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Nonexistence Results for Harmonic Maps between Noncompact Complete Riemannian Manifolds

Kazuo AKUTAGAWA and Atsushi TACHIKAWA

Shizuoka University

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§1. Introduction.

In this paper, we prove two kinds of nonexistence results for harmonic maps. The one is to prove nonexistence of a harmonic map with a rotational nondegeneracy at infinity, from a simple Riemannian manifold to an Hadamard manifold of negative sectional curvature bounded away from zero. The other is to prove nonexistence of a nonconstant harmonic map with a polynomial growth dilatation, from a complete Riemannian manifold of nonnegative Ricci curvature to a Riemannian manifold of negative sectional curvature bounded away from zero.

Let $M=(M^m, h)$ and $N=(N^n, g)$ be Riemannian manifolds of dimension m and n ($m, n\geq 2$) respectively. Throughout this paper we denote by $x=(x^1, \cdots, x^m)$ and $y=(y^1, \cdots, y^n)$ local coordinates on M and N respectively. We shall write $(h_{\alpha\beta}(x))$ and $(g_{ij}(y))$ for the metric tensors with respect to the local coordinates on M and N respectively. Moreover, $(h^{\alpha\beta}(x))=(h_{\alpha\beta}(x))^{-1}$, $(g^{ij}(y))=(g_{ij}(y))^{-1}$ and $h(x)$ denotes the determinant of $(h_{\alpha\beta})$. The Christoffel symbols on M and N will be denoted by $\Gamma^{\alpha}_{\beta\gamma}$ and Γ^{i}_{jk} respectively.

For a map $U\in C^1(M, N)$ we define the *energy density* $e(U)(x)$ of U at $x\in M$ by

$$e(U)(x)=\frac{1}{2}\|dU(x)\|^2=\frac{1}{2}h^{\alpha\beta}(x)D_\alpha u^i(x)D_\beta u^i(x)g_{ij}(u(x)),$$

where $u(x)$ is the expression of $U(x)$ with respect to the local coordinates $(y^1, \cdots, y^n)$ and D_α denotes $\partial/\partial x^\alpha$. For a bounded domain $\Omega\subset M$, we define the *energy* of U on Ω by

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$$E(U\,;\,\Omega)=\int_{\Omega} e(U)d\mu\,,$$

where $d\mu=\sqrt{h(x)}dx$ stands for the volume element on M. A map $U: M\to N$ is said to be *harmonic* if it is of class C^2 and is a critical point of the energy functional on any bounded domain $\Omega\subset M$. The Euler-Lagrange equation for the energy functional is given by

$$\Delta u^i(x)+h^{\alpha\beta}(x)\Gamma^i_{jk}(u(x))D_\alpha u^j(x)D_\beta u^k(x)=0 \qquad \text{for} \quad 1\le i\le n\,, \tag{1.1}$$

where Δ denotes the Laplace-Beltrami operator on M, i.e.

$$\Delta=h^{\alpha\beta}(x)D_\alpha D_\beta-h^{\alpha\beta}(x)\Gamma^\gamma_{\alpha\beta}(x)D_\gamma\,. \tag{1.2}$$

A Reimannian manifold is said to be *simple* if it is diffeomorphic to the Euclidean m-space $\boldsymbol{R}^m$ and furnished with a metric for which associated Laplace-Beltrami operator is uniformly elliptic, and $p_0\in M$ is said to be a *pole* of M if the exponential map at $p_0\in M$ gives a diffeomorphism between M and the Euclidean space. Moreover, a Riemannian manifold N is said to be an *Hadamard manifold* if it is a complete simply connected Riemannian manifold with nonpositive sectional curvature.

In [24], the second author proved that there exists no harmonic map U from $\boldsymbol{R}^m$ to an Hadamard n-manifold N with negative sectional curvature K_N, $K_N\le-\kappa^2<0$, satisfying the following uniform rotational nondegeneracy condition (see also [25]): There exists a positive constant ε such that

$$\sum_{i=1}^{n}\sum_{\alpha=1}^{m}\left(D_\alpha\frac{u^i}{|u|}\right)^2\ge\frac{\varepsilon}{|x|^2} \qquad \text{for all} \quad x\in\boldsymbol{R}^m\,,$$

where $x=(x^1,\cdots,x^m)$ is the canonical normal coordinate system on $\boldsymbol{R}^m$, $u=(u^1,\cdots,u^n)$ a normal coordinate system centered at $U(0)$, $|x|=\sqrt{\sum_{\alpha=1}^m(x^\alpha)^2}$ and $|u(x)|=\sqrt{\sum_{i=1}^n(u^i(x))^2}$. The nonexistence results of this type can be found in [20] also.

Now we can state the first main result.

THEOREM 1.1. *Let M be a simple Riemannian m-manifold with a pole $p_0\in M$, $(x^1,\cdots,x^m)$ a normal coordinate system centered at p_0 and $k_M(x)$ the minimum of the sectional curvature of M at x. Assume that $k_M(x)$ satisfies*

$$-\min\{k_M(x),0\}\le O(r^{-2}) \qquad \text{as} \quad r=|x|\to\infty\,. \tag{1.3}$$

Let N be an Hadamard n-manifold whose sectional curvature is bounded above by a negative constant $-\kappa^2$. Then there exists no harmonic map $U: M\to N$ which satisfies the following condition:

$$\liminf_{|x|\to\infty}(\log|x|)\left\{|x|^2\left(\frac{\kappa}{\sinh(\kappa\rho)}\right)^2(e(U)(x)-e(\rho)(x))\right\}>0\,, \tag{1.4}$$

where $\rho(x)=\mathrm{dist}_N(U(x), q_0)$ *for an arbitrarily fixed point* $q_0\in N$.

Especially, for the case that $M=\boldsymbol{R}^2$ we get the following result.

COROLLARY 1.1. *Let N be as in Theorem* 1.1. *Then, there is no harmonic map* $U:\boldsymbol{R}^2\to N$ *satisfying the following condition*:

$$\liminf_{|x|\to\infty}(\log|x|)^2\left\{|x|^2\left(\frac{\kappa}{\sinh(\kappa\rho)}\right)^2(e(U)(x)-e(\rho)(x))\right\}>0\,. \tag{1.5}$$

A C^1 map $\varphi: M\to N$ is said to have *bounded (first) dilatation* if there exists a constant $K\geq 1$ such that $\lambda_1(x)\leq K^2\lambda_2(x)$ for all $x\in M$, where $\lambda_1(x)\geq\lambda_2(x)\geq\cdots\geq\lambda_m(x)\geq 0$ are the eigenvalues of the pull-back quadratic form φ^*g on T_xM. Goldberg-Har'El [11] and Sealey [21] proved (improving on a succession of earlier result [12]) that harmonic maps with bounded dilatation, from a complete Riemannian manifold of nonnegative Ricci curvature to a Riemannian manifold of negative sectional curvature bounded away from zero, are constant maps. Kendall [16], [17] also obtained similar results by using stochastic methods. We shall generalize the concept of bounded dilatation. For each $s\geq 0$, we say that a C^1 map $\varphi: M\to N$ has *polynomial growth dilatation of order at most s* if there exist a constant $K\geq 1$ and some point $p_0\in M$ such that

$$\lambda_1(x)\leq K^2(1+r(x)^2)^{s/2}\lambda_2(x)\qquad\text{for all}\quad x\in M\,, \tag{1.6}$$

where $r(x)=\mathrm{dist}_M(p_0, x)$ and $\lambda_1(x)\geq\lambda_2(x)\geq\cdots\geq\lambda_m(x)\geq 0$ are the eigenvalues of φ^*g on T_xM. It turns out that this definition is independent on the choice of the point $p_0\in M$.
The second main result can be stated as follows.

THEOREM 1.2. *Let M be a complete Riemannian manifold of nonnegative Ricci curvature* $\mathrm{Ric}_M\geq 0$ *and N a Riemannian manifold of negative sectional curvature* K_N, $K_N\leq -\kappa^2<0$. *Let* $U: M\to N$ *be a harmonic map with polynomial growth dilatation of order at most s. If* $s=2$, *then the energy density* $e(U)$ *of U is bounded on M. Furthermore, if* $s<2$, *then U is a constant map.*

COROLLARY 1.2. *Under the same assumptions as in Theorem* 1.2, *if* $m=\dim M=2$, *then every harmonic map* $U: M^2\to N$ *with polynomial growth dilatation of order at most* 2 *is a constant map.*

In [5] Choi and Treibergs constructed an explicit one-parameter family of harmonic diffeomorphisms of the hyperbolic plane $\boldsymbol{H}^2$ onto itself (See also [19]). Recently, constructions of harmonic maps from the hyperbolic m-space $\boldsymbol{H}^m$ to $\boldsymbol{H}^n$ with unbounded image have been obtained in [1] (for $m=n=2$) and Li-Tam [18], [19] (for all $m, n\geq 2$), by solving the asymptotic Dirichlet problem for harmonic maps between $\boldsymbol{H}^m$ and $\boldsymbol{H}^n$.
On the other hand, not much has been known about harmonic maps from $\boldsymbol{R}^m$ to $\boldsymbol{H}^n$ with unbounded images. In fact, Eells-Lemaire proposed the following problem.

PROBLEM. ((7.4) in [7]) *Is there a harmonic map from* $\boldsymbol{R}^2$ *to* $\boldsymbol{H}^2$ *of rank 2 almost* *everywhere? (Certainly such a map φ must have infinite energy $E(\varphi\,;\,\boldsymbol{R}^2)=\infty$. Further, φ can not have bounded dilatation.)*

For an affirmative answer of this question, Choi-Treibergs [6] also constructed a class of harmonic diffeomorphisms from $\boldsymbol{R}^2$ into $\boldsymbol{H}^2$ with unbounded images. Corollary 1.1 implies that these harmonic diffeomorphisms never satisfy the rotational non-degeneracy condition (1.5) at infinity. Also Corollary 1.2 implies that these harmonic maps have never polynomial growth dilatation of order at most 2. These facts, together with Theorem 1.1 and Theorem 1.2, suggest a special aspect of harmonic maps from $\boldsymbol{R}^m$ to $\boldsymbol{H}^n$.

Finally, we would like to mention that Liouville-type theorems have been obtained, for examples, in [2], [8], [9], [13], [14] and [22].

2. Proof of Theorem 1.1.

First of all, we prove some differential geometric estimates which are based on Lemma 6 of [15].

LEMMA 2.1. *Let M_0 be a Riemannian p-manifold with a pole p_0, $(y^1,\cdots,y^p)$ a normal coordinate system centered at p_0, $(\gamma_{ij}(y))$ the metric tensor with respect to the normal coordinate system and $K(y)$ and $k(y)$ the maximum and minimum of the sectional curvatures of M_0 at y respectively. Let f_0, f_1 be functions of class $C^2(\boldsymbol{R}_+, \boldsymbol{R}_+)$ which satisfy*

$$\lim_{t\to 0}\frac{f_a(t)}{t}=1\,,\quad f_a'(t)>0\quad \forall t\in(0,\infty)\,,\qquad a=0,1\,. \tag{2.1}$$

Assume that

$$\begin{aligned}\max\left\{-\frac{f_0''(t)}{f_0(t)},\frac{1-\{f_0'(t)\}^2}{f_0^2(t)}\right\}&\le k(y)\\ \le K(y)&\le\min\left\{-\frac{f_1''(t)}{f_1(t)},\frac{1-\{f_1'(t)\}^2}{f_1^2(t)}\right\},\end{aligned} \tag{2.2}$$

where $t=|y|=\sqrt{\sum_{j=1}^p(y^j)^2}$. Then we have the following estimates

$$|\zeta|^2+t\frac{f_0'(t)}{f_0(t)}\gamma_{ij}(y)\xi^i\xi^j\ge\gamma_{ij}(y)(X^iX^j+y^k\Gamma^i_{kl}(y)X^jX^l)\ge|\zeta|^2+t\frac{f_1'(t)}{f_1(t)}\gamma_{ij}(y)\xi^i\xi^j\,, \tag{2.3}$$

$$|\zeta|^2+\frac{f_0^2(t)}{t^2}|\xi|^2\ge\gamma_{ij}(y)X^iX^j\ge|\zeta|^2+\frac{f_1^2(t)}{t^2}|\xi|^2\,, \tag{2.4}$$

for all $y, X\in\boldsymbol{R}^p$, where $t=|y|$, $\zeta=(X,y)y/t^2$ and $\xi=X-\zeta$.

PROOF. Let P_a $(a=0,1)$ be a warped product manifold $\boldsymbol{R}_+\times_{f_a}S^{p-1}$. Then the metric tensor of P_a with respect to a normal coordinate system $(y^1,\cdots,y^p)$ centered at

$\{0\}\times S^{p-1}$ is given by

$$\gamma^a_{ij}(y)=\frac{y^iy^j}{|y|^2}+\frac{f_a^2(|y|)}{|y|^2}\left(\delta_{ij}-\frac{y^iy^j}{|y|^2}\right).$$

Moreover the maximum and the minimum of sectional curvatures at y are attained by

$$-\frac{f_a''(t)}{f_a(t)}\quad\text{and}\quad\frac{1-(f_a'(t))^2}{f_a^2(t)},$$

where $t=|y|$ (see [22]). Now, applying Rauch's comparison theorem to M_0 and P_a, we get the following estimates as in the proof of Lemma 6 of [15]:

$$t\frac{f_0'(t)}{f_0(t)}\geq\frac{\gamma_{ij}(y)(\xi^i\xi^j+y^k\Gamma^i_{kl}(y)\xi^l\xi^j)}{\gamma_{ij}(y)\xi^i\xi^j}\geq t\frac{f_1'(t)}{f_1(t)},\tag{2.5}$$

$$\frac{f_0^2(t)}{t^2}|\xi|^2\geq\gamma_{ij}(y)\xi^i\xi^j\geq\frac{f_1^2(t)}{t^2}|\xi|^2\tag{2.6}$$

for all $y,\ \xi\in\boldsymbol{R}^p$ with $(\xi,y)=\sum_{j=1}^p\xi^jy^j=0$, where $|\xi|=\sqrt{\sum_{j=1}^p(\xi^j)^2}$.

Since (y^i) is a normal coordinate system, we have the following relations (cf. [15])

$$\begin{cases}\text{(a)}\quad \gamma_{ij}(y)y^i=\sum_{i=1}^p\gamma^{ij}y^i=y^j\,,\\[4pt] \text{(b)}\quad \Gamma^i_{jk}(y)y^jy^k=0\,,\\[4pt] \text{(c)}\quad \gamma_{ij}(y)y^k\Gamma^i_{kl}(y)y^j=\Gamma_{kjl}(y)y^ky^j=-\Gamma_{klj}(y)y^ky^j+y^ky^jD_k\gamma_{jl}\\ \qquad\qquad =0+y^kD_k(\gamma_{jl}y^j)-\gamma_{jl}\delta^j_ky^k=y^kD_ky^l-y^l=0\,.\end{cases}\tag{2.7}$$

Remarking the above relations we get

$$\gamma_{ij}(y)(X^iX^j+y^k\Gamma^i_{kl}(y)X^lX^j)=\gamma_{ij}(y)\zeta^i\zeta^j+\gamma_{ij}(y)\{\xi^i\xi^j+y^k\Gamma^i_{kl}(y)\xi^l\xi^j\}\,.$$

Using above equality, from (2.5), (2.6) and (2.7) we get (2.3) and (2.4). □

LEMMA 2.2 *Let M be as in Theorem 1.1. Let c_0 and c_1 be constants which satisfy*

$$\begin{aligned}c_0^2&\geq-\inf\{k_M(x):x\in M,\ |x|\leq1\}\,,\\ c_1^2&\geq-\inf\{|x|^2k_M(x):x\in M,\ |x|\geq1\}\,.\end{aligned}\tag{2.8}$$

Then there exists a positive constant $c_2=c_2(c_0,c_1)$ such that

$$h_{\alpha\beta}(x)(\xi^\alpha\xi^\beta+x^\gamma\Gamma^\alpha_{\gamma\delta}(x)\xi^\delta\xi^\beta)\leq c_2h_{\alpha\beta}(x)\xi^\alpha\xi^\beta\tag{2.9}$$

for all x and $\xi\in\boldsymbol{R}^m$.

PROOF. For a constant k, put $\varphi_k(t)=c_3t+c_4t^a$, and choose the constants c_3, c_4 and a so that

$$(2.10)\qquad \varphi_k(1)=\frac{1}{k}\sinh k\,,\quad \varphi_k'(1)=\cosh k\,,\quad \varphi_k''(1)=k\sinh k\,.$$

Moreover, put

$$(2.11)\qquad \tilde{\varphi}_k(t)=-t^2\frac{\varphi_k''(t)}{\varphi_k(t)}\,,\qquad \hat{\varphi}_k(t)=t^2\frac{1-\{\varphi_k'(t)\}^2}{\varphi_k^2(t)}\,.$$

It is easy to see that if c_3, $c_4>0$ and $a>0$, then $\tilde{\varphi}_k'$, $\hat{\varphi}_k'<0$.

On the other hand, from (2.10), we have

$$(2.12)\qquad \begin{cases} a=a_k=k^2(e^k-e^{-k})/\{(k-1)e^k+(k+1)e^{-k}\}\,, \\ c_3=\dfrac{1}{2k}\left\{\left(1-\dfrac{k-1}{a-1}\right)e^k+\left(1+\dfrac{k+1}{a-1}\right)e^{-k}\right\}, \\ c_4=\dfrac{1}{2k(a-1)}\{(k-1)e^k+(k+1)e^{-k}\}\,. \end{cases}$$

From the above equalities, we can see that if k is sufficiently large then $a>1$ and c_3, $c_4>0$. Thus we can take a constant k_0 so that $\tilde{\varphi}_k$ and $\hat{\varphi}_k$ are monotone decreasing functions for $k>k_0$. Now take $k>\max\{k_0, c_0, c_1\}$ and put

$$f_k(t)=\begin{cases} \dfrac{1}{k}\sinh kt & 0<t\le 1\,, \\ \varphi_k(t) & t\ge 1\,. \end{cases}$$

Then, by a direct calculation, we see that

$$(2.13)\qquad -\frac{f_k''(t)}{f_k(t)}\,,\ \frac{1-\{f_k'(t)\}^2}{f_k^2(t)}=-k^2\le -c_0^2 \qquad \text{for}\quad 0\le t\le 1\,.$$

Moreover, remarking the monotonicity of $\tilde{\varphi}_k$ and $\hat{\varphi}_k$, we obtain

$$(2.14)\qquad \tilde{\varphi}_k,\ \hat{\varphi}_k\le -k^2\le -c_1^2 \qquad \text{for}\quad 1\le t\,.$$

Now, by (2.8), (2.13) and (2.14), we can apply Lemma 2.1 with $M_0=M$ and $f_0=f_k$. On the other hand tf_k'/f_k is bounded. Thus, putting $c_2=\sup_{0<t} tf_k'/f_k+1$, we get (2.9).

□

Let $u=(u^1(x), \cdots, u^n(x))$ be the expression of a harmonic map $U: M\to N$ in terms of a normal coordinate system centered at any fixed point q_0 in N. Then u satisfies the following equation of weak form:

$$(2.15)\qquad \int_{\mathbf{R}^m} h^{\alpha\beta}g_{ij}\{D_\alpha u^i D_\beta\varphi^j+\varphi^k\Gamma_{kl}^i D_\alpha u^l D_\beta u^j\}\sqrt{h}\,dx=0\,,\quad \forall\varphi\in C_0^\infty(\mathbf{R}^m, \mathbf{R}^n)\,.$$

PROPOSITION 2.1. *Let N be as in Theorem 1.1 and u the expression of a harmonic map $U: M \to N$ with respect to a normal coordinate system on N centered at an arbitrary fixed point $q_0 \in N$. Then we have the following differential inequality for $|u|$.*

$$\Delta|u|(x) - \frac{\sinh(2\kappa|u|)}{2\kappa}\left\{\frac{\kappa}{\sinh(\kappa|u|)}\right\}^2 \{e(u)(x) - e(\rho)(x)\} \geq 0 , \tag{2.16}$$

where Δ denotes the Laplace-Beltrami operator on M defined by (1.2) *and* $|u| = \sqrt{\sum_{i=1}^n (u^i)^2}$, $\|Du\|_h^2 = \sum_{i=1}^n h^{\alpha\beta} D_\alpha u^i D_\beta u^i$, $\|(D|u|)\|_h^2 = h^{\alpha\beta} D_\alpha |u| D_\beta |u|$.

Moreover, if M is simple and u satisfies (1.4), *then we get*

$$\Delta|u| - \frac{\varepsilon_0}{2\kappa|x|^2 \log|x|}\sinh(2\kappa|u|) \geq 0 \qquad on \quad M \diagdown B_{R_0}(0) \tag{2.17}$$

for some $\varepsilon_0 > 0$ and $R_0 > 0$.

PROOF. Taking $\varphi = u\eta$, $\eta \in C_0^\infty(\boldsymbol{R}^m, \boldsymbol{R})$ in (2.15) and using (2.7a), we get

$$\int_{\boldsymbol{R}^m} h^{\alpha\beta}\left\{\frac{1}{2} D_\alpha |u|^2 D_\beta \eta + \eta g_{ij}(D_\alpha u^i D_\beta u^j + u^k \Gamma_{kl}^i D_\alpha u^l D_\beta u^j)\right\}\sqrt{h}\, dx = 0 . \tag{2.18}$$

For a fixed $x \in M$, let $\{e_\alpha\}_{\alpha=1,\cdots,m}$ be an orthonormal basis of $T_x M$, and write $e_\alpha = \sum_{\gamma=1}^m e_\alpha^\gamma (\partial/\partial x^\gamma)$ in terms of the local coordinate system $(x^1, \cdots, x^m)$ on M. In (2.3), take $f_1 = \frac{1}{\kappa_0}\sinh(\kappa_0 r)$, $M_0 = N$, $X^i = e_\gamma^\alpha D_\alpha u^i$, and sum up with respect to γ, then we get the following inequality

$$h^{\alpha\beta} g_{ij}(u)(D_\alpha u^i D_\beta u^j + u^k \Gamma_{kl}^i D_\alpha u^l D_\beta u^j) \geq |\zeta|^2 + \sum_{\gamma=1}^m |u| \frac{\cosh(\kappa|u|)}{\sinh(\kappa|u|)} g_{ij}(u)\xi_\gamma^i \xi_\gamma^j , \tag{2.19}$$

where

$$\zeta = (\zeta_\gamma^i) , \quad \zeta_\gamma^i = \frac{\sum_{j=1}^n u^j e_\gamma^\alpha D_\alpha u^j}{|u|^2} u^i \quad \text{and} \quad \xi = (\xi_\gamma^i) = (e_\gamma^\alpha D_\alpha u^i - \zeta_\gamma^i) .$$

Here, we used the fact that

$$\sum_{\gamma=1}^m e_\gamma^\alpha e_\gamma^\beta = h^{\alpha\beta}(x) . \tag{2.20}$$

Moreover, using (2.20) again, we can see that

$$\begin{aligned} |\zeta|^2 &= \sum_{\gamma=1}^m \sum_{i=1}^n (\zeta_\gamma^i)^2 = \frac{1}{4|u|^2} h^{\alpha\beta} D_\alpha |u|^2 D_\beta |u|^2 = \frac{\|D|u|^2\|_h^2}{4|u|^2} , \\ &\sum_{\gamma=1}^m g_{ij}(u)\xi_\gamma^i \xi_\gamma^j = e(u)(x) - \|(D|u|)\|_h(x) = e(u)(x) - e(\rho)(x) . \end{aligned} \tag{2.21}$$

From (2.18), (2.19) and (2.21), we can deduce that $|u|$ satisfies the differential inequality

(2.16).

Now, assume that u satisfies (1.4) and that M is simple. Then $(h^{\alpha\beta})$ satisfies

$$\lambda|X|^2 \le h^{\alpha\beta}(x)X_\alpha X_\beta \qquad \forall x,\ X\in \boldsymbol{R}^m$$

for some positive constant λ, and therefore (1.4) implies

$$\left\{\frac{\kappa}{\sinh(\kappa\rho)}\right\}^2 \{e(u)(x)-e(\rho)(x)\} \ge \frac{\varepsilon_0}{|x|^2\log|x|}$$

for some ε_0 and sufficiently large $|x|$. Combining (2.16) and the above inequality, we get the differential inequality (2.17). □

Now, we can prove Theorem 1.1 by comparing $|u|$ with a suitable supersolution of (2.17).

PROOF OF THEOREM 1.1. We first give supersolutions of the elliptic equation corresponding to (2.17) which tend to infinity on a bounded region.

Let M be as in Theorem 1.1 and $(x^1, \cdots, x^m)$ a normal coordinate system on M. Let $\rho_\delta(t)=\log(1+t^\delta)-\log(1-t^\delta)$, $B_s=\{x\in M : |x|<s\}$ and $\varphi_{\delta,\kappa,R}(x) : B_{R/\kappa}\to \boldsymbol{R}$ be a function defined by

$$\varphi_{\delta,\kappa,R}(x)=\frac{1}{\kappa}\rho_\delta\left(\frac{\kappa|x|}{R}\right).$$

Denote $r=|x|$ and $t=\kappa r/R$. Then

$$\begin{aligned}\Delta\varphi_{\delta,\kappa,R}(x)&=\frac{1}{\kappa}h^{\alpha\beta}D_\alpha D_\beta\varphi_{\delta,\kappa,R}-\frac{1}{\kappa}h^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}D_\gamma\varphi_{\delta,\kappa,R}\\ &=\frac{\kappa}{R^2}\rho''_\delta(t)+\frac{1}{Rr}\left\{\left(\mathrm{tr}(h^{\alpha\beta})-\sum_{\gamma=1}^m h^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}x^\gamma\right)-1\right\}\rho'_\delta.\end{aligned} \tag{2.22}$$

For a fixed $x\in M$, let $\{e_\alpha\}_{\alpha=1,\cdots,m}$, $e_\alpha=(e^1_\alpha, \cdots, e^m_\alpha)$ be an orthonormal basis of T_xM. Using (2.20), we can see that

$$\begin{aligned}\mathrm{tr}(h^{\alpha\beta})-\sum_{\gamma=1}^m h^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}x^\gamma&=\mathrm{tr}(h^{\alpha\beta})-h^{\alpha\beta}\sum_{\eta=1}^m\Gamma_{\alpha\gamma\beta}h^{\gamma\eta}x^\eta\\ &=\mathrm{tr}(h^{\alpha\beta})-\sum_{\eta=1}^m e^\alpha_\eta e^\beta_\eta\Gamma_{\alpha\gamma\beta}x^\gamma\\ &=\mathrm{tr}(h^{\alpha\beta})+\sum_{\eta=1}^m\{x^\gamma e^\alpha_\eta e^\beta_\eta\Gamma_{\alpha\beta\gamma}-(D_\alpha h_{\beta\gamma})x^\gamma e^\alpha_\eta e^\beta_\eta\}\\ &=\mathrm{tr}(h^{\alpha\beta})+\sum_{\eta=1}^m\{x^\gamma e^\alpha_\eta e^\beta_\eta\Gamma_{\alpha\beta\gamma}-D_\alpha(h_{\beta\gamma}x^\gamma)e^\alpha_\eta e^\beta_\eta+h_{\beta\gamma}(D_\alpha x^\gamma)e^\alpha_\eta e^\beta_\eta\}\end{aligned}$$

$$=\operatorname{tr}(h^{\alpha\beta})+\sum_{\eta=1}^{m}\{x^{\gamma}e_{\eta}^{\alpha}e_{\eta}^{\beta}\Gamma_{\alpha\beta\gamma}-\delta_{\alpha\beta}e_{\eta}^{\alpha}e_{\eta}^{\beta}+h_{\beta\gamma}\delta_{\alpha}^{\gamma}e_{\eta}^{\alpha}e_{\eta}^{\beta}\}$$

$$=\operatorname{tr}(h^{\alpha\beta})+\sum_{\eta=1}^{m}\{x^{\gamma}e_{\eta}^{\alpha}e_{\eta}^{\beta}\Gamma_{\alpha\beta\gamma}+h_{\alpha\beta}e_{\eta}^{\alpha}e_{\eta}^{\beta}\}-\delta_{\alpha\beta}h^{\alpha\beta}$$

$$=\sum_{\eta=1}^{m}\{h_{\alpha\beta}e_{\eta}^{\alpha}e_{\eta}^{\beta}+x^{\gamma}e_{\eta}^{\alpha}e_{\eta}^{\beta}\Gamma_{\alpha\beta\gamma}\}=\sum_{\eta=1}^{m}h_{\alpha\beta}(e_{\eta}^{\alpha}e_{\eta}^{\beta}+x^{\gamma}\Gamma_{\delta\gamma}^{\alpha}e_{\eta}^{\delta}e_{\eta}^{\beta})\,.$$

Thus, using Lemma 2.3, we get the following estimate

$$\operatorname{tr}(h^{\alpha\beta})-\sum_{\gamma=1}^{m}h^{\alpha\beta}\Gamma_{\alpha\beta}^{\gamma}x^{\gamma}\le c_2\sum_{\eta=1}^{m}h_{\alpha\beta}e_{\eta}^{\alpha}e_{\eta}^{\beta}=mc_2\,. \tag{2.23}$$

Combining (2.22) and (2.23), we obtain

$$\Delta\varphi_{\delta,\kappa,R}(x)\le\frac{\kappa}{R^2}\rho_{\delta}''(t)+\frac{c_2m-1}{Rr}\rho_{\delta}'(t)\,. \tag{2.24}$$

Now, a direct calculation leads us to the following inequality for $|x|>1$.

$$\begin{aligned}
&\Delta\varphi_{\delta,\kappa,R}(x)-\frac{\varepsilon}{2\kappa|x|^2\log|x|}\sinh(2\kappa\varphi_{\delta,\kappa,R}(x))\\
&\le\frac{2\kappa}{R^2(1-t^{2\delta})^2\log(Rt/\kappa)}\Bigg[t^{\delta-2}\left\{\delta(\delta+c_2m-2)\log\left(\frac{R}{\kappa}t\right)-\varepsilon\right\}\\
&\qquad\qquad+t^{3\delta-2}\left\{\delta(\delta-c_2m+2)\log\left(\frac{R}{\kappa}t\right)-\varepsilon\right\}\Bigg]\\
&\le\frac{2\kappa(t^{\delta-2}+t^{3\delta-2})}{R^2(1-t^{2\delta})^2\log(Rt/\kappa)}\left\{c_5\delta\log\left(\frac{R}{\kappa}\right)-\varepsilon\right\},
\end{aligned} \tag{2.25}$$

for $0<\delta\le1$ and some $c_5=c(c_2,m)>0$. Thus, for any $\varepsilon>0$, if we take $\delta=\delta(R)=\min\{\varepsilon/(c_5\log(R/\kappa)),1\}$, we obtain

$$\Delta\varphi_{\delta,\kappa,R}(x)-\frac{\varepsilon}{2\kappa|x|^2\log|x|}\sinh(2\kappa\varphi_{\delta,\kappa,R}(x))\le0\qquad\text{on}\quad B_{R/\kappa}\setminus B_1. \tag{2.26}$$

In the following let us assume that R is sufficiently large so that $\varepsilon/(c_5\log(R/\kappa))<1$. For any fixed $x\in\boldsymbol{R}^m\setminus B_1$,

$$\delta(R)\log t=\frac{\varepsilon\log(\kappa|x|/R)}{c_5\log(R/\kappa)}\searrow-\frac{\varepsilon}{c_5}\qquad\text{as}\quad R\nearrow+\infty\,,$$

and therefore

$$t^{\delta(R)}=e^{\delta(R)\log t}\searrow e^{-\varepsilon/c_5}\qquad\text{as}\quad R\nearrow+\infty\,.$$

Thus, putting

$$\xi := \frac{1}{\kappa} \log \frac{1+e^{-\varepsilon/c_5}}{1-e^{-\varepsilon/c_5}},$$

we can see that

$$\phi_{\kappa,R}(x) := \phi_{\delta(R),\kappa,R}(x) \searrow \xi \qquad \text{as} \quad R \nearrow +\infty. \tag{2.27}$$

Let $u(x)$ be the expression of a harmonic map $U: M \to N$ with respect to a normal coordinate system $x=(x^1, \cdots, x^m)$ on M centered at the pole of $p_0 \in M$ and a normal coordinate system $y=(y^1, \cdots, y^n)$ on N centered at arbitrary fixed point $q_0 \in N$. Take $R_0 > 1$ and put $\mu = \sup_{B_{R_0}} |u|$. Assume that U is not a constant map. Then $|u|$ can not remain bounded because of a Liouville-type theorem due to [14]. Thus, there exists a compact set $D \subset \boldsymbol{R}^m \setminus B_{R_0}$ on which $|u| \geq \mu + \xi + 2$.

Let R_1 be sufficiently large so that $R_1/\kappa > R_0$ and

$$\phi_{\kappa,R_1}(x) \leq \xi + 1 \qquad \text{on} \quad D.$$

This choice is the case because of (2.27). On the other hand it is easy to see that

$$\psi_{R_1}(x) := \phi_{\kappa,R_1}(x) + \mu$$

satisfies (2.26) on $B_{R_1/\kappa} \setminus B_1$. Thus we get

$$\Delta\psi_{R_1} - \frac{\varepsilon_0}{2\kappa |x|^2 \log|x|} \sinh(2\kappa\psi_{R_1}) \leq 0 \qquad \text{on} \quad B_{R_1/\kappa} \setminus B_1,$$

$$\Delta|u| - \frac{\varepsilon_0}{2\kappa |x|^2 \log|x|} \sinh(2\kappa|u|) \geq 0 \qquad \text{on} \quad \boldsymbol{R}^m \setminus B_{R_0},$$

and

$$\begin{cases} \psi_{R_1} \geq \mu + \xi > |u| & \text{on} \quad \partial B_{R_0}, \\ \lim_{|x| \to R_1/\kappa} \psi_{R_1} = +\infty. \end{cases}$$

Now, taking R_1 sufficiently large so that $B_{R_1/\kappa} \supset\supset B_{R_0}$ $(\supset\supset B_1)$ and that $B_{R_1/\kappa} \setminus B_{R_0} \supset D$, we can use a comparison theorem for elliptic equations (see for example [10] Theorem 10.1) and get

$$\psi_{R_1} \geq |u| \qquad \text{on} \quad B_{R_1/\kappa} \setminus B_{R_0},$$

especially

$$|u| \leq \psi_{R_1} \leq \xi + \mu + 1 \qquad \text{on} \quad D.$$

It contradicts our choice of D. Thus Theorem 1.1 is proved. □

PROOF OF COROLLARY 1.1. Assume that $M=\boldsymbol{R}^2$ and a harmonic map $U: \boldsymbol{R}^2 \to N$ satisfies (1.5). Then we have

$$\Delta|u|-\frac{\varepsilon_0}{2\kappa|x|^2(\log|x|)^2}\sinh(2\kappa|u|)\geq 0 \qquad \text{on} \quad M\setminus B_{R_0}(0) \tag{2.28}$$

instead of (2.17). Moreover, for this case $m=2$ and $c_2=1$ in (2.24). Thus we get

$$\begin{aligned}&\Delta\varphi_{\delta,\kappa,R}(x)-\frac{\varepsilon}{2\kappa|x|^2(\log|x|)^2}\sinh(2\kappa\varphi_{\delta,\kappa,R}(x))\\ &\leq\frac{2\kappa(t^{\delta-2}+t^{3\delta-2})}{R^2(1-t^{2\delta})^2(\log(Rt/\kappa))^2}\left[\delta^2\left\{\log\left(\frac{R}{\kappa}\right)\right\}^2-\varepsilon\right],\end{aligned} \tag{2.29}$$

instead of (2.25). Thus for any $\varepsilon>0$, if we take $\delta=\min\{\varepsilon/(\log(R/\kappa)), 1\}$, we obtain

$$\Delta\varphi_{\delta,\kappa,R}(x)-\frac{\varepsilon}{2\kappa|x|^2(\log|x|)^2}\sinh(2\kappa\varphi_{\delta,\kappa,R}(x))\leq 0 \qquad \text{on} \quad B_{R/\kappa}\setminus B_1 . \tag{2.30}$$

Now (2.28) and (2.30) enable us to proceed as in the proof of Theorem 1.1 and get the assertion of Corollary 1.1. □

3. Proof of Theorem 1.2.

LEMMA 3.1. *Let $U: M\to N$ be a smooth map with polynominal growth dilatation of order at most s and u an expression of U. Then*

$$h^{\alpha\gamma}h^{\beta\delta}D_\alpha u^i D_\beta u^j D_\gamma u^k D_\delta u^l(g_{ik}g_{jl}-g_{il}g_{jk})\geq\frac{8}{K^2L^2(1+r^2)^{s/2}}e(U)^2 , \tag{3.1}$$

where K is the same constant as in (1.6) *and $L=\min\{m, n\}$.*

PROOF. For any $x\in M$, we may choose orthonormal frames at x so that $(\varphi^*g)_{\alpha\beta}(x)=\lambda_\alpha(x)\delta_{\alpha\beta}$. We first note

$$e(U)(x)=\frac{1}{2}\sum_{1\leq\alpha\leq L}\lambda_\alpha(x) \quad \text{and} \quad \lambda_{L+1}(x)=\cdots=\lambda_m(x)=0 .$$

Since u has polynomial growth dilatation of order at most s, we then have at x

$$\begin{aligned}&h^{\alpha\gamma}h^{\beta\delta}D_\alpha u^i D_\beta u^j D_\gamma u^k D_\delta u^l(g_{ik}g_{jl}-g_{il}g_{jk})\\ &=2\sum_{1\leq\alpha<\beta\leq L}\lambda_\alpha(x)\lambda_\beta(x)\geq 2\lambda_1(x)\lambda_2(x)\\ &\geq\frac{2}{K^2(1+r(x)^2)^{s/2}}\lambda_1(x)^2\geq\frac{8}{K^2L^2(1+r(x)^2)^{s/2}}(e(U)(x))^2 .\end{aligned}$$

□

LEMMA 3.2. *Under the same assumption as in Theorem* 1.2,

$$\Delta e(U) \geq \frac{8\kappa^2}{K^2L^2(1+r^2)^{s/2}} e(U)^2 . \tag{3.2}$$

PROOF. If u is an expression of a harmonic map $U: M \to N$, then the following Weitzenböck formula (cf. [7]) holds

$$\begin{aligned} \Delta e(U) = \|\nabla dU\|^2 + h^{\alpha\gamma} h^{\beta\delta} {}^M R_{\alpha\beta} D_\gamma u^i D_\delta u^j g_{ij} \\ - h^{\alpha\gamma} h^{\beta\delta} {}^N R_{ijkl}(u) D_\alpha u^i D_\beta u^j D_\gamma u^k D_\delta u^l , \end{aligned} \tag{3.3}$$

where ∇ is the covariant derivative on the bundle $T^*M \otimes u^{-1}TN$, $\mathrm{Ric}_M = ({}^M R_{\alpha\beta})$ the Ricci tensor of M and $R_N = ({}^N R_{ijkl})$ the curvature tensor of N. From (3.1), (3.3) and the assumptions that $\mathrm{Ric}_M \geq 0$ and $K_N \leq -\kappa^2 < 0$, we then obtain

$$\begin{aligned} \Delta e(U) &\geq h^{\alpha\gamma} h^{\beta\delta} {}^M R_{\alpha\beta} D_\gamma u^i D_\delta u^j g_{ij} - h^{\alpha\gamma} h^{\beta\delta} {}^N R_{ijkl}(u) D_\alpha u^i D_\beta u^j D_\gamma u^k D_\delta u^l \\ &\geq \kappa^2 h^{\alpha\gamma} h^{\beta\delta} D_\alpha u^i D_\beta u^j D_\gamma u^k D_\delta u^l (g_{ik} g_{jl} - g_{il} g_{jk}) \\ &\geq \frac{8\kappa^2}{K^2L^2(1+r^2)^{s/2}} e(U)^2 . \end{aligned}$$

□

From (3.2) and the maximum principle, Theorem 1.2 is immediate provided M is compact.

We now assume that M is noncompact. We will modify the maximum principle argument as in [4]. Take the point $p_0 \in M$ as in (1.6) and for $a > 0$, let $B_a(p_0) = \{x \in M ; r(x) = \mathrm{dist}_M(p_0, x) < a\}$ be the geodesic ball of radius a and centered at p_0.

LEMMA 3.3. *Under the same assumptions as in Theorem* 1.2, *for any* $a > 0$, *we have*

$$e(U)(x) \leq \frac{(m+6)K^2L^2a^2(1+a^2)^{s/2}}{2\kappa^2(a^2 - r(x)^2)^2} \tag{3.4}$$

for all $x \in B_a(p_0)$.

PROOF. Assuming that $e(U)$ is not identically zero on $B_a(p_0)$, we consider the function

$$f(x) = (a^2 - r(x)^2)^2 e(U)(x) , \qquad x \in B_a(p_0) .$$

Since M is complete, the closure of $B_a(p_0)$ is compact and then f attains a nonzero maximum at some point $p \in B_a(p_0)$. As in §2 of [3], we may assume that p is not on the cut locus of p_0 and then f is C^2 in a neighborhood of p. Then we have

$$\nabla f(p) = 0 , \qquad \Delta f(p) \leq 0 .$$

Hence at p

$$\frac{\nabla e(U)}{e(U)}=\frac{4r\nabla r}{a^2-r^2},$$

$$\frac{\Delta e(U)}{e(U)}\leq\frac{\|\nabla e(U)\|^2}{e(U)^2}+\frac{8r^2}{(a^2-r^2)^2}+\frac{4(1+r\Delta r)}{a^2-r^2},$$

from which we obtain

$$\frac{\Delta e(U)(p)}{e(U)(p)}\leq\frac{24r(p)^2}{(a^2-r(p)^2)^2}+\frac{4(1+r(p)\Delta r(p))}{a^2-r(p)^2}. \tag{3.5}$$

On the other hand, according to Lemma 9 of [26], the following inequality holds

$$\Delta r(p)\leq\min_{0\leq k<r(p)}\left[\frac{m-1}{r(p)-k}-\frac{1}{(r(p)-k)^2}\int_k^{r(p)}(t-k)^2\operatorname{Ric}_M(\dot\sigma(t),\dot\sigma(t))dt\right], \tag{3.6}$$

where $\dot\sigma(t)$ is the tangent vector of the unique minimizing geodesic $\sigma:[0,r(p)]\to M$ from p_0 to p. Since the Ricci curvature of M is nonnegative, (3.6) implies the estimate

$$r(p)\Delta r(p)\leq m-1. \tag{3.7}$$

It follows from (3.5) and (3.7) that

$$(a^2-r(p)^2)^2\frac{\Delta e(U)(p)}{e(U)(p)}\leq 4(m+6)a^2.$$

Hence from (3.2) we obtain

$$f(p)=(a^2-r(p)^2)^2e(U)(p)\leq\frac{(m+6)K^2L^2a^2(1+r(p)^2)^{s/2}}{2\kappa^2}$$

$$\leq\frac{(m+6)K^2L^2a^2(1+a^2)^{s/2}}{2\kappa^2}.$$

Since p is the maximum point of f in $B_a(p_0)$, this implies

$$(a^2-r(x)^2)^2e(U)(x)\leq\frac{(m+6)K^2L^2a^2(1+a^2)^{s/2}}{2\kappa^2},$$

for all $x\in B_a(p_0)$ and then $e(U)$ satisfies the inequality (3.4). □

PROOF OF THEOREM 1.2. Take any point $x\in M$ and fix it. Letting $a\to\infty$ in the inequality (3.4), we then obtain

$$e(U)(x)\leq\frac{(m+6)K^2L^2}{2\kappa^2}\quad\text{when}\quad s=2$$

and $e(U)(x)=0$ when $s<2$, i.e. U is a constant map. This completes the proof of

Theorem 1.2. □

PROOF OF COROLLARY 1.2. By Theorem 1.2, $e(U)$ is bounded on M. It is well-known that a bounded subharmonic function on a 2-dimensional complete Riemannian manifold with nonnegative Ricci curvature is constant. From (3.2) we obtain that the energy density $e(U)$ of U is constant. Using this result also in (3.2) then gives that u is a constant map. □

A map $\varphi : M \to N$ is said to be *linear growth* if there exist points $p_0 \in M$ and $q_0 \in N$ such that

$$\limsup_{t\to\infty} \frac{\mu(\varphi, t)}{t} < \infty ,$$

where $\mu(\varphi, t) = \sup\{\mathrm{dist}_N(q_0, \varphi(x))\,;\, x \in M,\ \mathrm{dist}_M(p_0, x) = t\}$. It turns out that this definition is independent of the choice of $p_0 \in M$ and $q_0 \in N$. By the argument as in [2], we see that a harmonic map u, from a complete manifold with nonnegative Ricci curvature to an Hadamard manifold is linear growth if and only if its energy density $e(U)$ is bounded on M. From this result together with Theorem 1.2, we then obtain the following corollary.

COROLLARY 3.1. *Let M be a complete Riemannian manifold with nonnegative Ricci curvature and N an Hadamard manifold with negative sectional curvature, $K_N \le -\kappa^2 < 0$. Then every harmonic map $U : M \to N$ with polynomial growth dilatation of order at most 2 is linear growth.*

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Present Address:
DEPARTMENT OF MATHEMATICS, FACULTY OF LIBERAL ARTS, SHIZUOKA UNIVERSITY
OHYA, SHIZUOKA 422, JAPAN