Foliations on Manifolds with Positive Constant Curvature

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§1. Introduction.

Geometric notions in the theory of Riemannian submanifolds have their counterparts for foliations on Riemannian manifolds. A foliation all of whose leaves are minimal (resp. totally geodesic) submanifolds is called a harmonic (resp. totally geodesic) foliation and has been extensively studied in recent years. Many harmonic foliations which are not totally geodesic are known. However, under some geometric restrictions, harmonicity implies totally-geodesicness. For example, Oshikiri [13] proves that any harmonic foliation of codimension 1 on a compact orientable Riemannian manifold with non-negative Ricci curvature is totally geodesic. This result fails to hold in the case of higher codimensions (see Takagi and Yorozu [15]). But for harmonic foliations on a space-form, the following conjecture has been known:

CONJECTURE 1. Any harmonic foliation with minimal normal plane field on a compact n-dimensional space-form $M^n(c)$ of constant sectional curvature c $(c \geq 0)$ is totally geodesic.

As an analogue when the ambient space is a complex space-form, the following is also conjectured:

CONJECTURE 2. Any harmonic foliation with minimal normal plane field on the complex projective space $P_n(C)$ with its standard metric is totally geodesic.

It should be mentioned that Escobales [5] proves that any totally geodesic foliation with bundle-like metric on a rank one symmetric space consists of fibres of a Hopf fibration. Nakagawa and Takagi [11] claimed that Conjecture 1 was true and Gotoh [6] successively gave a proof of Conjecture 2 by using similar technique. However, Li [10] pointed out an error in their proof and these conjectures still remain unsolved.

In this paper, we shall obtain partial affirmative answers to these conjectures.

Specifically, we prove the following two theorems as partial answers to Conjecture 1.

THEOREM 1. *If $\mathscr{F}$ is a p-dimensional harmonic foliation on $M^n(c)$ $(c \geq 0)$ such that the Ricci curvature of each leaf of $\mathscr{F}$ is not less than $c(p-2)$ $(p \geq 4)$, then $\mathscr{F}$ is totally geodesic.*

THEOREM 2. *Let $\mathscr{F}$ be a p-dimensional harmonic foliation with minimal normal plane field on $M^n(c)$ $(c \geq 0)$. Suppose further that $\mathscr{F}$ satisfies the following conditions:*
1) *the normal connection is flat along $\mathscr{F}$;*
2) *the sectional curvature of each leaf is non-negative;*
3) *each leaf is simply connected.*
Then $\mathscr{F}$ is totally geodesic.

A general idea for the proof of these theorems is as follows. We first determine the possible types of each leaf by using Simons' type formulas concerning the second fundamental form of leaves. Next we examine whether such types can actually appear as leaves of a foliation.

We also prove the following theorem, which is a partial answer to Conjecture 2:

THEOREM 3. *Let $\mathscr{F}$ be a p-dimensional harmonic foliation on the complex projective space $P_n(C)$ of complex dimension n and of constant holomorphic sectional curvature 4. If the normal plane field $\mathscr{F}^{\perp}$ is minimal and the scalar curvature of each leaf of $\mathscr{F}$ is not less than $p(p+2)-(p+1)/(2-1/(2n-p))$, then $\mathscr{F}$ is totally geodesic.*

This theorem is proved by using the Hopf fibration $\phi : S^{2n+1}(1) \to P_n(C)$ and a result of Li [10] on harmonic foliations on the sphere which satisfy the conditions similar to those in Theorem 3. In relation to Conjecture 2, Gotoh [7] obtains an interesting result in the case that all the leaves are complex submanifolds.

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2. Preliminaries.

Let $\mathscr{F}$ be a foliation of codimension q on an n-dimensional Riemannian manifold (M, g). Consider $\mathscr{F}$ as an $(n-q)$-dimensional integrable distribution on M and denote the orthogonal distribution of $\mathscr{F}$ by $\mathscr{F}^{\perp}$. Then the tangent bundle TM of M is decomposed as the direct sum $\mathscr{F} \oplus \mathscr{F}^{\perp}$. We denote two natural projections associated to the decomposition $TM = \mathscr{F} \oplus \mathscr{F}^{\perp}$ as follows:

$$\pi : TM \to \mathscr{F}^{\perp}, \qquad \pi^{\perp} : TM \to \mathscr{F} .$$

Using these projections π and $\pi^{\perp}$, we define two tensor fields A and h of type $(1, 2)$ on M as follows:

$$A(X, Y) = -\pi^\perp(\nabla^M_{\pi(Y)}\pi(X)),$$

$$h(X, Y) = \pi(\nabla^M_{\pi^\perp(X)}\pi^\perp(Y)), \qquad X, Y \in \Gamma(TM),$$

(2.1)

where ∇^M denotes the Riemannian connection with respect to g, and $\Gamma(TM)$ the space of sections of the tangent bundle TM, that is, the space of vector fields on M. Note that the restriction of h to each leaf of $\mathscr{F}$ is identified with the second fundamental form of the leaf.

We will use, throughout this paper, the following convention on the range of indices unless otherwise stated:

$$A, B, C, \cdots = 1, \cdots, n,$$

$$i, j, k, \cdots = 1, \cdots, p,$$

$$\alpha, \beta, \gamma, \cdots = p+1, \cdots, n,$$

$$a, b, c, \cdots = n+1, \cdots, n+s,$$

(2.2)

where $p = n - q$ is the dimension of $\mathscr{F}$ (indices $a, b, c, \cdots$ will be used in §4 for a Riemannian submersion $\phi: (\tilde{M}, \tilde{g}) \to (M, g)$, where we put $n = \dim M$, $p = \dim \mathscr{F}$ and $n + s = \dim \tilde{M}$).

Let $\{E_1, E_2, \cdots, E_n\}$ be a local orthonormal frame field on M such that $E_1, \cdots, E_p$ are tangent to $\mathscr{F}$. The components h^A_{BC} (resp. A^B_{CD}) of h (resp. A) with respect to $\{E_A\}$ are given by

$$h^\alpha_{ij} = g(h(E_i, E_j), E_\alpha) \qquad (\text{resp. } A^i_{\alpha\beta} = g(A(E_\alpha, E_\beta), E_i)).$$

(2.3)

It is an easy matter to see that all other components vanish.

The tension vector field (or the mean curvature vector field) τ of $\mathscr{F}$ on M is defined by

$$\tau = \sum_i h(E_i, E_i).$$

(2.4)

The restriction of τ to each leaf of $\mathscr{F}$ coincides with the usual mean curvature vector field of the leaf except for a factor $1/p$. The foliation $\mathscr{F}$ is said to be *harmonic* (resp. *totally geodesic*) if $\tau \equiv 0$ (resp. $h \equiv 0$). Moreover, following Reinhart [14], we say the normal plane field $\mathscr{F}^\perp$ is *minimal* if $\tau^\perp \equiv 0$, where

$$\tau^\perp = \sum_\alpha A(E_\alpha, E_\alpha).$$

(2.5)

$\tau^\perp$ is the mean curvature vector field of $\mathscr{F}^\perp$. The Riemannian metric g is said to be *bundle-like* if

$$A(X, Y) + A(Y, X) = 0 \qquad \text{for } X, Y \in \Gamma(\mathscr{F}^\perp).$$

(2.6)

In particular, if g is bundle-like, then the normal plane filed $\mathscr{F}^\perp$ is minimal.

We define the normal connection $\nabla^\perp$ of $\mathscr{F}^\perp$ as follows:

$$\nabla^\perp_X Z = \pi(\nabla^M_X Z), \qquad X \in \Gamma(TM), \ Z \in \Gamma(\mathscr{F}^\perp). \tag{2.7}$$

A vector field $Z \in \Gamma(\mathscr{F}^\perp)$ is called parallel along $\mathscr{F}$ with respect to $\nabla^\perp$ if

$$\nabla^\perp_X Z \equiv 0 \qquad \text{for all } X \in \Gamma(\mathscr{F}).$$

The curvature tensor field R^M of ∇^M is defined by

$$R^M(X, Y)Z = \nabla^M_X \nabla^M_Y Z - \nabla^M_Y \nabla^M_X Z - \nabla^M_{[X,Y]} Z \qquad \text{for } X, Y, Z \in \Gamma(TM)$$

and the curvature tensor field $R^\perp$ of $\nabla^\perp$ is defined similarly by

$$R^\perp(X, Y)Z = \nabla^\perp_X \nabla^\perp_Y Z - \nabla^\perp_Y \nabla^\perp_X Z - \nabla^\perp_{[X,Y]} Z \qquad \text{for } X, Y \in \Gamma(TM), \ Z \in \Gamma(\mathscr{F}^\perp).$$

Moreover, we define for $X, Y, Z \in \Gamma(\mathscr{F})$,

$$R^{\mathscr{F}}(X, Y)Z = \pi^\perp(\nabla^M_X \pi^\perp(\nabla^M_Y Z) - \nabla^M_Y \pi^\perp(\nabla^M_X Z) - \nabla^M_{[X,Y]} Z),$$

$$Ric^{\mathscr{F}}(X, Y) = \sum_i g(R^{\mathscr{F}}(E_i, X)Y, E_i),$$

$$r^{\mathscr{F}} = \sum_i Ric^{\mathscr{F}}(E_i, E_i), \tag{2.8}$$

$$K^{\mathscr{F}}(X, Y) = g(R^{\mathscr{F}}(X, Y)Y, X)/\{g(X, X)g(Y, Y) - g(X, Y)^2\}.$$

Note that the restrictions of $R^{\mathscr{F}}$, $Ric^{\mathscr{F}}$, $r^{\mathscr{F}}$, $K^{\mathscr{F}}$ and $R^\perp$ to each leaf of $\mathscr{F}$ are identical with the curvature tensor field, the Ricci curvature tensor field, the scalar curvature, the sectional curvature and the normal curvature of the leaf, respectively. We also define the covariant derivative $\nabla^M T$ of a tensor field T of type $(1, r)$ on M by

$$(\nabla^M T)(X; Y_1, \cdots, Y_r) = (\nabla^M_X T)(Y_1, \cdots, Y_r)$$

$$= \nabla^M_X T(Y_1, \cdots, Y_r) - \sum_{a=1}^{r} T(Y_1, \cdots, \nabla^M_X Y_a, \cdots, Y_r), \tag{2.9}$$

where $X, Y_1, \cdots, Y_r \in \Gamma(TM)$. Then we have the following equations:

$$R^l_{ijk} = (R^{\mathscr{F}})^l_{ijk} - \sum_\alpha h^\alpha_{ik} h^\alpha_{jl} + \sum_\alpha h^\alpha_{ij} h^\alpha_{kl}, \tag{2.10}$$

$$\sum_{i,j} R^i_{jij} = r^{\mathscr{F}} - \|\tau\|^2 + \|h\|^2, \tag{2.11}$$

$$(R^\perp)^\alpha_{\beta ij} = \sum_k (h^\alpha_{ik} h^\beta_{kj} - h^\beta_{ik} h^\alpha_{kj}), \tag{2.12}$$

$$h^\alpha_{ijk} - h^\alpha_{ikj} = R^\alpha_{ikj}, \tag{2.13}$$

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_\beta h^\beta_{ij} R^\alpha_{\beta lk} + \sum_m h^\alpha_{mj} R^m_{ikl} + \sum_m h^\alpha_{im} R^m_{jkl}, \tag{2.14}$$

$$h^\alpha_{ijkl} = h^\alpha_{jikl} ,$$

(2.15)

$$h^\alpha_{ijkl} - h^\alpha_{ikjl} = R^\alpha_{ikj,l} - \sum_m h^\alpha_{lm} R^m_{ikj} - \sum_\beta h^\beta_{lk} R^\alpha_{ij\beta} - \sum_\beta h^\beta_{li} R^\alpha_{\beta jk} - \sum_\beta h^\beta_{lj} R^\alpha_{i\beta k}$$

$$- \sum_{m,\beta} h^\alpha_{lm}(h^\beta_{lj}h^\beta_{km} - h^\beta_{lk}h^\beta_{jm}) - \sum_{m,\beta} h^\beta_{li}(h^\alpha_{mj}h^\beta_{km} - h^\alpha_{mk}h^\beta_{jm})$$

(2.16)

$$- \sum_{m,\beta} h^\alpha_{lm}(h^\beta_{lj}h^\beta_{km} - h^\beta_{lk}h^\beta_{jm}) - \sum_\beta h^\beta_{lk}h^\alpha_{ij\beta} + \sum_\beta h^\beta_{lj}h^\alpha_{ik\beta} ,$$

where the components $R^A_{BCD} = (R^M)^A_{BCD}$ of R^M, $R^A_{BCD,F}$ of $\nabla^M R^M$, $(R^{\mathscr{F}})^i_{jkl}$ of $R^{\mathscr{F}}$, $(R^\perp)^\alpha_{\beta AB}$ of $R^\perp$, h^A_{BCD} of $\nabla^M h$ and h^A_{BCDF} of $\nabla^M \nabla^M h$ are given respectively by

$$R^A_{BCD} = g(R^M(E_C, E_D)E_B, E_A) ,$$

$$R^A_{BCD,F} = g((\nabla^M_{E_F} R^M)(E_C, E_D)E_B, E_A) ,$$

$$(R^{\mathscr{F}})^i_{jkl} = g(R^{\mathscr{F}}(E_k, E_l)E_j, E_i) ,$$

(2.17)

$$(R^\perp)^\alpha_{\beta AB} = g(R^\perp(E_A, E_B)E_\beta, E_\alpha) ,$$

$$h^A_{BCD} = g((\nabla^M_{E_D} h)(E_B, E_C), E_A) ,$$

$$h^A_{BCDF} = g((\nabla^M_{E_F} \nabla^M h)(E_D; E_B, E_C), E_A) .$$

3. Divergence formula.

Let (M, g) be a locally symmetric Riemannian manifold (i.e., $\nabla^M R^M = 0$). Let $\mathscr{F}$ be a foliation on M, and assume that τ is parallel along $\mathscr{F}$ with respect to $\nabla^\perp$ (i.e., $\nabla^\perp_X \tau = 0$ for $X \in \Gamma(\mathscr{F})$). We denote by $\|h\|^2$ the squared norm of the tensor field h, that is,

$$\|h\|^2 = \sum_{i,j,\alpha} (h^\alpha_{ij})^2 ,$$

(3.18)

which is a function globally defined on M. We define a vector field V on M by

$$V = \frac{1}{2} \pi^\perp(\mathrm{grad}_M \|h\|^2) ,$$

where $\mathrm{grad}_M \|h\|^2$ denotes the gradient vector field of $\|h\|^2$. This vector field V was first used by Nakagawa and Takagi [11].

By using (2.14), (2.15) and (2.16), we obtain the following formula for the divergence $\mathrm{div}_M V$ of V:

$$\operatorname{div}_M V = g(\tau^\perp, V) + \|\nabla h\|^2 - \sum_{\alpha,\beta} (\operatorname{tr} H^\alpha H^\beta)^2 + \sum_{\alpha,\beta} \operatorname{tr}[H^\alpha, H^\beta]^2$$

$$+ \sum_{\alpha,\beta} \operatorname{tr} H^\beta \left(\sum_{i,j} h_{ij}^\alpha R_{i\beta j}^\alpha + \operatorname{tr}(H^\alpha)^2 H^\beta \right) \tag{3.19}$$

$$+ 2 \sum_{i,j,k,l,\alpha} h_{ij}^\alpha h_{kl}^\alpha R_{ijk}^l + 2 \sum_{i,j,k,l,\alpha} h_{ij}^\alpha h_{il}^\alpha R_{kjk}^l$$

$$+ 4 \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{ik}^\beta R_{\beta kj}^\alpha - \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{ij}^\beta R_{k\beta k}^\alpha ,$$

where H^α denotes the $p \times p$-matrix (h_{ij}^α), and $\|\nabla h\|^2 = \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2$. It should be noted that this formula is also obtained in Li [10] when $\mathscr{F}$ is harmonic.

We now let M be an n-dimensional space-form with constant sectional curvature c (≥ 0). Then, since the components of R^M are given by

$$R_{BDE}^A = c(\delta_{AD}\delta_{BE} - \delta_{AE}\delta_{BD}) , \tag{3.20}$$

it follows from (3.19) that

$$\operatorname{div}_M V = g(\tau^\perp, V) + \|\nabla h\|^2 - \sum_{\alpha,\beta} (\operatorname{tr} H^\alpha H^\beta)^2$$

$$\tag{3.21}$$

$$+ \sum_{\alpha,\beta} \operatorname{tr}[H^\alpha, H^\beta]^2 + \sum_{\alpha,\beta} \operatorname{tr} H^\beta \operatorname{tr}(H^\alpha)^2 H^\beta + cp\|h\|^2 - c\|\tau\|^2 .$$

By making use of (2.11) and (3.20), we can further write (3.21) as follows:

$$\operatorname{div}_M V = g(\tau^\perp, V) + \|\nabla h\|^2 + \sum_{i,j,k,l,\alpha} (h_{ij}^\alpha h_{kl}^\alpha (R^{\mathscr{F}})_{ijk}^l + h_{ij}^\alpha h_{il}^\alpha (R^{\mathscr{F}})_{kjk}^l)$$

$$\tag{3.22}$$

$$+ \frac{1}{2} \sum_{\alpha,\beta} \operatorname{tr}[H^\alpha, H^\beta]^2 .$$

Finally we note that the divergence formula (3.21) restricted to each leaf L of $\mathscr{F}$ gives the following formula (see Chern, doCarmo and Kobayashi [2]):

$$\frac{1}{2} \Delta_L \|h\|^2 = \|\nabla h\|^2 - \sum_{\alpha,\beta} (\operatorname{tr} H^\alpha H^\beta)^2 + \sum_{\alpha,\beta} \operatorname{tr}[H^\alpha, H^\beta]^2$$

$$+ \sum_{\alpha,\beta} \operatorname{tr} H^\beta \operatorname{tr}(H^\alpha)^2 H^\beta + cp\|h\|^2 - c\|\tau\|^2 \tag{3.23}$$

$$= \|\nabla h\|^2 + \sum_{i,j,k,l,\alpha} (h_{ij}^\alpha h_{kl}^\alpha (R^L)_{ijk}^l + h_{ij}^\alpha h_{il}^\alpha (R^L)_{kjk}^l) - \frac{1}{2} \|R^\perp\|^2 ,$$

where Δ_L denotes the Laplacian on L (i.e., $\Delta_L f = \operatorname{div}_L \operatorname{grad}_L f$ for $f \in \Gamma(L \times R)$), and $\|R^\perp\|^2 = -\sum_{\alpha,\beta} \operatorname{tr}[H^\alpha, H^\beta]^2$.

4. Riemannian submersion.

Let $(\tilde{M}, \tilde{g})$ and (M, g) be Riemannian manifolds and $\phi: (\tilde{M}, \tilde{g}) \to (M, g)$ be a Riemannian submersion of $(\tilde{M}, \tilde{g})$ to (M, g). Recall that Riemannian coverings and the Hopf fibrations are typical examples of Riemannian submersions.

We denote two projections $\mathscr{V}$ and $\mathscr{H}$ naturally induced by the decomposition of the tangent bundle $T\tilde{M} = \mathscr{V} \oplus \mathscr{H}$ as follows:

$$\mathscr{V}: T\tilde{M} \to \mathscr{V}, \qquad \mathscr{H}: T\tilde{M} \to \mathscr{H},$$

where $\mathscr{V}$, which is called "vertical" (resp. $\mathscr{H}$, which is called "horizontal"), denotes the subbundle of $T\tilde{M}$ which is tangent to fibres of ϕ (resp. the orthogonal complement of $\mathscr{V}$). We denote the Riemannian connection of $(\tilde{M}, \tilde{g})$ (resp. (M, g)) by $\tilde{\nabla}$ (resp. ∇). If X is a vector field on M, we denote its horizontal lift by $\tilde{X}$.

Let $\mathscr{F}$ be a foliation of codimention q on (M, g). Then there is associated a foliation $\tilde{\mathscr{F}}$ of codimension q (which is called the "pull-back" foliation) on $(\tilde{M}, \tilde{g})$, which is defined as follows:

$$\tilde{\mathscr{F}} = \phi^{-1}(\mathscr{F}),$$

considering $\mathscr{F}$ as a system of submanifolds.

From now on, let $\phi: (\tilde{M}, \tilde{g}) \to (M, g)$ be a Riemannian submersion satisfying the condition that all the fibres are totally geodesic (i.e., $\mathscr{H}(\tilde{\nabla}_{\tilde{V}}\tilde{W}) = 0$ for $\tilde{V}, \tilde{W} \in \mathscr{V}$).

Let $\{E_1, \cdots, E_p; E_{p+1}, \cdots, E_n\}$ be a local orthonormal frame field on M such that $E_1, \cdots, E_p$ are tangent to $\mathscr{F}$. Also, let $\{\tilde{E}_1, \cdots, \tilde{E}_n; \tilde{V}_{n+1}, \cdots, \tilde{V}_{n+s}\}$ be a local orthonormal frame field on $\tilde{M}$ such that

$$\left.\begin{array}{r}\tilde{E}_{p+1}, \cdots, \tilde{E}_n \\ \tilde{E}_1, \cdots, \tilde{E}_p \end{array}\right\} \text{ are the horizontal lifts of } E_1, \cdots, E_n, \text{ and}$$
$$\left.\begin{array}{r}\phantom{\tilde{E}_1, \cdots, \tilde{E}_p} \\ \tilde{V}_{n+1}, \cdots, \tilde{V}_{n+s}\end{array}\right\} \text{ are tangent to } \tilde{\mathscr{F}}.$$

Then the following formulas are given in O'Neil [12]:

LEMMA 4.1.

$$\tilde{g}(\tilde{\nabla}_{\tilde{X}}\tilde{Y}, \tilde{V}) + \tilde{g}(\tilde{\nabla}_{\tilde{Y}}\tilde{X}, \tilde{V}) = 0 \qquad \text{for } \tilde{X}, \tilde{Y} \in \mathscr{H} \text{ and } \tilde{V} \in \mathscr{V}. \tag{4.24}$$

In particular, $\mathscr{V}(\tilde{\nabla}_{\tilde{X}}\tilde{X}) = 0$ *for* $\tilde{X} \in \mathscr{H}$.

LEMMA 4.2. *For* $\tilde{X} \in \mathscr{H}$ *and* $\tilde{V} \in \mathscr{V}$,

$$\tilde{K}(\tilde{X}, \tilde{V}) = \sum_A \tilde{g}(\tilde{\nabla}_{\tilde{X}}\tilde{V}, \tilde{E}_A)^2 = \sum_A \tilde{g}(\tilde{\nabla}_{\tilde{X}}\tilde{E}_A, \tilde{V})^2, \tag{4.25}$$

where $\tilde{K}(\tilde{X}, \tilde{V})$ *denotes the sectional curvature of* $\tilde{M}$ *for the plane spanned by* $\tilde{X}$ *and* $\tilde{V}$.

For $\tilde{\mathscr{F}}$ and $\mathscr{F}$, when Ω is a quantity defined for $\mathscr{F}$, we denote by $\tilde{\Omega}$ the correspond-

ing quantity defined for $\tilde{\mathscr{F}}$. Then the following results hold (see Lawson [9]):

PROPOSITION 4.1.

$$\|\tilde{\tau}\|^2 = \|\tau\|^2 \circ \phi .$$

(4.26)

In particular, $\tilde{\mathscr{F}}$ is harmonic if and only if $\mathscr{F}$ is harmonic.

PROPOSITON 4.2.

$$\|\tilde{h}\|^2 \geq \|h\|^2 \circ \phi .$$

(4.27)

Hence, if $\tilde{\mathscr{F}}$ is totally geodesic, $\mathscr{F}$ is totally geodesic.

LEMMA 4.3.

$$r^{\mathscr{F}} \circ \phi = \tilde{r}^{\mathscr{F}} + \sum_{a,i,j} \tilde{g}(\tilde{\nabla}_{\tilde{E}_i} \tilde{E}_j, \tilde{V}_a)^2 - \sum_{a,b} \tilde{K}(\tilde{V}_a, \tilde{V}_b) .$$

(4.28)

For the normal plane fields of $\mathscr{F}$ and $\tilde{\mathscr{F}}$, the following proposition holds.

PROPOSITION 4.3. $\tilde{\mathscr{F}}^\perp$ is minimal if and only if $\mathscr{F}^\perp$ is minimal.

PROOF.   Denote the mean curvature of $\tilde{\mathscr{F}}^\perp$ (resp. $\mathscr{F}^\perp$) by $\tilde{\tau}^\perp$ (resp. $\tau^\perp$). Then we have

$$\|\tilde{\tau}^\perp\|^2 = \sum_a \left(\sum_\beta \tilde{g}(\tilde{\nabla}_{\tilde{E}_\beta} \tilde{E}_\beta, \tilde{V}_a) \right)^2 + \sum_i \left(\sum_\alpha \tilde{g}(\tilde{\nabla}_{\tilde{E}_\alpha} \tilde{E}_\alpha, \tilde{E}_i) \right)^2$$

$$= \sum_i \left(\sum_\alpha g(\nabla_{E_\alpha} E_\alpha, E_i) \right)^2 \quad \text{(by (4.24))}$$

$$= \|\tau^\perp\|^2 \circ \phi .$$

This proves the proposition.//

5.   Proof of main results.

In this section, we shall prove Theorem 1 and Theorem 2. If $c = 0$, it is easy to see that the theorems hold. So we suppose, in the following, that $c > 0$. We may also assume that $M^n(c) = S^n(c)$, the n-dimensional sphere with constant sectional curvature c, since if a foliation on $M^n(c)$ satisfies the assumptions of each theorem then its pull-back to $S^n(c)$ also satisfies the same assumptions (by Propositions 4.1 and 4.3) and totally-geodesicness of the pull-back foliation on $S^n(c)$ implies that of the foliation on $M^n(c)$ (by Proposition 4.2).

PROOF OF THEOREM 1.   Let $\mathscr{F}$ be a p-dimensional harmonic foliation on $S^n(c)$. We regard each leaf L of $\mathscr{F}$ as a p-dimensional submanifold of $S^n(c)$. Under the assumptions of Theorem 1, L is compact since L is complete and the Ricci curvature of $L \geq c(p-2) > 0$. According to Ejiri [3], L is either $S^p(c)$, $S^m(2c) \times S^m(2c)$ or $P_2(C)$, the 2-dimensional

complex projective space.

In general, for a foliation $\mathscr{F}$ of codimension q on a manifold M, we denote the leaf space (i.e., the space of all leaves) of $\mathscr{F}$ by $M/\mathscr{F}$. If all leaves of $\mathscr{F}$ are compact and simply connected, then by Reeb's stability theorem, the leaf space $M/\mathscr{F}$ has a structure of a q-dimensional manifold, and the natural projection $M \to M/\mathscr{F}$ is a fibration.

According to Browder [1], the fibres of a fibration from spheres to a connected manifold (but not a point) are homotopic either to S^1, S^3 or S^7. Thus, Theorem 1 is proved.//

Theorem 2 is an immediate consequence of the following theorem:

THEOREM 5.1.   *Let $\mathscr{F}$ be a p-dimensional foliation on $S^n(c)$ which satisfies the followng conditions*:
1)   *the tension field τ is parallel along $\mathscr{F}$ (i.e., $\nabla^{\perp}_X \tau = 0$, $X \in \Gamma(\mathscr{F})$)*;
2)   *the normal plane field $\mathscr{F}^{\perp}$ is minimal (i.e., $\tau^{\perp} = 0$)*;
3)   *the normal connection $\nabla^{\perp}$ is flat along $\mathscr{F}$ (i.e., $R^{\perp}(X, Y) = 0$, $X, Y \in \Gamma(\mathscr{F})$)*;
4)   *the sectional curvature of each leaf is non-negative*;
5)   *each leaf is simply connected.*
Then $\mathscr{F}$ is umbilic (i.e., all leaves of $\mathscr{F}$ are umbilic submanifolds).

PROOF.   By the condition 1), the divergence formula (3.23) is valid. By the conditions 2) and 3), it follows that $-\mathrm{tr}[H^{\alpha}, H^{\beta}]^2 = \sum_{i,j} g(R^{\perp}(E_i, E_j)E_{\alpha}, E_{\beta})^2 = 0$ and $\tau^{\perp} = 0$. Therefore the formula (3.23) now reads as follows:

$$\mathrm{div}_{S^n(c)} V = \|\nabla h\|^2 + \sum_{i,j,k,l,\alpha} (h^{\alpha}_{ij} h^{\alpha}_{kl}(R^{\mathscr{F}})^l_{ijk} + h^{\alpha}_{ij} h^{\alpha}_{il}(R^{\mathscr{F}})^l_{kjk}).$$

Since $R^{\perp} = 0$ along $\mathscr{F}$, $[H^{\alpha}, H^{\beta}] = 0$ ($\alpha, \beta = p+1, \cdots, n$) and so H^{α} ($\alpha = p+1, \cdots, n$) are simultaneously diagonalizable. We choose a local orthonormal frame field $\{E_1, \cdots, E_p; E_{p+1}, \cdots, E_n\}$ with respect to which each H^{α} is diagonal. For each α, we denote the eigenvalues of H^{α} by λ^{α}_i ($i = 1, \cdots, p$), that is,

$$H^{\alpha} = \begin{pmatrix} \lambda^{\alpha}_1 & & & 0 \\ & \lambda^{\alpha}_2 & & \\ & & \ddots & \\ 0 & & & \lambda^{\alpha}_p \end{pmatrix}.$$

Then it holds that (see Erbacher [4], Yano and Ishihara [16])

$$\mathrm{div}_{S^n(c)} V = \|\nabla h\|^2 + \frac{1}{2}\sum_{i,k}(\lambda^{\alpha}_i - \lambda^{\alpha}_k)^2 (R^{\mathscr{F}})^k_{iki}. \tag{5.29}$$

Moreover, by the condition 4), $(R^{\mathscr{F}})^k_{iki} \geq 0$ ($i, j = 1, \cdots, p$). Therefore

$$\mathrm{div}_{S^n(c)} V \geq \|\nabla h\|^2 \geq 0. \tag{5.30}$$

By integrating this, we get $\|\nabla h\|^2 = 0$ on $S^n(c)$ and hence $\|h\|^2$ is constant on each leaf.

Since each leaf L of $\mathcal{F}$ is complete and satisfies the conditions 1), 3), 4), we can use a result of Erbacher [4], Yano and Ishihara [16] to conclude that possible types of each leaf are following:

$$L = S^{p_1}(c_1) \times S^{p_2}(c_2) \times \cdots \times S^{p_l}(c_l),$$

where c_i ($i = 1, \cdots, l$) satisfy $1/c_1 + 1/c_2 + \cdots + 1/c_l \leq 1/c$. Note that each leaf of $\mathcal{F}$ is compact. Since L is simply connected, the same argument as in the proof of Theorem 1 (Reeb's stability and Browder's result) shows that each leaf of $\mathcal{F}$ is homotopic to either S^1, S^3 or S^7. Hence $\mathcal{F}$ is umbilic.//

6. Some other results and remarks.

First, we recall a result due to Li on harmonic foliations on the sphere $S^n(c)$ (see Li [10]).

THEOREM 6.1.   *Let $\mathcal{F}$ be a p-dimensional harmonic foliation on $S^n(c)$. If the normal plane field $\mathcal{F}^{\perp}$ is minimal and the squared norm $\|h\|^2$ of the tensor field h is not more than $cp/(2 - 1/(n-p))$, then $\mathcal{F}$ is either totally geodesic, or $\|h\|^2 = cp/(2 - 1/(n-p))$ and $p = n - p = 2$. Furthermore, if the standard metric g is bundle-like, $\mathcal{F}$ is totally geodesic.*

Theorem 6.1 may be refined in the following fashion:

COROLLARY 6.1.   *Let $\mathcal{F}$ be a p-dimensional harmonic foliation on $M^n(c)$ ($c \geq 0$). If the normal plane field $\mathcal{F}^{\perp}$ is minimal and the squared norm $\|h\|^2$ of the tensor field h is not more than $cp/(2 - 1/(n-p))$, then $\mathcal{F}$ is totally geodesic.*

PROOF.   By the same reason as in the beginning of §5, we may assume that $M^n(c) = S^n(c)$. Suppose that $\|h\|^2 = cp/(2 - 1/(n-p))$ and $p = n - p = 2$. Then the scalar curvature r^L of L is given by:

$$r^L = cp(p-1) - \|h\|^2 = \frac{2}{3}c.$$

Since $p = 2$, the sectional curvature K^L of L is given by $K^L = \frac{1}{2}r^L = \frac{1}{3}c > 0$. Thus L is compact.

According to Chern, doCarmo and Kobayashi [2], L is isometric to the 2-dimensional real projective space $P_2(R)$. On the other hand, since $\mathcal{F}$ is orientable as a vector bundle (by simply connectedness of $S^n(c)$), the tangent bundle TL of L is orientable, that is, L is orientable. This contradiction shows that the case $\|h\|^2 = cp/(2-1/(n-p))$ and $p = n - p = 2$ can not happen. Thus we conclude the proof of the corollary.//

By making use of the Hopf fibration $\phi: S^{2n+1}(1) \to P_n(C)$, we can also obtain a

partial answer to Conjecture 2 (see Gotoh [6]). That is, the following holds:

THEOREM 3. *Let $\mathcal{F}$ be a p-dimensional harmonic foliation on the complex projective space $P_n(C)$ of complex dimension n and of constant holomorphic sectional curvature 4. If the normal plane field $\mathcal{F}^{\perp}$ is minimal and the scalar curvature of each leaf of $\mathcal{F}$ is not less than $p(p+2)-(p+1)/(2-1/(2n-p))$, then $\mathcal{F}$ is totally geodesic.*

PROOF. Let $\mathcal{F}$ be a p-dimensional harmonic foliation on $P_n(C)$ with minimal normal plane field. Recall that the Hopf fibration $\phi: S^{2n+1}(1) \to P_n(C)$ is a Riemannian submersion. Let $\tilde{\mathcal{F}}$ be the pull-back foliation of $\mathcal{F}$ by ϕ. Then by Propositions 4.1, 4.3, $\tilde{\mathcal{F}}$ is a $(p+1)$-dimensional harmonic foliation on $S^{2n+1}(1)$ whose normal plane field $\tilde{\mathcal{F}}^{\perp}$ is minimal. Using the same notations as in §4, we have the following:

$$\|\tilde{h}\|^2 = \sum_{i,j} \tilde{K}(\tilde{E}_i, \tilde{E}_j) + 2\sum_i \tilde{K}(\tilde{E}_i, \tilde{V}) - \tilde{r}^{\mathcal{F}} \qquad \text{(by (2.11))}$$

$$= p(p-1) + 2p - r^{\mathcal{F}} + \sum_{i,j} \tilde{g}(\tilde{\nabla}_{\tilde{E}_i}\tilde{E}_j, \tilde{V})^2 \qquad \text{(by (4.28))}$$

$$\leq p(p+1) - r^{\mathcal{F}} + \sum_{i,A} \tilde{g}(\tilde{\nabla}_{\tilde{E}_i}\tilde{E}_A, \tilde{V})^2$$

$$= p(p+1) - r^{\mathcal{F}} + \sum_i \tilde{K}(\tilde{E}_i, \tilde{V}) \qquad \text{(by (4.25))}$$

$$= p(p+2) - r^{\mathcal{F}}.$$

Therefore, if $p(p+2) - r^{\mathcal{F}} \leq (p+1)/(2-1/(2n-p))$, $\tilde{\mathcal{F}}$ is totally geodesic, and so is $\mathcal{F}$ (by Proposition 4.2).//

REMARK. If Conjecture 1 is true, Conjecture 2 is also true. In fact, if $\mathcal{F}$ is a harmonic foliation on $P_n(C)$ with minimal normal plane field, the pull-back foliation $\tilde{\mathcal{F}}$ on $S^{2n+1}(1)$ by the Hopf fibration is also a harmonic foliation, with minimal normal plane field. Therefore, supposing that Conjecture 1 is true, $\tilde{\mathcal{F}}$ is totally geodesic, and hence $\mathcal{F}$ is also totally geodesic.

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