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Twisting of Knots along Knotted Solid Tori

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Abstract. We twist a knot by twisting some solid tori, each of which contains the knot, one by one to the meridional direction. Suppose the solid tori have mutually disjoint essential toral boundaries in the knot exterior.

M. Kouno, K. Motegi and T. Shibuya have shown that, by twisting a knot along one or two solid tori in an inclusion relation, the resulting knot type is different from the original one (see [5] and [6]). In this paper, we investigate whether twisting along various solid tori also produces a different knot type or not.

§1. Introduction.

Let K_1 and K_2 be unoriented knots in the oriented sphere S^3. If there exists an orientation preserving homeomorphism of S^3 carrying K_1 to K_2, then we write $K_1 \cong K_2$. This is equivalent to saying that K_1 and K_2 are ambient isotopic in S^3.

Let K be a knot in S^3, and V a solid torus with a preferred framing such that $K \subset \mathit{int} V$. The *wrapping number* of K in V, denoted by $w_V(K)$, is the minimal geometric intersection number of K and a meridian disk of V. Suppose ∂V is oriented. Let $f^{(n)}: S^3 \to S^3$ be a map, which may be discontinuous on T, such that $f^{(n)}|_{S^3-V}$ is the identity, and $f^{(n)}|_V$ is an orientation preserving self-homeomorphism of V satisfying $f_*^{(n)}(m)=m$ and $f_*^{(n)}(l)=l+nm$, where $f_*^{(n)}: H_1(\partial V) \to H_1(\partial V)$ is an isomorphism induced by $f^{(n)}$, m and l are homology classes of a meridian and a preferred longitude of ∂V with the intersection number $m \cdot l = 1$ respectively. $f^{(n)}$ is not homeomorphic on S^3, but it gives the orientation preserving self-homeomorphisms of $S^3 - V$ and V. Note that for a given knot K, a solid torus V, an orientation of ∂V and an integer n determine the unique knot type $f^{(n)}(K)$. We call $f^{(n)}$ an *n-twist* along V.

Suppose K is fixed by an orientation preserving homeomorphism φ of S^3 of order $n > 1$. When φ is a rotation about an unknotted circle disjoint from K, we say K has a *semifree period*. When φ has no fixed point set, we say K has a *free period*.

Let V_1 and V_2 be arbitrary solid tori containing K with mutually disjoint boundaries, and $f_i^{(n)}$ an n-twist along V_i. Since $f_2^{(n_2)}$ gives a homeomorphism either on

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V_1 or on S^3-V_1, $V_1'=f_2^{(n_2)}(V_1)$ is a solid torus bounded by $f_2^{(n_2)}(T_1)$. We do not distinguish notationally between $f_1^{(n_1)}$ and n_1-twist along V_1'. Similarly $f_2^{(n_2)}$ also denotes the n_2-twist along $f_1^{(n_1)}(V_2)$. So we have $f_1^{(n_1)}\circ f_2^{(n_2)}=f_2^{(n_2)}\circ f_1^{(n_1)}$ up to isotopies on S^3.

When $w_V(K)\leq 1$, knot type K is invariant under any twisting along V. So in this paper, we consider a solid torus V, to carry out a twisting of K, is always knotted and satisfies $w_V(K)\geq 2$. Note that ∂V is essential in the exterior of K.

The following theorem was shown by M. Kouno, K. Motegi and T. Shibuya.

THEOREM 1 ([5]). *Let K be a knot in S^3, V a knotted solid torus containing K with $w_V(K)\geq 2$, and $f^{(n)}$ an n-twist along V. For any non-zero integer n, $f^{(n)}(K)\not\cong K$.*

They showed the following theorem in their succeeding paper.

THEOREM 2 ([6]). *Let K be a knot in S^3, V_i a knotted solid torus containing K, and $f_i^{(n)}$ an n-twist along V_i $(i=1, 2)$. Suppose that $V_1\subset int V_2$, $w_{V_2}(core V_1)\geq 2$ where $core V_1$ denotes a core of V_1, and $w_{V_1}(K)\geq 2$. Then $f_1^{(m)}(K)\not\cong f_2^{(n)}(K)$ for any pair $(m, n)\neq(0, 0)$.*

In Theorem 2, either $m=0$ or $n=0$ implies Theorem 1.

Throughout this paper, $intX$ and $N(X, Y)$ denotes the *interior* of X and the *neighbourhood* of X in Y respectively.

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§2. Main results.

Let K be a non-trivial knot in S^3. Consider the torus decomposition of $E=S^3-intN(K, S^3)$ (see [3] and [4]). By a finite set $\mathcal{T}'=\{T_1', \cdots, T_k'\}$ of mutually disjoint, non-parallel, essential tori imbedded in E, we can decompose E uniquely into the pieces each of which is Seifert fibered or admits a complete hyperbolic structure of finite volume in its interior (see [10]). Moreover each Seifert piece is one of a torus knot space, a cable space and a composing space (see [3]).

Let V_i' be the solid torus bounded by T_i'. Define a subset $\mathcal{T}$ of $\mathcal{T}'$ by $\mathcal{T}=\{T_i'\in\mathcal{T}'; w_{V_i'}(K)\geq 2\}$. Let $\mathcal{T}=\{T_1, \cdots, T_l\}$ and V_i the solid torus bounded by T_i. Suppose T_i has an orientation decided by that of S^3 and a normal of T_i oriented to the exterior of V_i. Note that if there is an orientation preserving homeomorphism of S^3 carrying T_i to T_j, it induces an orientation preserving homeomorphism from T_i to T_j.

We consider the composition of the twisting along $V_1, \cdots, V_l$, and show the following theorem.

THEOREM 3. *If $\sum_{i=1}^{l} n_i\neq 0$, then $f_1^{(n_1)}\circ\cdots\circ f_l^{(n_l)}(K)\not\cong K$.*

Here $f_1^{(n_1)}\circ\cdots\circ f_l^{(n_l)}: S^3\to S^3$ may be discontinuous on $T_1, \cdots, T_l$, but it induces an orientation preserving homeomorphism on the interior of each decomposing piece.

The following corollary of Theorem 3 answers all the case of knotted solid tori to carry out the twisting of K.

COROLLARY 4. *Let V_i' $(1\leq i\leq k)$ be knotted solid tori containing K, and $g_i^{(n)}$ an n-twist along V_i'. Suppose $\partial V_i'$ are mutually disjoint and $w_{V_i'}(K)\geq 2$. If $\sum_{i=1}^k n_i\neq 0$, then $g_1^{(n_1)}\circ\cdots\circ g_k^{(n_k)}(K)\not\cong K$.*

When $k=1$, Corollary 4 is the same as Theorem 1.

Next, we consider two knotted solid tori V_1' and V_2' with mutually disjoint boundaries, containing a knot K in their interiors. By [8, Satz 1], we may assume one of the following occurs:

(1) $V_1'\subset V_2'$ or $V_2'\subset V_1'$.

(2) $V_1'\cup V_2'=S^3$.

(3) There exists a solid torus $W\subset int V_1'\cap int V_2'$ such that $w_{V_1'}(core W)=w_{V_2'}(core W)=1$.

Theorem 2 corresponds to the case (1), and the following corollary of Theorem 3 corresponds to the case (2).

COROLLARY 5. *Let V_i' $(i=1, 2)$ be in the case (2), and $g_i^{(n)}$ an n-twist along V_i'. If $n_1+n_2\neq 0$, then $g_1^{(n_1)}\circ g_2^{(n_2)}(K)\not\cong K$.*

In a special case when all the solid tori to twist are in an inclusion relation, we get the following theorem, which is a generalization of Theorem 1 and 2.

THEOREM 6. *Let K be a knot in S^3, V_i' $(1\leq i\leq k)$ knotted solid tori such that:*

(1) $K\subset int V_1'$ and $w_{V_1'}(K)\geq 2$.

(2) $V_i'\subset int V_{i+1}'$ and $w_{V_{i+1}'}(core V_i')\geq 2$ for $1\leq i\leq k-1$.

Let $g_i^{(n)}$ be an n-twist along V_i'. Then $g_1^{(n_1)}\circ\cdots\circ g_k^{(n_k)}(K)\not\cong K$ for any $(n_1, \cdots, n_k)\neq(0, \cdots, 0)$.

Theorem 3 requires the condition $\sum_{i=1}^l n_i\neq 0$. The next problem is what kind of knot admits both $(n_1, \cdots, n_l)\neq(0, \cdots, 0)$, $\sum_{i=1}^l n_i=0$ and $f_1^{(n_1)}\circ\cdots\circ f_l^{(n_l)}(K)\cong K$. We give a partial answer of this problem in the following theorem.

THEOREM 7. *Suppose $(n_1, \cdots, n_l)\neq(0, \cdots, 0)$ and $\sum_{i=1}^l n_i=0$. If there is an orientation preserving periodic homeomorphism $\varphi: S^3\to S^3$ carrying $f_1^{(n_1)}\circ\cdots\circ f_l^{(n_l)}(K)$ to K, then for some $(n_1', \cdots, n_l')\neq(0, \cdots, 0)$, $f_1^{(n_1')}\circ\cdots\circ f_l^{(n_l')}(K)$ is a periodic knot.*

Here a periodic knot means a knot with a free or semifree period.

§3. Twisting of *K*.

Let m_i and l_i be the homology classes of a meridian and a preferred longitude of

T_i with $m_i \cdot l_i = 1$ respectively.

First we give a lemma for the decomposing piece M_i attaching T_i in V_i. In the statement and the proof of the following lemma, double signs are in the same order.

LEMMA 8. *Let $F_i: M_i \to M_i$ be an orientation preserving homeomorphism carrying T_i to T_i, $F_{i*}: H_1(T_i) \to H_1(T_i)$ an isomorphism induced by F_i. If M_i is not a composing space and $F_{i*}(m_i) = \pm m_i$, then $F_{i*}(l_i) = \pm l_i$.*

PROOF. We have $F_{i*}(l_i) = \pm(l_i + am_i)$ for some a. Since ∂M_i contains T_i and a torus separating T_i and K, M_i is either a hyperbolic piece or a cable space.

Assume M_i is a hyperbolic piece. By Mostow's rigidity theorem ([10, 5.7.4]), $Isom(intM_i) \cong Out(\pi_1(intM_i))$ is a finite group. So $F_i|_{intM_i}$ is homotopic to a unique isometry G_i of M_i and G_i^N is the identity for some integer $N > 0$. Then F_i^N is homotopic to the identity. So we get $F_{i*}^N(l_i) = (\pm 1)^N(l_i + Nam_i) = l_i$. Hence we get $a = 0$ and $F_{i*}(l_i) = \pm l_i$.

Assume M_i is a cable space. Since the Seifert fibering of M_i is unique [2, Lemma VI.17], F_i is isotopic to a fiber preserving homeomorphism. Then we have $F_*(pm_i + ql_i) = \pm(pm_i + ql_i)$, where $pm_i + ql_i \in H_1(T_i)$ is a class of a regular fiber on T_i. Here p and q are coprime non-zero integers such that $q \neq 1$. So we get $\pm(pm_i + q(l_i + am_i)) = \pm(pm_i + ql_i)$. Hence we get $a = 0$ and $F_{i*}(l_i) = \pm l_i$. □

LEMMA 9. *Let M_i be a composing space, $T_j \subset \partial M_i$ a torus separating T_i and K, T an essential torus in $intM_i$, V a solid torus bounded by T, and $f^{(n)}$ an n-twist along V. Then we have $f_i^{(n)}(K) \cong f_j^{(n)}(K) \cong f^{(n)}(K)$.*

PROOF. By [1, Proposition 13], $coreV_j$, which is a companion of K, is a composite knot. Since $w_{V_i}(coreV_j) = 1$, $f_i^{(n)}$ twists V_j n times to the meridional direction and $f_i^{(n)}(coreV_j)$ is ambient isotopic to $coreV_j$ in V_i. So we get $f_i^{(n)}(K) \cong f_j^{(n)}(K)$. Other cases are similar. □

By Lemma 9, $f_i^{(n)}$, $f_j^{(n)}$ and $f^{(n)}$ are mutually replaceable.

PROOF OF THEOREM 3. When $f_1^{(n_1)} \circ \cdots \circ f_l^{(n_l)}(K) \cong K$ for some $(n_1, \cdots, n_l)$, there is an orientation preserving homeomorphism $\varphi: S^3 \to S^3$ carrying $f_1^{(n_1)} \circ \cdots \circ f_l^{(n_l)}(K)$ to K. Each torus $\varphi \circ f_1^{(n_1)} \circ \cdots \circ f_l^{(n_l)}(T_i)$ is essential in E and the bounded solid torus V_i' satisfies $w_{V_i'}(K) \geq 2$. Modify φ so as to carry $\bigcup_{i=1}^{l} f_1^{(n_1)} \circ \cdots \circ f_l^{(n_l)}(T_i)$ to $\bigcup \mathscr{T}$.

Assume M_i with $n_i \neq 0$ is a composing space for some i. Let $T_j \subset \partial M_i$ be a torus separating T_i and K. By the condition of the torus decomposition of E, M_j is not a composing space. Change $f_i^{(n_i)}$ for $f_j^{(n_i)}$ by Lemma 9. Then we can assume M_i with $n_i \neq 0$ is not a composing space for any i.

For convenience, we write $\varphi_\mu = \varphi \circ f_1^{(n_1)} \circ \cdots \circ f_l^{(n_l)}$ where $\mu = (n_1, \cdots, n_l)$. Then $\varphi_\mu: S^3 \to S^3$, which may be discontinuous, fixes $\bigcup \mathscr{T}$ and induces an orientation preserving homeomorphism on the interior of each decomposing piece. Then φ_μ induces a permutation on a finite set $\mathscr{T}$. Since the degree of the permutation is finite, there is an integer $N>0$ such that $\varphi_\mu^N(T)=T$ for any $T\in\mathscr{T}$. Let $T\in\mathscr{T}$, and $N_T>0$ a minimal integer such that $\varphi_\mu^{N_T}(T)=T$. We call the set $\{T, \varphi_\mu(T), \cdots, \varphi_\mu^{N_T-1}(T)\}\subset\mathscr{T}$ the *orbit* of T under φ_μ. Next, we show the following lemma.

LEMMA 10. *Suppose* $f_1^{(n_1)}\circ\cdots\circ f_l^{(n_l)}(K)\cong K$ *and* φ_μ *is as above, and any* M_i *with* $n_i\neq 0$ *is not a composing space. Let* $\mathscr{T}_i=\{T_{p(i,1)}, T_{p(i,2)}, \cdots, T_{p(i,N_i)}\}$ $(1\le i\le r)$ *be the orbits under* φ_μ, *which are mutually disjoint, and satisfy* $\bigcup_{i=1}^r\mathscr{T}_i=\mathscr{T}$, $\varphi_\mu(T_{p(i,j)})=T_{p(i,j+1)}$ *for* $1\le j\le N_i-1$ *and* $\varphi_\mu(T_{p(i,N_i)})=T_{p(i,1)}$. *Then we have* $v_i=\sum_{j=1}^{N_i}n_{p(i,j)}=0$ *for any* i.

Since $\sum_{i=1}^l n_i=\sum_{i=1}^r v_i$, Lemma 10 completes the proof of Theorem 3.

PROOF. Let $N=LCM\{N_1, \cdots, N_r\}$ and $v_i'=v_iN/N_i$. We calculate φ_μ^N in the following. First we calculate it in the case when $r=1$ and $N=N_1=l$. Since $f_{p(1,j)}^{(n)}\circ\varphi_\mu=\varphi_\mu\circ f_{p(1,j-1)}^{(n)}$ for $2\le j\le N_1$ and $f_{p(1,1)}^{(n)}\circ\varphi_\mu=\varphi_\mu\circ f_{p(1,l)}^{(n)}$, we have

$$\begin{aligned}\varphi_\mu^l&=\varphi\circ f_{p(1,1)}^{(n_{p(1,1)})}\circ f_{p(1,2)}^{(n_{p(1,2)})}\circ\cdots\circ f_{p(1,l)}^{(n_{p(1,l)})}\circ\varphi_\mu^{l-1}\\&=\varphi\circ\varphi_\mu\circ f_{p(1,l)}^{(n_{p(1,1)})}\circ f_{p(1,1)}^{(n_{p(1,2)})}\circ\cdots\circ f_{p(1,l-1)}^{(n_{p(1,l)})}\circ\varphi_\mu^{l-2}\\&=\varphi^2\circ f_{p(1,1)}^{(n_{p(1,1)}+n_{p(1,2)})}\circ f_{p(1,2)}^{(n_{p(1,2)}+n_{p(1,3)})}\circ\cdots\circ f_{p(1,l)}^{(n_{p(1,l)}+n_{p(1,1)})}\circ\varphi_\mu^{l-2}\\&=\varphi^3\circ f_{p(1,1)}^{(n_{p(1,1)}+n_{p(1,2)}+n_{p(1,3)})}\circ f_{p(1,2)}^{(n_{p(1,2)}+n_{p(1,3)}+n_{p(1,4)})}\circ\cdots\circ f_{p(1,l)}^{(n_{p(1,l)}+n_{p(1,1)}+n_{p(1,2)})}\circ\varphi_\mu^{l-3}\\&=\varphi^l\circ f_{p(1,1)}^{(v_1)}\circ f_{p(1,2)}^{(v_1)}\circ\cdots\circ f_{p(1,l)}^{(v_1)}\,.\end{aligned}$$

Similarly, in the general case, we have

$$\varphi_\mu^N=\varphi^N\circ\{f_{p(1,1)}^{(v_1')}\circ\cdots\circ f_{p(1,N_1)}^{(v_1')}\}\circ\cdots\circ\{f_{p(r,1)}^{(v_r')}\circ\cdots\circ f_{p(r,N_r)}^{(v_r')}\}\,.$$

The pieces $M_{p(i,j)}$ $(1\le j\le N_i)$ are mutually homeomorphic. So we have $v_i'=v_i=0$ if $M_{p(i,1)}$ is a composing space.

Assume that $M_{p(i,1)}$ is not a composing space. We have $\varphi_\mu^N(T_{p(i,1)})=T_{p(i,1)}$ and φ_μ^N induces an orientation preserving self-homeomorphism of $M_{p(i,1)}$. Note that when $V_{p(i,1)}\subset int V_j$ for some j, $V_{p(i,1)}$ can be automatically twisted by $f_j^{(n_j)}$. So φ_μ^N induces an isomorphism of $H_1(T_{p(i,1)})$ carrying $m_{p(i,1)}$ to $\pm m_{p(i,1)}$ and $l_{p(i,1)}$ to $\pm(l_{p(i,1)}+(v_i'+\alpha_i)m_{p(i,1)})$. Here α_i depends on the twisting along the solid tori which contain $V_{p(i,1)}$. Applying Lemma 8 to $\varphi_\mu^N|_{M_{p(i,1)}}$, we get $v_i'+\alpha_i=0$. If either there is no solid torus containing $V_{p(i,1)}$ or any of the twisting along the solid tori containing $V_{p(i,1)}$ is 0-twist, then we have $\alpha_i=0$. So we get $\alpha_i=v_i'=0$ in order of the inclusion relation of the solid tori. Then we get $v_i=0$ for any i. This completes the proof of Lemma 10 and Theorem 3. □

PROOF OF COROLLARY 4. Let T_j, V_j and $f_j^{(n)}$ $(1\le j\le l)$ be as above. Note that $\partial V_i'$ is an essential torus in E, and each T_j represents an isotopy class of essential tori in E. If the class of $\partial V_i'$ is represented by some T_j, change $g_i^{(n_i)}$ for $f_j^{(n_i)}$. Otherwise, it is represented by an essential torus in a composing space (see [1]). Then by Lemma 9, we can change $g_i^{(n_i)}$ for some $f_j^{(n_i)}$. Hence the result holds by Theorem 3. □

PROOF OF THEOREM 6. As in the proof of Corollary 4, change V_i' for some V_{r_i} and $g_i^{(n_i)}$ for $f_{r_i}^{(n_i)}$. By Lemma 9, we can assume that T_{r_i} does not bound a composing space for $1 \leq i \leq k$. New solid tori V_{r_i} $(1 \leq i \leq k)$ also satisfy (1) and (2). Since T_{r_i} $(1 \leq i \leq k)$ are ordered by the inclusion relation of V_{r_i}, their orbits are mutually different. So $f_{r_1}^{(n_1)} \circ \cdots \circ f_{r_k}^{(n_k)}(K) \cong K$ implies $n_i = 0$ for $1 \leq i \leq k$ by Lemma 10. This completes the proof. □

Theorem 6 gives a proof of Theorem 2 which is different from that in [6].

§4. Relations to the periodicity.

In this section, we observe the case when $(n_1, \cdots, n_l) \neq (0, \cdots, 0)$, $\sum_{i=1}^{l} n_i = 0$ and $f_1^{(n_1)} \circ \cdots \circ f_l^{(n_l)}(K) \cong K$.

REMARK 11. We construct a new knot K' from a periodic knot K in the following. Theorem 3 without the condition $\sum_{i=1}^{l} n_i \neq 0$ does not hold for K'.

Let K be a periodic knot in S^3 with the period $N \geq 2$, $\varphi : S^3 \to S^3$ an orientation preserving homeomorphism with the period N which fixes K. Suppose $\mathscr{T} = \{T_1, \cdots, T_l\}$ is as in §3, and φ fixes $\bigcup \mathscr{T}$. As in Lemma 10, assume the orbits under φ are $\{T_{p(i,1)}, \cdots, T_{p(i,N_i)}\}$ where $1 \leq i \leq r$. We have $N = LCM\{N_1, \cdots, N_r\}$. Suppose $\sum_{j=1}^{N_i} n_{p(i,j)} = 0$ for $1 \leq i \leq r$. Note that this implies $\sum_{i=1}^{l} n_i = 0$. Define a knot K' by

$$
\begin{aligned}
K' = &\{f_{p(1,2)}^{(\theta(1,1))} \circ f_{p(1,3)}^{(\theta(1,2))} \circ \cdots \circ f_{p(1,N_1)}^{(\theta(1,N_1-1))}\} \\
&\circ \cdots \circ \{f_{p(r,2)}^{(\theta(r,1))} \circ f_{p(r,3)}^{(\theta(r,2))} \circ \cdots \circ f_{p(r,N_r)}^{(\theta(r,N_r-1))}\}(K),
\end{aligned}
$$

where $\theta(i,j) = \sum_{k=1}^{j} n_{p(i,k)}$. Denote $\varphi \circ f_1^{(n_1)} \circ \cdots \circ f_l^{(n_l)}$ by φ_μ where $\mu = (n_1, \cdots, n_l)$. Note that $\theta(i, 1) = n_{p(i,1)}$, $\theta(i, N_i) = 0$, and $f_{p(i,j+1)}^{(\theta(i,j+1))} \circ f_{p(i,j+1)}^{(-n_{p(i,j+1)})} = f_{p(i,j+1)}^{(\theta(i,j))}$ for any $1 \leq i \leq r$ and $1 \leq j \leq N_i - 1$. Then we have

$$
\begin{aligned}
\varphi_\mu(K') &= \varphi_\mu \circ \{f_{p(1,2)}^{(\theta(1,1))} \circ \cdots \circ f_{p(1,N_1)}^{(\theta(1,N_1-1))}\} \circ \cdots \circ \{f_{p(r,2)}^{(\theta(r,1))} \circ \cdots \circ f_{p(r,N_r)}^{(\theta(r,N_r-1))}\}(K) \\
&= \varphi_\mu \circ \{(f_{p(1,1)}^{\theta(1,1)} \circ f_{p(1,1)}^{(-n_{p(1,1)})}) \circ (f_{p(1,2)}^{(\theta(1,2))} \circ f_{p(1,2)}^{(-n_{p(1,2)})}) \circ \cdots \circ (f_{p(1,N_1)}^{(\theta(1,N_1))} \circ f_{p(1,N_1)}^{(-n_{p(1,N_1)})})\} \\
&\quad \circ \cdots \circ \{(f_{p(r,1)}^{(\theta(r,1))} \circ f_{p(r,1)}^{(-n_{p(r,1)})}) \circ \cdots \circ (f_{p(r,N_r)}^{(\theta(r,N_r))} \circ f_{p(r,N_r)}^{(-n_{p(r,N_r)})})\}(K) \\
&= \varphi_\mu \circ \{f_{p(1,1)}^{(\theta(1,1))} \circ \cdots \circ f_{p(1,N_1-1)}^{(\theta(1,N_1-1))}\} \circ \cdots \circ \{f_{p(r,1)}^{(\theta(r,1))} \circ \cdots \circ f_{p(r,N_r-1)}^{(\theta(r,N_r-1))}\} \\
&\quad \circ \{f_{p(1,1)}^{(-n_{p(1,1)})} \circ \cdots \circ f_{p(1,N_1)}^{(-n_{p(1,N_1)})}\} \circ \cdots \circ \{f_{p(r,1)}^{(-n_{p(r,1)})} \circ \cdots \circ f_{p(r,N_r)}^{(-n_{p(r,N_r)})}\}(K) \\
&= \varphi_\mu \circ \{f_{p(1,1)}^{(\theta(1,1))} \circ \cdots \circ f_{p(1,N_1-1)}^{(\theta(1,N_1-1))}\} \circ \cdots \circ \{f_{p(r,1)}^{(\theta(r,1))} \circ \cdots \circ f_{p(r,N_r-1)}^{(\theta(r,N_r-1))}\} \\
&\quad \circ f_1^{(-n_1)} \circ f_2^{(-n_2)} \circ \cdots \circ f_l^{(-n_l)}(K).
\end{aligned}
$$

Since $\varphi_\mu \circ f_{p(i,j)}^{(n)} = f_{p(i,j+1)}^{(n)} \circ \varphi_\mu$ for $1 \leq i \leq r$ and $1 \leq j \leq N_i - 1$, we get

$$
\begin{aligned}
\varphi_\mu(K') = &\{f_{p(1,2)}^{(\theta(1,1))} \circ \cdots \circ f_{p(1,N_1)}^{(\theta(1,N_1-1))}\} \circ \cdots \circ \{f_{p(r,2)}^{(\theta(r,1))} \circ \cdots \circ f_{p(r,N_r)}^{(\theta(r,N_r-1))}\} \\
&\circ \varphi_\mu \circ f_1^{(-n_1)} \circ f_2^{(-n_2)} \circ \cdots \circ f_l^{(-n_l)}(K)
\end{aligned}
$$

$$=\{f_{p(1,2)}^{(\theta(1,1))}\circ\cdots\circ f_{p(1,N_1)}^{(\theta(1,N_1-1))}\}\circ\cdots\circ\{f_{p(r,2)}^{(\theta(r,1))}\circ\cdots\circ f_{p(r,N_r)}^{(\theta(r,N_r-1))}\}\circ\varphi(K)$$
$$=K'.$$

So we have $f_1^{(n_1)}\circ\cdots\circ f_l^{(n_l)}(K')\cong K'$.

When $r=1$ and $N_1=2$, we show an example in Figure 1. The knot K' is constructed from a periodic knot $K=f_2^{(-1)}(K')$, and φ has a period 2. T_1 and T_2 are essential tori in the same orbit under $\varphi_{(1,-1)}$. In the sense of Remark 11, a knot type related to the periodicity is invariant under twisting.

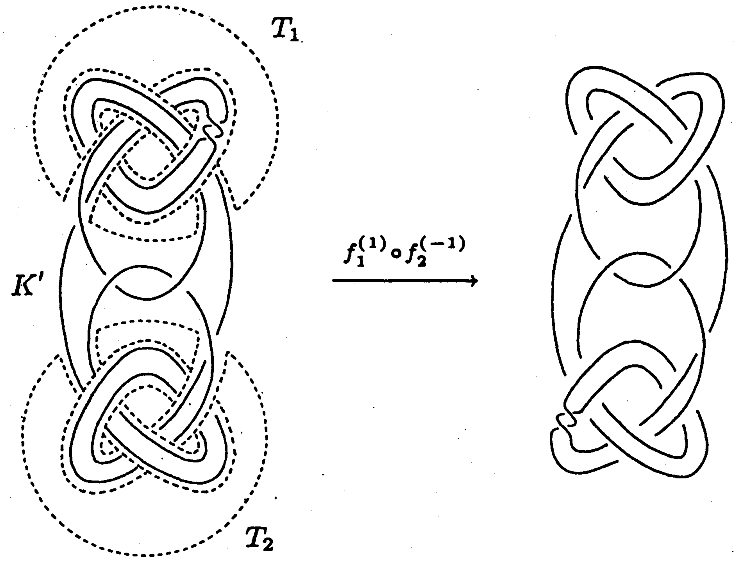

FIGURE 1

Assume the case $(n_1,\cdots,n_l)\neq(0,\cdots,0)$, $\sum_{i=1}^{l}n_i=0$ and $f_1^{(n_1)}\circ\cdots\circ f_l^{(n_l)}(K)\cong K$. If the equivalence of these knots is given by an orientation preserving periodic homeomorphism $\varphi: S^3\to S^3$, the above observation is formulated as Theorem 7.

PROOF OF THEOREM 7. Let $\mathscr{T}=\{T_1,\cdots,T_l\}$ and M_i be as in §3. We denote $\varphi\circ f_1^{(n_1)}\circ\cdots\circ f_l^{(n_l)}$ by φ_μ where $\mu=(n_1,\cdots,n_l)$. Modify φ so as to satisfy $\varphi_\mu(\bigcup\mathscr{T})=\bigcup\mathscr{T}$. Let the orbits under φ_μ be $\{T_{p(i,1)},T_{p(i,2)},\cdots,T_{p(i,N_i)}\}$ where $1\leq i\leq r$. By Lemma 9, we can assume M_i with $n_i\neq 0$ is not a composing space for any i. Then we have $\sum_{j=1}^{N_i}n_{p(i,j)}=0$ for $1\leq i\leq r$ by Lemma 10. Define the knot K' by

$$K'=\{f_{p(1,2)}^{(-\theta(1,1))}\circ f_{p(1,3)}^{(-\theta(1,2))}\circ\cdots\circ f_{p(1,N_1)}^{(-\theta(1,N_1-1))}\}$$
$$\circ\cdots\circ\{f_{p(r,2)}^{(-\theta(r,1))}\circ f_{p(r,3)}^{(-\theta(r,2))}\circ\cdots\circ f_{p(r,N_r)}^{(-\theta(r,N_r-1))}\}(K),$$

where $\theta(i,j)=\sum_{k=1}^{j}n_{p(i,k)}$. Then we get

$$\varphi(K')=\varphi\circ\{f_{p(1,2)}^{(-\theta(1,1))}\circ\cdots\circ f_{p(1,N_1)}^{(-\theta(1,N_1-1))}\}\circ\cdots\circ\{f_{p(r,2)}^{(-\theta(r,1))}\circ\cdots\circ f_{p(r,N_r)}^{(-\theta(r,N_r-1))}\}(K)$$

$$
\begin{aligned}
&= \varphi_\mu \circ f_1^{(-n_1)} \circ f_2^{(-n_2)} \circ \cdots \circ f_l^{(-n_l)} \circ \{ f_{p(1,2)}^{(-\theta(1,1))} \circ \cdots \circ f_{p(1,N_1)}^{(-\theta(1,N_1-1))} \} \\
&\qquad \circ \cdots \circ \{ f_{p(r,2)}^{(-\theta(r,1))} \circ \cdots \circ f_{p(r,N_r)}^{(-\theta(r,N_r-1))} \}(K) \\
&= \varphi_\mu \circ \{ f_{p(1,1)}^{(-n_{p(1,1)})} \circ \cdots \circ f_{p(1,N_1)}^{(-n_{p(1,N_1)})} \} \circ \cdots \circ \{ f_{p(r,1)}^{(-n_{p(r,1)})} \circ \cdots \circ f_{p(r,N_r)}^{(-n_{p(r,N_r)})} \} \\
&\qquad \circ \{ f_{p(1,2)}^{(-\theta(1,1))} \circ \cdots \circ f_{p(1,N_1)}^{(-\theta(1,N_1-1))} \} \circ \cdots \circ \{ f_{p(r,2)}^{(-\theta(r,1))} \circ \cdots \circ f_{p(r,N_r)}^{(-\theta(r,N_r-1))} \}(K) .
\end{aligned}
$$

Since we have

$$
\begin{aligned}
&\{ f_{p(i,1)}^{(-n_{p(i,1)})} \circ \cdots \circ f_{p(i,N_i)}^{(-n_{p(i,N_i)})} \} \circ \{ f_{p(i,2)}^{-\theta(i,1)} \circ \cdots \circ f_{p(i,N_i)}^{(-\theta(i,N_i-1))} \} \\
&\quad = f_{p(i,1)}^{(-n_{p(i,1)})} \circ (f_{p(i,2)}^{(-\theta(i,1))} \circ f_{p(i,2)}^{(-n_{p(i,2)})}) \circ \cdots \circ (f_{p(i,N_i)}^{(-\theta(i,N_i-1))} \circ f_{p(i,N_i)}^{(-n_{p(i,N_i)})}) \\
&\quad = f_{p(i,1)}^{(-\theta(i,1))} \circ \cdots \circ f_{p(i,N_i-1)}^{(-\theta(i,N_i-1))} ,
\end{aligned}
$$

then

$$
\begin{aligned}
\varphi(K') &= \varphi_\mu \circ \{ f_{p(1,1)}^{(-\theta(1,1))} \circ \cdots \circ f_{p(1,N_1-1)}^{(-\theta(1,N_1-1))} \} \circ \cdots \circ \{ f_{p(r,1)}^{(-\theta(r,1))} \circ \cdots \circ f_{p(r,N_r-1)}^{(-\theta(r,N_r-1))} \}(K) \\
&= \{ f_{p(1,2)}^{(-\theta(1,1))} \circ \cdots \circ f_{p(1,N_1)}^{(-\theta(1,N_1-1))} \} \circ \cdots \circ \{ f_{p(r,2)}^{(-\theta(r,1))} \circ \cdots \circ f_{p(r,N_r)}^{(-\theta(r,N_r-1))} \} \circ \varphi_\mu(K) \\
&= K' .
\end{aligned}
$$

By [7] and [9], the fixed point set of an orientation preserving periodic homeomorphism on S^3 is either empty or a trivial knot. So according as the fixed point set of φ is empty or a trivial knot, K' has a free or semifree period. Therefore it has the required property. □

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