\left(\frac{1}{4}+\frac{1}{12z}\right)\right\}.$$

Further, for $\omega_1,\omega_2\in \boldsymbol{C}$, $\omega_1/\omega_2\notin \boldsymbol{C}-(-\infty,0]$, we define

(4.4)

$$\Gamma_2(w\,;\,\omega_1,\omega_2)=\hat{\Gamma}_2(w/\omega_1\,;\,\omega_2/\omega_1)\exp\left[\left\{\frac{1}{2}\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)w-\frac{w^2}{2\omega_1\omega_2}-1\right\}\log\omega_1\right],$$

and

(4.5) $$\rho_2(\omega_1,\omega_2)=\hat{\rho}_2(\omega_2/\omega_1)\exp\left\{\left(\frac{\omega_2}{12\omega_1}+\frac{\omega_1}{12\omega_2}-\frac{3}{4}\right)\log\omega_1\right\}.$$

Hence it follows easily that

(4.6)

$$\Gamma_2(tw\,;\,t\omega_1,t\omega_2)=\Gamma_2(w\,;\,\omega_1,\omega_2)\exp\left[\left\{\frac{1}{2}\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)w-\frac{w^2}{2\omega_1\omega_2}-1\right\}\log t\right],$$

and

(4.7) $$\rho_2(t\omega_1,t\omega_2)=\rho_2(\omega_1,\omega_2)\exp\left\{\left(\frac{\omega_2}{12\omega_1}+\frac{\omega_1}{12\omega_2}-\frac{3}{4}\right)\log t\right\}$$

hold for $t>0$. Since $P(0, z)=\rho(z)$, we see that for $w\to 0$,

$$w\Gamma_2(w\ ;\ (1, z))=P(w, z)^{-1}\rho(z)(2\pi)^{w/2}w\Gamma(w)\exp\left[\left(\frac{w^2-w}{2z}-\frac{w}{2}\right)\log z+\frac{\gamma(w^2-w)}{2z}\right]$$

tends to 1: namely

$$\lim_{w\to 0} w\Gamma_2(w\ ;\ \omega_1, \omega_2)=1\ . \tag{4.8}$$

Then Theorem 2 can be written in the following form:

$$\begin{aligned}&\Gamma_2(w+\omega_1; \omega_1, \omega_2)\\ &\quad=\Gamma_2(w\ ;\ \omega_1, \omega_2)(2\pi)^{1/2}\Gamma\left(\frac{w}{\omega_2}\right)^{-1}\exp\left\{\left(\frac{1}{2}-\frac{w}{\omega_2}\right)\log\omega_2\right\},\end{aligned} \tag{4.9}$$

$$\begin{aligned}&\Gamma_2(w+\omega_2\ ;\ \omega_1, \omega_2)\\ &\quad=\Gamma_2(w\ ;\ \omega_1, \omega_2)(2\pi)^{1/2}\Gamma\left(\frac{w}{\omega_1}\right)^{-1}\exp\left\{\left(\frac{1}{2}-\frac{w}{\omega_1}\right)\log\omega_1\right\}.\end{aligned} \tag{4.10}$$

From (4.8), (4.9), (4.10), we easily deduce

$$\begin{cases}\Gamma_2(\omega_1\ ;\ \omega_1, \omega_2)=\sqrt{\dfrac{2\pi}{\omega_2}}, \qquad \Gamma_2(\omega_2\ ;\ \omega_1, \omega_2)=\sqrt{\dfrac{2\pi}{\omega_1}},\\ \Gamma_2(\omega_1+\omega_2\ ;\ \omega_1, \omega_2)=\dfrac{2\pi}{\sqrt{\omega_1\omega_2}}.\end{cases} \tag{4.11}$$

Now Shintani's Proposition 2 in [3] asserts that for $\mathrm{Re}\,w>0$, $\mathrm{Re}\,\omega_1>0$, $\mathrm{Re}\,\omega_2>0$, there exists a function $\lambda_2(\omega_1, \omega_2)$, not depending on w, such that

$$\begin{aligned}\log\frac{\Gamma_2(w\ ;\ \omega_1, \omega_2)}{\lambda_2(\omega_1, \omega_2)}&=\frac{1}{2\pi i}\int_{I(\varepsilon,\infty)}\frac{e^{-wt}}{(1-e^{-\omega_1 t})(1-e^{-\omega_2 t})}\frac{\log t}{t}dt\\ &\quad+(\gamma-\pi i)\,{}_2S_1'(w\ ;\ \omega_1, \omega_2)\,, \qquad 0<\varepsilon<|2\pi/\omega_1|, |2\pi/\omega_2|\,.\end{aligned} \tag{4.12}$$

The point of this proof is the analyticity of both hand sides and the difference equations (4.9), (4.10) satisfied by Γ_2 and the right hand side of (4.12).

From (4.12) or Shintani's proof of (4.12), it follows that there exists one and only one function Γ_2 satisfying (4.9), (4.10) and (4.8). Thus our $\Gamma_2(w\ ;\ \omega_1, \omega_2)$ is nothing but the double gamma function in the sense of Barnes [1].

Let t be positive. Differentiating

$$\zeta_2(s\ ;\ tw, t\omega_1, t\omega_2)=t^{-s}\zeta_2(s\ ;\ w, \omega_1, \omega_2)$$

with respect to s and putting $s=0$, we have

$$\text{(4.13)}\qquad \log\frac{\Gamma_2(tw\,;\,t\omega_1,t\omega_2)}{\lambda_2(t\omega_1,t\omega_2)}=-\log t\,\zeta_2(0\,;\,w,\omega_1,\omega_2)+\log\frac{\Gamma_2(w\,;\,\omega_1,\omega_2)}{\lambda_2(\omega_1,\omega_2)}\,.$$

Put $w=\omega_1$ in (4.13). Then by (4.11) and Prop. 1, §3, it follows

$$\log\sqrt{\frac{2\pi}{t\omega_2}}-\log\lambda_2(t\omega_1,t\omega_2)=-\log t\,{}_2S_1'(t\omega_1\,;\,\omega_1,\omega_2)+\log\sqrt{\frac{2\pi}{\omega_2}}-\log\lambda_2(\omega_1,\omega_2)\,.$$

From this and (3.4), the formula

$$\text{(4.14)}\qquad \lambda_2(t\omega_1,t\omega_2)=\lambda_2(\omega_1,\omega_2)\exp\left[\left\{\frac{\omega_1}{12\omega_2}+\frac{\omega_2}{12\omega_1}-\frac{3}{4}\right\}\log t\right]$$

is obtained. Denote by $U(\varepsilon)$ the circle of radius $\varepsilon>0$, rounding 0 in the positive sense. Let $l(\varepsilon)$ denote any quantity which tends to 0 as $\varepsilon\to 0$.

LEMMA 1. *Let w and z be positive. Then*

$$\frac{1}{2\pi i}\int_{U(\varepsilon)}\frac{e^{-wt}}{(1-e^{-t})(1-e^{-zt})}\frac{\log t}{t}dt$$
$$={}_2S_1'(w\,;\,1,z)\log\varepsilon-\frac{1}{2z\varepsilon^2}+\frac{{}_2S_0'(w\,;\,1,z)}{\varepsilon}+{}_2S_1'(w\,;\,1,z)\pi i+l(\varepsilon)\,.$$

PROOF. For $t=\varepsilon e^{i\theta}$, we have, by (3.2)

$$\frac{e^{-wt}}{(1-e^{-t})(1-e^{-zt})}=\frac{1}{z\varepsilon^2e^{2i\theta}}-\frac{{}_2S_0'(w\,;\,1,z)}{\varepsilon e^{i\theta}}+{}_2S_1'(w\,;\,1,z)+l(\varepsilon)\,.$$

Then our Lemma follows easily from this. ■

LEMMA 2. *Let w and z be positive. Then*

(i) $$\int_\varepsilon^\infty\frac{e^{-wt}}{1-e^{-zt}}dt=-\frac{1}{z}\frac{\Gamma'(w/z)}{\Gamma(w/z)}-\frac{\log\varepsilon}{z}-\frac{\pi i}{z}-\frac{\log z}{z}-\frac{\gamma-\pi i}{z}+l(\varepsilon)\,,$$

(ii) $$\int_\varepsilon^\infty\frac{e^{-wt}}{1-e^{-zt}}\frac{dt}{t}=\log\Gamma\left(\frac{w}{z}\right)-\frac{1}{2}\log(2\pi)-B_1\left(1-\frac{w}{z}\right)\log\varepsilon-B_1\left(1-\frac{w}{z}\right)\pi i$$
$$+\frac{1}{\varepsilon z}-B_1\left(1-\frac{w}{z}\right)\log z-(\gamma-\pi i)\left(\frac{1}{2}-\frac{w}{z}\right)+l(\varepsilon)\,,$$

(iii) $$\int_\varepsilon^\infty\frac{e^{-wt}}{1-e^{-zt}}\frac{dt}{t^2}=z\,\mathrm{LG}\left(\frac{w}{z}\right)-\frac{1}{2}B_2\left(1-\frac{w}{z}\right)z\log\varepsilon+\frac{1}{\varepsilon}B_1\left(1-\frac{w}{z}\right)$$
$$-\frac{1}{2}B_2\left(1-\frac{w}{z}\right)z\pi i+\frac{1}{2z\varepsilon^2}-\frac{1}{2}B_2\left(1-\frac{w}{z}\right)z\log z$$

$$-\frac{1}{2}(\gamma-\pi i)B_2\left(\frac{w}{z}\right)z+l(\varepsilon)\,,$$

where we denote by LG(u), *for* $u>0$, *the function defined in Shintani* [3]:

$$\mathrm{LG}(u)=\frac{1}{2\pi i}\int_{I(\varepsilon,\infty)}\frac{e^{-ut}}{1-e^{-t}}\frac{\log t}{t^2}dt+\frac{\gamma-\pi i}{2}B_2(u)\,. \tag{4.15}$$

PROOF. Hurwitz zeta-function is defined by

$$\zeta(s,u)=\sum_{n=0}^{\infty}(n+u)^{-s}\qquad \mathrm{Re}\,s>1\,,\quad \mathrm{Re}\,u>0\,.$$

Then it is well known that $\zeta(s,u)$ has the following contour integral representation:

$$\zeta(s,u)=\frac{e^{-\pi is}\Gamma(1-s)}{2\pi i}\int_{I(\varepsilon,\infty)}\frac{e^{-ut}}{1-e^{-t}}t^{s-1}dt\qquad(0<\varepsilon<2\pi)\,.$$

Differentiating this formula at $s=0$ and noting that ([5], p. 271)

$$\zeta'(s,u)\,|_{s=0}=\log\frac{\Gamma(u)}{\sqrt{2\pi}}\,,$$

we have

$$\log\frac{\Gamma(u)}{\sqrt{2\pi}}=\frac{1}{2\pi i}\int_{I(\varepsilon,\infty)}\frac{e^{-ut}}{1-e^{-t}}\frac{\log t}{t}dt+(\gamma-\pi i)\left(\frac{1}{2}-u\right). \tag{4.16}$$

We put $t=\varepsilon e^{i\theta}$ in the formula:

$$\frac{1}{2\pi i}\int_{I(\varepsilon,\infty)}\frac{e^{-ut}}{1-e^{-t}}\frac{\log t}{t}dt=\frac{1}{2\pi i}\int_{U(\varepsilon)}\frac{e^{-ut}}{1-e^{-t}}\frac{\log t}{t}dt+\int_{\varepsilon}^{\infty}\frac{e^{-ut}}{1-e^{-t}}\frac{dt}{t}\,.$$

Then

$$\begin{aligned}\frac{1}{2\pi i}\int_{U(\varepsilon)}\frac{e^{-ut}}{1-e^{-t}}\frac{\log t}{t}dt&=\frac{1}{2\pi i}\int_{U(\varepsilon)}\frac{e^{-(u-1)t}}{e^{t}-1}\frac{\log t}{t}dt\\&=\frac{1}{2\pi}\int_0^{2\pi}(\log\varepsilon+i\theta)\left\{\frac{1}{\varepsilon e^{i\theta}}+B_1(1-u)+\frac{B_2(1-u)}{2}\varepsilon e^{i\theta}+\cdots\right\}d\theta\\&=B_1(1-u)\log\varepsilon-\frac{1}{\varepsilon}+B_1(1-u)\pi i+l(\varepsilon)\end{aligned}$$

and

$$\int_{\varepsilon}^{\infty}\frac{e^{-ut}}{1-e^{-t}}\frac{dt}{t}=\int_{\varepsilon/z}^{\infty}\frac{e^{-uzt}}{1-e^{-zt}}\frac{dt}{t}=\int_{\varepsilon}^{\infty}-\int_{\varepsilon}^{\varepsilon/z}$$

$$=\int_{\varepsilon}^{\infty}\frac{e^{-uzt}}{1-e^{-zt}}\frac{dt}{t}+\frac{1}{z}\left(\frac{z}{\varepsilon}-\frac{1}{\varepsilon}\right)+B_1(1-u)\log z+l(\varepsilon)\,.$$

Put these formulas into (4.16) and $u=w/z$. Then (ii) is obtained.

Differentiating (4.16) with respect to u, we have

$$\frac{\Gamma'(u)}{\Gamma(u)}=-\frac{1}{2\pi i}\int_{I(\varepsilon,\infty)}\frac{e^{-ut}}{1-e^{-t}}\log t\,dt-(\gamma-\pi i)\,. \tag{4.17}$$

Now, since

$$\frac{1}{2\pi i}\int_{I(\varepsilon,\infty)}\frac{e^{-ut}}{1-e^{-t}}\log t\,dt=\frac{1}{2\pi i}\int_{U(\varepsilon)}\frac{e^{(1-u)t}}{e^{t}-1}\log t\,dt+\int_{\varepsilon}^{\infty}\frac{e^{-ut}}{1-e^{-t}}dt$$

$$=\frac{1}{2\pi}\int_0^{2\pi}(\log\varepsilon+i\theta)\{1+B_1(1-u)\varepsilon e^{i\theta}+\cdots\}d\theta$$

$$+z\int_{\varepsilon}^{\infty}\frac{e^{-uzt}}{1-e^{-zt}}dt-\log z+l(\varepsilon)$$

$$=\log\varepsilon+\pi i+z\int_{\varepsilon}^{\infty}\frac{e^{-uzt}}{1-e^{-zt}}dt-\log z+l(\varepsilon)$$

holds, put this into (4.17) and $u=w/z$. Then (i) is obtained.

Finally, we shall transform the contour integral for LG(u) to a line integral. So in the formula

$$\mathrm{LG}(u)=\frac{1}{2\pi i}\int_{U(\varepsilon)}\frac{e^{-ut}}{1-e^{-t}}\frac{\log t}{t^2}dt+\int_{\varepsilon}^{\infty}\frac{e^{-ut}}{1-e^{-t}}\frac{dt}{t^2}+\frac{(\gamma-\pi i)}{2}B_2(u)\,, \tag{4.18}$$

compute both integrals as

$$\frac{1}{2\pi i}\int_{U(\varepsilon)}=\frac{1}{2\pi}\int_0^{2\pi}(\log\varepsilon+i\theta)\left\{\frac{1}{\varepsilon^2e^{2i\theta}}+\frac{B_1(1-u)}{\varepsilon e^{i\theta}}+\frac{1}{2}B_2(1-u)+\cdots\right\}d\theta$$

$$=\frac{1}{2}B_2(1-u)\log\varepsilon-\frac{1}{2\varepsilon^2}-\frac{B_1(1-u)}{\varepsilon}+\frac{B_2(1-u)}{2}\pi i+\cdots$$

and

$$\int_{\varepsilon}^{\infty}\frac{e^{-ut}}{1-e^{-t}}\frac{dt}{t^2}=\frac{1}{z}\int_{\varepsilon/z}^{\infty}\frac{e^{-uzt}}{1-e^{-zt}}\frac{dt}{t^2}$$

$$=\frac{1}{z}\int_{\varepsilon}^{\infty}\frac{e^{-uzt}}{1-e^{-zt}}\frac{dt}{t^2}-\frac{1}{z}\int_{\varepsilon}^{\varepsilon/z}\frac{e^{-uzt}}{1-e^{-zt}}\frac{dt}{t^2}\,.$$

The last integral is equal to

$$-\frac{1}{z}\left\{\left(-\frac{z}{2\varepsilon^2}+\frac{1}{2z\varepsilon^2}\right)-\frac{B_2(1-u)}{2}\left(\frac{z}{\varepsilon}-\frac{1}{\varepsilon}\right)-\frac{B_2(1-u)}{2}\log z\right\}+l(\varepsilon)\,.$$

Put these in (4.18) and $u=w/z$. Then (iii) is obtained. ∎

PROPOSITION 2. *For* $\mathrm{Re}\,w>0$, $\omega_1>0$, $\omega_2>0$, *we have*

$$\log\frac{\Gamma_2(w\,;\,\omega_1,\omega_2)}{\lambda_2(\omega_1,\omega_2)}=\int_0^\infty\frac{e^{-(w/\omega_1-1)t}}{1-e^{-(\omega_2/\omega_1)t}}\left\{\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}-\frac{1}{12}t\right\}\frac{dt}{t}$$
$$+\frac{\omega_2}{\omega_1}\mathrm{LG}\left(\frac{w}{\omega_1}-\frac{\omega_1}{\omega_2}\right)-\frac{\omega_1}{12\omega_2}\frac{\Gamma'(w/\omega_2-\omega_1/\omega_2)}{\Gamma(w/\omega_2-\omega_1/\omega_2)}-\frac{1}{2}\log\Gamma\left(\frac{w}{\omega_2}-\frac{\omega_1}{\omega_2}\right)$$
$$-\log\frac{\omega_2}{\omega_1}\left\{\frac{\omega_2}{2\omega_1}B_2\left(\frac{\omega_1+\omega_2-w}{\omega_2}\right)-\frac{1}{2}B_1\left(\frac{\omega_1+\omega_2-w}{\omega_2}\right)+\frac{\omega_1}{12\omega_2}\right\}$$
$$+\frac{1}{4}\log(2\pi)-\left\{\frac{\omega_2}{12\omega_1}+\frac{\omega_1}{12\omega_2}+\frac{1}{4}-\frac{1}{2}\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)w+\frac{w^2}{2\omega_1\omega_2}\right\}\log\omega_1\,.$$

PROOF. By (4.6), (4.7) and (4.14), we have, for $t>0$,

$$\log\frac{\Gamma_2(w\,;\,\omega_1,\omega_2)}{\lambda_2(\omega_1,\omega_2)}=\log\frac{\Gamma_2(tw\,;\,t\omega_1,t\omega_2)}{\lambda_2(t\omega_1,t\omega_2)}$$
$$+\left\{\frac{\omega_2}{12\omega_1}+\frac{\omega_1}{12\omega_2}-\frac{3}{4}-\frac{1}{2}\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)w+\frac{w^2}{2\omega_1\omega_2}+1\right\}\log t\,.$$

Hence, we shall compute

$$\log\frac{\Gamma_2(w^*\,;\,1,z)}{\lambda_2(1,z)}\,,$$

where $t=1/\omega_1$, $z=\omega_2/\omega_1$, $w^*=w/\omega_1$. By (4.12)

$$\log\frac{\Gamma_2(w^*\,;\,1,z)}{\lambda_2(1,z)}=\frac{1}{2\pi i}\int_{I(\varepsilon,\infty)}\frac{e^{-w^*t}}{(1-e^{-t})(1-e^{-zt})}\frac{\log t}{t}dt$$
$$+(\gamma-\pi i)\,{}_2S_1'(w^*\,;\,\omega_1,\omega_2)\,.$$

We divide the contour integral to

$$\frac{1}{2\pi i}\int_{I(\varepsilon,\infty)}=\frac{1}{2\pi i}\int_{U(\varepsilon)}+\int_\varepsilon^\infty$$

and observe that

$$\int_\varepsilon^\infty=\int_\varepsilon^\infty\frac{e^{-(w^*-1)t}}{(1-e^{-zt})}\left\{\frac{1}{e^t-1}+\frac{1}{2}-\frac{1}{t}-\frac{t}{12}\right\}\frac{dt}{t}-\frac{1}{2}\int_\varepsilon^\infty\frac{e^{-(w^*-1)t}}{(1-e^{-zt})}\frac{dt}{t}$$

$$+\int_{\varepsilon}^{\infty}\frac{e^{-(w^*-1)t}}{(1-e^{-zt})}\frac{dt}{t^2}+\frac{1}{12}\int_{\varepsilon}^{\infty}\frac{e^{-(w^*-1)t}}{(1-e^{-zt})}dt\,.$$

Then our proposition is obtained by applying Lemmas 1 and 2 and letting ε tend to 0. ∎

PROPOSITION 3. *For $\omega_1, \omega_2 \in \boldsymbol{C}$ with $\omega_2/\omega_1 \notin (-\infty, 0]$,*

$$\lambda_2(\omega_1, \omega_2)=\rho_2(\omega_1, \omega_2)$$

holds. In particular for $z \in \boldsymbol{C}-(\infty, 0]$,

$$\lambda_2(1, z)=\rho_2(1, z).$$

PROOF. By the principle of analytic continuation, it suffices to prove our proposition for $\omega_1>0$, $\omega_2>0$. Further, taking $t=\omega_1^{-1}$, $z=\omega_2/\omega_1$ in $\lambda_2(\omega_1, \omega_2)=\rho_2(\omega_1, \omega_2)$, we get $\lambda_2(1, z)=\rho_2(1, z)$. Hence by (4.7) and (4.14), it suffices to prove our proposition for $\omega_1=1$, $\omega_2=z$.

In the formula of Proposition 2, put $\omega_1=1$, $\omega_2=z$ and $w=1+z$. Then by (4.11), the left hand side is equal to $\log 2\pi/\sqrt{z}-\log\lambda_2(1, z)$. The right hand side is equal to

$$\int_0^{\infty}\frac{e^{-zt}}{1-e^{-zt}}\left\{\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}-\frac{t}{12}\right\}\frac{dt}{t}+z\,\mathrm{LG}(1)-\frac{1}{12z}\Gamma'(1)$$
$$-\log z\left\{\frac{z}{2}B_2-\frac{1}{2}B_1+\frac{1}{12z}\right\}+\frac{1}{4}\log(2\pi)\,.$$

Here note that (Shintani [3])

$$\mathrm{LG}(1)=\frac{1}{12}-\zeta'(-1)$$

and

$$\Gamma'(1)=-\gamma\,.$$

Then we have

$$\log\lambda_2(1, z)=\frac{3}{4}\log(2\pi)-\frac{\gamma}{12z}+z\zeta'(-1)-\frac{z}{12}+\left(\frac{1}{12z}+\frac{z}{12}-\frac{1}{4}\right)\log z$$
$$-\int_0^{\infty}\frac{1}{e^{zt}-1}\left(\frac{1}{e^t-1}+\frac{1}{2}-\frac{1}{t}-\frac{t}{12}\right)\frac{dt}{t}\,. \tag{4.19}$$

We employ, here, Binet's first formula for $\log\Gamma(z)$ (Whittaker-Watson [5], p. 249):

$$\log\Gamma(z)=\left(z-\frac{1}{2}\right)\log z-z+\frac{1}{2}\log(2\pi)+\phi(z)\,,$$

$$\phi(z)=\int_0^\infty\left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)\frac{e^{-zt}}{t}dt \qquad (\mathrm{Re}\, z>0)\,.$$

Put

$$f(z)=(2\pi)^{1/2}\Gamma(1+z)^{-1}\exp\left\{\frac{1}{12z}+\left(\frac{1}{2}+z\right)\log z-z\right\}.$$

Then

$$f(z)=\exp\left\{\frac{1}{12z}-\phi(z)\right\}.$$

Observing that

$$\frac{1}{z}=\int_0^\infty e^{-zt}dt\,,$$

we have

$$\frac{1}{12z}-\phi(z)=\int_0^\infty\left(\frac{t}{12}-\frac{1}{2}+\frac{1}{t}-\frac{1}{e^t-1}\right)\frac{e^{-zt}}{t}dt$$

and by (1.10)

$$\rho(z)=\prod_{n=1}^\infty f(nz)\,.$$

Now,

$$\begin{aligned}\log\prod_{n=1}^\infty f(nz)&=\sum_{n=1}^\infty\left(\frac{1}{12nz}-\phi(nz)\right)\\&=\sum_{n=1}^\infty\int_0^\infty\left(\frac{t}{12}-\frac{1}{2}+\frac{1}{t}-\frac{1}{e^t-1}\right)e^{-nzt}\frac{dt}{t}\\&=\int_0^\infty\left(\frac{t}{12}-\frac{1}{2}+\frac{1}{t}-\frac{1}{e^t-1}\right)\frac{1}{e^{zt}-1}\frac{dt}{t}\end{aligned}$$

and the last integral converges since

$$\frac{1}{e^t-1}+\frac{1}{2}-\frac{1}{t}-\frac{t}{12}=O(t^2)\,,$$

$$\frac{1}{e^{zt}-1}=\frac{1}{zt}+\cdots\,.$$

Thus

$$\log \rho(z)=\int_0^{\infty}\left(\frac{t}{12}-\frac{1}{2}+\frac{1}{t}+\frac{1}{e^t-1}\right)\frac{1}{e^{zt}-1}\frac{dt}{t}$$

and our proposition follows from this, (4.2), (4.5) and (4.19). ∎

We summarize the above result in the following:

THEOREM 3. *For* $\operatorname{Re} w>0$, $\omega_1>0$, $\omega_2>0$,

$$\log\frac{\Gamma_2(w\,;\,\omega_1,\omega_2)}{\rho_2(\omega_1,\omega_2)}=\zeta_2'(0\,;\,w\,;\,\omega_1,\omega_2)\,.$$

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