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Index and Flat Ends of Minimal Surfaces

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§1. Introduction.

The purpose of this paper is to study the index of a complete orientable minimal surface in $\boldsymbol{R}^3$ with finite total curvature.

In 1964, Osserman [O] proved that a complete minimal surface in $\boldsymbol{R}^3$ with finite total curvature is conformally a Riemann surface with finitely many punctures. Fisher-Colbrie [F] showed that the statement remains true even if the condition of 'finite total curvature' is replaced by that of 'finite Morse index'. This theorem of Fisher-Colbrie gave us a new direction of research on those surfaces, that is, the study of the Morse index of a complete minimal surface.

Let us briefly review the history of research on the indices of complete immersed orientable minimal surfaces in $\boldsymbol{R}^3$. It has been classically known that the plane is the only such surface with index zero, i.e. stable. Fisher-Colbrie showed in [F] that the catenoid and Enneper's surface have index one. Conversely, it was shown in [L-R] that a complete immersed orientable minimal surface in $\boldsymbol{R}^3$ with index one is either the catenoid or Enneper's surface. This result had been proved by [C-T] when all the ends of the surface are embedded. In 1990, Choe [C] and Nayatani [N] independently proved, under the assumption that its genus is zero, that a complete immersed orientable minimal surface in $\boldsymbol{R}^3$ has index less than three if and only if it is one of these three; the plane, the catenoid, Enneper's surface.

As for other examples of minimal surfaces, including Jorge-Meeks' and Hoffman-Meeks' surfaces, upper and lower bounds of their indices were found in [C], [N] and [T1]. The indices of various individual minimal surfaces have been determined. Here we are interested in determining the index of a '*generic*' minimal surface in $\boldsymbol{R}^3$ with finite total curvature $4\pi d$. The first author and Micallef [E-M] proved the following estimate for an arbitrary complete orientable minimal surface in $\boldsymbol{R}^3$ with finite total curvature.

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THEOREM (Ejiri-Micallef [E-M]). *Let M be a complete orientable minimal surface of genus g in $\boldsymbol{R}^3$ with total curvature $\int|K|<+\infty$. It is known that the number $d=(1/4\pi)\int|K|$ is an integer. For any such M, we have*

$$Index(M)+Nullity(M)\leq 2(2d+g-1)-1+3=4d+2g\,.$$

REMARK. For estimates of the Morse index of a minimal submanifold of higher dimension or higher codimension, see [E], [E-M], [C-T2], [T2] and [B-B].

In this paper we will compute the index of a 'generic' minimal surface of genus zero. (The precise definition of 'generic' will be given in §4.)

THEOREM A. *Let M be a generic complete orientable finitely branched minimal surface of genus zero in $\boldsymbol{R}^3$ with finite total curvature $4\pi d$. Then we have*

$$Index(M)=2d-1 \quad and \quad Nullity(M)=3\,.$$

REMARK. In [C], Choe conjectured that $\mathrm{Index}(M)\leq 2d-1$ for any complete orientable minimal surface in $\boldsymbol{R}^3$ with finite total curvature $4\pi d$. Our theorem proves his conjecture for any generic surface of genus zero.

The following is an idea of our proof of the main theorem. Let us consider an arbitrary end p of a complete orientable minimal surface M in $\boldsymbol{R}^3$ with finite total curvature. Then, by [O], the surface M has the limiting tangent plane at the end p. It implies that the surface M, around p, is a local graph over the limiting tangent plane. Suppose that the end p is embedded. The embedded end p is called a *catenoid end* if M is locally a graph of logarithmic growth. It is called a *flat end* if M is locally a bounded graph. Bryant ([B]) proved that the embedded end p is either a catenoid end or a flat end. The current definition of flat end can be applied only when the end is embedded. We will first generalize the definition of flat end to the non-embedded ends. We say that a minimal surface is *flat-ended* if all of its ends are flat. Then the flat-ended minimal surfaces allow a number of characterizations. These characterizations will turn out to be very useful. By making use of them, we will first show that the space of all Gauss maps of flat-ended minimal surfaces is a union of proper algebraic subvarieties of the space of all meromorphic functions of degree d. It implies, in particular, that it has codimension greater than one. In other words, flat-ended minimal surfaces are *exceptional* among all minimal surfaces. Next we will prove

THEOREM B. *A complete orientable finitely branched minimal surface in $\boldsymbol{R}^3$ with finite total curvature has nullity ≥ 4 if and only if its Gauss map can be the Gauss map of a flat-ended minimal surface.*

The argument combining those two facts leads us to the proof of Theorem A as we see in §4.

As a by-product of this proof, we can show that there exists a branched Willmore surface of an arbitrary genus in $S^3(1)$. There we will use the technique developed by

Bryant in [B].

§2. Gauss parametrization.

In this section, we will define a '*Gauss parametrization*' of a complete orientable finitely branched minimal surface in $\boldsymbol{R}^3$. This concept enables us to reduce our problem to a study of meromorphic functions on a compact Riemann surface.

Let $X: M \to \boldsymbol{R}^3$ be a complete orientable finitely branched minimal surface with finite total curvature $\int |K| = 4\pi d$. Then there exist a compact Riemann surface $\hat{M}$ and a finite number of points $\{p_1, \cdots, p_k\}$ of $\hat{M}$ such that M is conformally diffeomorphic to $\hat{M} \setminus \{p_1, \cdots, p_k\}$. In other words, M can be compactified by attaching a finite number of points $p_1, \cdots, p_k$. Those points $p_1, \cdots, p_k$ are called the ends of M ([O], [R-T]).

Let $\Phi: M \to S^2(1)$ be the Gauss map assigning the unit normal vectors. Then Φ extends to a meromorphic function over $\hat{M}$ of degree d, which is also denoted by $\Phi: \hat{M} \to S^2(1)$ by abuse of notation. In this fashion, we can associate a meromorphic function Φ on $\hat{M}$ with any complete orientable finitely branched minimal surface $X: M \to \boldsymbol{R}^3$ with finite total curvature.

Give a compact Riemann surface $\hat{M}$ and a meromorphic function $\Phi: \hat{M} \to S^2(1)$ on $\hat{M}$. Now we construct a complete orientable minimal surface with Φ as its Gauss map. Let g_* be the Riemannian metric on $\hat{M} \setminus \{\text{finite points}\}$ induced from the canonical metric on $S^2(1)$. Let ∇^* and Δ^* denote the Levi-Civita connection and the Laplacian defined by g_*, respectively. By using an isothermal coordinate on $\hat{M}$, they are given by

$$g_* = 2\left|\Phi_*\left(\frac{\partial}{\partial z}\right)\right|^2 dz d\bar{z}\,,$$

$$\nabla^*_{\partial/\partial z}\frac{\partial}{\partial z} = \frac{\partial}{\partial z}\left(\log\left|\Phi_*\left(\frac{\partial}{\partial z}\right)\right|^2\right)\frac{\partial}{\partial z}\,,$$

$$\nabla^*_{\partial/\partial z}\frac{\partial}{\partial \bar{z}} = 0\,,$$

$$\Delta^* = \frac{2}{\left|\Phi_*\left(\frac{\partial}{\partial z}\right)\right|^2}\frac{\partial^2}{\partial z \partial \bar{z}}\,.$$

Take a smooth function F on $\hat{M}$ satisfying $\Delta^* F + 2F = 0$. We define a map X: $\hat{M} \setminus \{\text{finite points}\} \to \boldsymbol{R}^3$ by

$$X = F\Phi + \operatorname{grad}_{S^2(1)} F\,.$$

For convenience, we abuse the notation for the map X and write simply $X: \hat{M} \to \boldsymbol{R}^3$.

By simple calculations we get

$$X=F\Phi+|\Phi_z|^{-2}(F_{\bar{z}}\Phi_z+F_z\Phi_{\bar{z}}),$$

$$X_z=|\Phi_z|^{-2}\mathrm{Hess}\,F\left(\frac{\partial}{\partial z},\frac{\partial}{\partial z}\right)\Phi_{\bar{z}}=|\Phi_z|^{-2}\left(F_{zz}-\frac{\partial}{\partial z}(\log|\Phi_z|^2)F_z\right)\Phi_{\bar{z}},$$

where we write $\Phi_z=\Phi_*(\partial/\partial z)$, $F_z=\partial F/\partial z$, etc.

Therefore we construct a branched immersion $X:\hat{M}\to\boldsymbol{R}^3$, which has Φ as its Gauss map and the metric

$$g=2|X_z|^2dzd\bar{z}=2|\Phi_z|^{-2}\left|F_{zz}-\frac{\partial}{\partial z}(\log|\Phi_z|^2)F_z\right|^2dzd\bar{z}$$

induced by X.

With these understood, we have the following.

LEMMA 2.1. *Let $\Omega=(F_{zz}-(\partial/\partial z)(\log|\Phi_z|^2)F_z)dz^2$ for Φ and F above. Then the following hold.*

(1) *Ω is a meromorphic differential on $\hat{M}$, which has poles of order* 1 *only at some zeros of Φ_z.*

(2) *Ω is identically zero if and only if there is a constant vector A in $\boldsymbol{R}^3$ such that $F=\langle X, A\rangle$.*

(3) *Φ_z has at least* 4 *distinct zeros if the genus of $\hat{M}$ is zero.*

PROOF. (1) and (2) are immediate.

(3) The last statement follows from the fact that

$$\#\{\text{the zeros of }\Omega\}-\#\{\text{the poles of }\Omega\}=-2\times\{\text{Euler number}\}=4(g-1).$$

PROPOSITION 2.2. *If F satisfies the equation $\Delta^*F+2F=0$ and $\Omega(F)\neq 0$, then $X: M\to\boldsymbol{R}^3$ gives a complete finitely branched minimal surface, which has a finite number of ends.*

In fact, it is easy to see that

$$X_{z\bar{z}}=0,$$

$$|X|^2=|F|^2+2|\Phi_z|^{-2}|F_z \quad \text{and}$$

$$|X_z|^2=|\Phi_z|^{-2}\left|F_{zz}-\frac{\partial}{\partial z}(\log|\Phi_z|^2)F_z\right|^2.$$

Thus X is minimal and has the ends at zeros of the meromorphic function Φ and X_z has the branch points at zeros of the meromorphic function $F_{zz}-(\partial/\partial z)(\log|\Phi_z|^2)F_z$, both of which are finite in number. We note that some zeros of Φ_z may not be the ends of X.

It should be remarked that any complete finitely branched minimal surface

$X: M \to \boldsymbol{R}^3$ can be written as

$$X = F\Phi + \operatorname{grad}_{S^2} F$$

if we take $F = \langle X, \Phi \rangle$, which satisfies $\Delta^* F + 2F = 0$ (F is not necessarily bounded on M). This construction of X is called *the Gauss parametrization* of X (cf. [D-G]). In §3 we shall see that the boundedness of F implies the flatness of all ends.

§3. Flat ends.

Throughout this section we keep the notation in §2. Let $X: M \to \boldsymbol{R}^3$ be a complete orientable finitely branched minimal surface with finite total curvature and $\hat{M} = M \cup \{p^1, \cdots, p^r\}$ its compactification by the Gauss map.

It has been known through recent investigation about minimal surfaces of finite total curvature that such a surface has the limiting tangent plane at each end p and is, around p, a local graph over the plane. If the end p is embedded, the graph has at most logarithmic growth. The embedded end p is called a *catenoid end* if M is a graph of logarithmic growth around p. It is called a *flat end* if M is locally a bounded graph. Here we will define a flat end when the end p is non-embedded. As seen in §2, the unit normal vector Φ of M converges to $\Phi(p)$ around p of M. We call a non-embedded end p *a flat end* when $\langle X, \Phi(p) \rangle$ is uniformly bounded near p. It should be remarked that $\langle X, \Phi(p) \rangle$ is a function of at most logarithmic growth when p is embedded, while it may have higher order growth if p is non-embedded. In this section, we will give a number of characterization of flat ends. We will use those characterization to prove Theorem A in §4.

We first characterize flat ends in terms of a series expansion;

$$X_z = \frac{1}{z^k} V_{-k} + \cdots + \frac{1}{z} V_{-1} + \text{holomorphic part}$$

where z is a local isothermal coordinate around p with $z(p) = 0$ and V_i's are constant vectors in $\boldsymbol{C}^3$.

PROPOSITION 3.1. *An end p is flat if and only if*

$$V_{-k} /\!/ \cdots /\!/ V_{-2} \quad \textit{and} \quad V_{-1} = 0 .$$

Note that X_z has the residue $V_{-1} = 0$, which comes from well definedness of the immersion $X = \operatorname{Re}\{\int X_z dz\}$.

Let $\underline{\boldsymbol{C}^3} = \hat{M} \times \boldsymbol{C}^3$ be the trivial bundle over M. $\underline{\boldsymbol{C}^3}$ is decomposed as $\{\Phi\} \oplus L_1 \oplus L_2$, where L_1 and L_2 stand for the line bundles spanned by Φ_z and $\Phi_{\bar{z}}$ respectively. Since

$$\langle \Phi, \Phi_z \rangle = \frac{\partial}{\partial z} |\Phi|^2 = 0 \quad \text{and} \quad \langle \Phi_z, \Phi_z \rangle = 0 ,$$

it follows that

$$L_1=\{\xi\in\underline{C}^3 : \langle\xi, \Phi\rangle=0 \text{ and } \langle\xi, \Phi_z\rangle=0\} .$$

Remark that

$$L=\{\xi\in\underline{C}^3 : \langle\xi, \Phi\rangle=0 \text{ and } \langle\zeta, \xi\rangle=0 \text{ for all } \zeta\in L_1\}$$

is one-dimensional.

PROOF OF PROPOSITION 3.1. Since $\Phi(p)$ is the limiting normal vector at p and X_z is a null vector, we get

$$\langle V_{-k}, \Phi(p)\rangle=0 \quad \text{and} \quad \langle V_{-k}, V_{-k}\rangle=0 .$$

So, together with the above remark, we have

$$\{\xi\in\underline{C}^3 : \langle\xi, \Phi(p)\rangle=0, \langle\xi, V_{-k}\rangle=0\}=CV_{-k} .$$

The function $\langle X, \Phi(p)\rangle$ is uniformly bounded near p if and only if

$$\langle V_{-k+l}, \Phi(p)\rangle=0 \qquad \text{for} \quad l=0, \cdots, k-1 .$$

Note that X_z is a null vector. We also get

$$\sum_{j=0}^{m}\langle V_{-k+j}, V_{-k+m-j}\rangle=0 \qquad \text{for} \quad m=0, \cdots, k-1 .$$

We prove inductively that V_{-k+l} is parallel to V_{-k} for $l=0, \cdots, k-1$. Since X_z has no real period, $\mathrm{Im}(V_{-1})=0$, which implies $V_{-1}=0$. q.e.d.

Next we shall characterize flat ends in terms of the Gauss parametrization of X.

PROPOSITION 3.2. *Let p be an end of X. Then p is flat if and only if $F=\langle X, \Phi\rangle$ is uniformly bounded near p.*

As in §2, the position vector X can be written as

$$X=\langle\Phi, X\rangle\Phi+|\Phi_z|^{-2}\{F_{\bar z}\Phi_z+F_z\Phi_{\bar z}\} .$$

Flatness, from the definition, requires that

$$\langle X, \Phi(p)\rangle=\langle X, \Phi\rangle\langle\Phi, \Phi(p)\rangle+|\Phi_z|^{-2}\{F_{\bar z}\langle\Phi_z, \Phi(p)\rangle+F_z\langle\Phi_{\bar z}, \Phi(p)\rangle\}$$

is bounded near p.

If $F=\langle X, \Phi\rangle$ is bounded, then, by elliptic regularity, F_z and $F_{\bar z}$ are also bounded and

$$|\langle\Phi, \Phi(p)\rangle|\le|\Phi||\Phi(p)|=1 .$$

Therefore $\langle X, \Phi(p)\rangle$ is bounded if $|\Phi_z|^{-2}\langle\Phi_z, \Phi(p)\rangle$ is bounded. As noted in §1, Φ_z is zero at every end. Hence we can write locally $\Phi_z=z^m\eta$ near p with $\eta(p)\neq0$ and $z(p)=0$.

First we shall see that all $\eta^{(k,0)}=(\partial^k/\partial z^k)\eta$ are parallel.

LEMMA 3.3. *$\eta^{(k,0)}$ are parallel to each other for $k=0, 1, \cdots$.*

Let L_1 be as before. We are going to see that all $\{\eta^{(k,0)}\}$ are local sections of the line bundle L_1. Note that $\eta\in L_1$. Assume that $\eta^{(k,0)}\in L_1$ for some k. Then we get

$$\langle\eta^{(k+1,0)}, \eta^{(k,0)}\rangle=\frac{1}{2}\frac{\partial}{\partial z}\langle\eta^{(k,0)}, \eta^{(k,0)}\rangle=0\,,$$

which implies that

$$\langle\eta^{(k+1,0)}, \Phi_z\rangle=0\,.$$

We also get

$$\langle\eta^{(k+1,0)}, \Phi\rangle=\frac{\partial}{\partial z}\langle\eta^{(k,0)}, \Phi\rangle-\langle\eta^{(k,0)}, \Phi_z\rangle=0\,.$$

These prove that $\eta^{(k+1,0)}\in L_1$.

Remark that $\langle\partial\eta/\partial\bar z, \Phi\rangle=-\langle\eta, \Phi_{\bar z}\rangle=-\bar z^m\langle\eta, \bar\eta\rangle$. By a similar computation as in Lemma 3.3, we have

LEMMA 3.4. *$\langle\eta^{(0,l)}, \Phi\rangle=\bar z^{m-l+1}A_{0,l}(z, \bar z)$ for $l=1, \cdots, m+1$, where $A_{0,1}(z, \bar z)$ is a non zero bounded function in z and $\bar z$.*

Using Lemmas 3.3–3.4, we will show the 'if' part of Proposition 3.2. We need to see that

$$|\Phi_z|^{-2}\langle\Phi_z, \Phi(p)\rangle=\bar z^{-m}\langle\eta, \bar\eta\rangle^{-1}\langle\eta, \Phi(p)\rangle$$

is bounded near p. On the other hand the Taylor expansion of $\langle\eta, \Phi(p)\rangle$ around p is given by

$$\langle\eta, \Phi(p)\rangle=\sum_{k,l}\langle\eta^{(k,l)}(p)z^k\bar z^l, \Phi(p)\rangle=\bar z^{m+1}C(z, \bar z)\,,$$

where $C(z, \bar z)$ is a non zero bounded function in z and $\bar z$. The last equation follows from Lemma 3.3 and Lemma 3.4. Therefore

$$|\Phi_z|^{-2}\langle\Phi_z, \Phi(p)\rangle=\bar z|\eta|^{-2}C(z, \bar z)\longrightarrow 0 \qquad \text{as} \quad z\to 0\,.$$

This completes the proof of "if" part. We leave "only if" part after Proposition 3.5.

We now give the most useful characterization of flat ends in the following:

PROPOSITION 3.5. *Let p be an end of X: $M\to\boldsymbol{R}^3$. Then p is a flat end if and only if the following inequality holds.*

The order of a zero of Φ_z at $p\geq$(The order of a pole of X_z at p)-1.

PROOF. We identify $G_{2,4}(\boldsymbol{R})\simeq\boldsymbol{R}^3$ under the isometry given by

$$(e^1 \wedge e^2)^* = e^3, \quad (e^2 \wedge e^3)^* = e^1 \quad \text{and} \quad (e^3 \wedge e^1)^* = e^2$$

for the orthonormal frames $\{e^1, e^2, e^3\}$ of $\boldsymbol{R}^3$. Then we get

$$\left| \left\{ \frac{X_z}{|X_z|} \wedge \left(\frac{X_z}{|X_z|} \right)_z \right\}^* \right| = |\Phi_{\bar{z}}| ,$$

since z is the isothermal coordinate.

On the other hand, using the Taylor expansion

$$X_z = z^{-k} V_{-k} + \cdots + z^{-1} V_{-1} + V_0 + \cdots ,$$

we get

$$\frac{X_z}{|X_z|} \wedge \left(\frac{X_z}{|X_z|} \right)_z = O\left(\sum_{i,j \geq -k} z^{2k+i+j-1} V_i \wedge V_j \right).$$

Thus the order of a zero of $|\Phi_z|$ is equal to $k+i-1$, where i is the smallest number of V_i which is not parallel to V_{-k}.

From Proposition 3.1, $i \geq 0$ is the condition for flatness. Thus we complete the proof.

PROOF OF PROPOSITION 3.2. We have proved 'if' part of the proposition and now prove 'only if' part. Namely, we prove that if p is a flat end, then $F = \langle X, \Phi \rangle$ is uniformly bounded near p.

From Proposition 3.1 X has the Taylor expansion

$$X = \mathrm{Re}\{(a_k z^{1-k} + a_{k-1} z^{2-k} + \cdots + a_2 z^{-1}) V + \text{holomorphic part}\}$$

around a flat end $z=0$. We also know from Proposition 3.5 that $\Phi_z = z^{k-1} \eta(z, \bar{z})$, where η may be zero at $z=0$. Therefore we get $\langle X, \Phi_z \rangle = (\partial / \partial z) \langle X, \Phi \rangle$ is bounded, which implies that F is bounded near $z=0$. q.e.d.

By Proposition 3.2, we obtain that a minimal surface X given in Proposition 2.2 is a flat-ended minimal surface, since F is a smooth function on a compact Riemann surface $\hat{M}$, and is bounded on $M = \hat{M} - \{\text{finite points}\}$.

Now we apply Propositions 3.1 and 3.2 to a study of Willmore surfaces. Let X be a complete orientable minimal surface in $\boldsymbol{R}^3$ and p be an embedded end. Bryant proved in [B] that $\langle X, \Phi \rangle$ is uniformly bounded near p if and only if p is a flat end, and X_z has no residues at embedded flat ends. By using those facts, he showed that we get a branched Willmore surface as the image of a flat-ended minimal surface in $\boldsymbol{R}^3$ of the stereographic projection from S^3 to $\boldsymbol{R}^3$ if all ends are embedded. By Propositions 3.1 and 3.2, we prove his argument still works even if we remove the condition that all ends are embedded.

COROLLARY 3.6. *There is a branched Willmore surface of an arbitrary genus g in $S^3(1)$.*

PROOF. It is enough to show the existence of a flat-ended minimal surface for each genus g. Costa and Hoffman-Meeks gave examples of complete orientable embedded minimal surfaces M_g of genus g in [H-M]. On the other hand, Choe ([C]) proved that there exists a smooth function F on $\hat{M}_g$ satisfying that $\Delta^* F + 2F = 0$ and $\Omega \neq 0$ with respect to the induced metric by the Gauss map of M_g. The construction in §2 offers flat-ended minimal surfaces of genus g.

§4. The space of meromorphic functions and the index.

First we will review briefly relevant results in [F]. Let M be a complete orientable finitely branched minimal immersion in $\boldsymbol{R}^3$ with finite total curvature. For a bounded domain Ω in M, we consider the quadratic form

$$I_\Omega(\psi) = \int_\Omega \{|\nabla\psi|^2 - |\nabla\Phi|^2\psi^2\} d(\mathrm{vol} M)$$

on $C_0^\infty(\Omega)$, where Φ is the Gauss map of M. Index(Ω) is, by definition, the number of negative eigenvalues of I_Ω. The least upper bound of Index(Ω) among all bounded domains is called the *index* of M and is denoted by Index(M). M has finite total curvature if and only if Index(M) is finite. In that case, we have a compact Riemann surface $\hat{M}$ with the metric g_* induced from the Gauss map Φ as we see in §2. In stead of the operator I_Ω, put

$$I_g(\psi) = \int_{\hat{M}} (|\nabla\psi|^2 - |\nabla g|^2\psi^2) d(\mathrm{vol}_{g_*} \hat{M})$$

for the meromorphic function g defined by the composition of the stereographic projection $S^2(1) \to \boldsymbol{C}$ and the Gauss map Φ of M. Identifying $S^2(1) \simeq \boldsymbol{C} \cup \{\infty\}$, we may call g the Gauss map of M. It is known that the index of I_g is equal to Index(M). It is easy to see $I_g(\psi)$ is conformally invariant and depends only on g. Therefore we can define, for each meromorphic function g on $\hat{M}$, the operator

$$L_g = -\Delta_{\hat{M}} - |\nabla_{\hat{M}} g|^2 = -\Delta_{\hat{M}} - 2\,,$$

Index(g) and Nullity(g), regardless of g being a Gauss map of some minimal surface or not. We remark that each element of the nullity space for L_g is a bounded Jacobi field on M. Via the Gauss map of M, eigenfunctions of $S^2(1)$ with eigenvalues 2 can be considered as nullity functions for L_g. Hence the coordinate functions X_1, X_2 and X_3 of $\boldsymbol{R}^3$ restricted to $S^2(1)$ give three linearly independent elements of the nullity space. We state

LEMMA 4.1. *Nullity$(g) \geq 3$ for any meromorphic function g on $\hat{M}$.*

A nullity function F is said to be *trivial* if $\Omega(F) \equiv 0$. Notice that the coordinate functions are trivial and all trivial nullity functions are linear combinations of X_1, X_2

and X_3 from the lemma in §2.

THEOREM B. *For a meromorphic function* $g: \hat{M} \to \boldsymbol{C}$, *Nullity*$(g) \geq 4$ *if and only if there is a complete finitely branched minimal surface all of whose ends are flat in* $\boldsymbol{R}^3$ *and whose Gauss map is* g.

PROOF. If M is flat-ended then $F = \langle X, \Phi \rangle$ yields a non-trivial nullity function. Conversely if g has nullity ≥ 4, then we have a non-trivial nullity function f. We can construct a finitely branched complete minimal surface which is flat-ended given by the map

$$X = f\Phi + \mathrm{grad}_{S^2} f$$

where Φ is a pull back of g by the stereographic projection.

Therefore we can prove the main theorem if we show that the space $\mathfrak{G}_F$ of all meromorphic functions on $\hat{M}$ corresponding to Gauss maps of some minimal branched flat-ended surfaces is a subset with co-dimension greater than 1 in the space $\mathfrak{G}$ of all meromorphic functions on $\hat{M}$.

From now on we assume that the genus of $\hat{M}$ *is zero.* Any $g(z) \in \mathfrak{G}$ with k poles and l zeros of degree d has the form

$$g(z) = \alpha \frac{\prod (z - b_i)^{l_i}}{\prod (z - a_j)^{k_j}}$$

for all distinct l points b_i and k points $a_j \in \boldsymbol{C}$ and $\alpha \in \boldsymbol{C}$. We may assume that $\sum_i l_i = d$ without loss of generality. By using the expression, we define parametrization of $\mathfrak{G}_{k_1 \cdots k_k l_1 \cdots l_l}$ of all such functions over a domain in $\boldsymbol{C}^{k+l+1}$ by

$$g(z) \longleftarrow (\alpha, a_1, \cdots, a_k, b_1, \cdots, b_l),$$

a_i and b_j being all distinct.

For brevity, we denote $\mathfrak{G}_{k_1 \cdots k_k l_1 \cdots l_l}$ simply by $\mathfrak{G}_{k,l}$.

Under this identification, $\mathfrak{G}_{F,k,l} = \mathfrak{G}_F \cap \mathfrak{G}_{k,l}$ is contained in an algebraic subvariety in this parameter space of $\mathfrak{G}_{k,l}$. In fact p is a flat end if and only if the Weierstrass representation

$$X_z = (f(1 - g^2),\ if(1 + g^2),\ 2fg)$$

has no residues and

$$\begin{aligned} &\{\text{the order of a zero of } |\Phi_z(z)| = |g'(z)|/(1 + |g(z)|^2) \text{ at } p\} \\ &\qquad \geq \{\text{the order of a pole of } |f(z)|(1 + |g(z)|^2) \text{ at } p\} - 1\,. \end{aligned}$$

Those follow from propositions in §3.

The conditions under which such an f exists are algebraic and $\mathfrak{G}_{F,k,l}$ is an algebraic subset in $\boldsymbol{C}^{k+l+1}$. However it does not cover the whole parameter space. In fact, LEMMA 4.2. *$g(z)=z^d$ cannot be the Gauss map of any minimal branched flat-ended surface. Moreover a little perturbation gives a family of meromorphic functions $g^0_{k,l}$ in $\mathfrak{G}_{k,l} \setminus \mathfrak{G}_{F,k,l}$ for any pair of (k, l). We also know that Index$(g^0_{k,l})=2d-1$ and Nullity$(g^0_{k,l})=3$.*

PROOF. The index and the nullity of $g^0_{k,l}$ is computed in [N]. If $g(z)=z^d$ is the Gauss map of a flat-ended minimal surface and $(f(z), g(z))$ is its Weierstrass pair, $f(z)$ has poles only at its ends. We know that each end occurs at a zero of Φ_z. In our case Φ_z has zeros of orders $d-1$ at both $z=0$ and $z=\infty$. At these points we apply the above inequality. More precisely, let

$$f(z)=z^{-k}(c_0+\ \cdots\ +c_l z^l)$$

where $c_0, \cdots, c_l$ ($c_0, c_l \neq 0$) denote constants and k and l are nonnegative integers. We get $d \geq k$ and $d \geq 2d+2+l-k \geq d+2+l$, which implies that such an $f(z)$ cannot exist. Therefore $g(z) \in \mathfrak{G} \setminus \mathfrak{G}_F$. The same argument holds for $g^0_{k,l}$.

THEOREM A. *Let M be a complete orientable finitely branched minimal surface of genus zero with finite total curvature $\int |K|=4\pi d$. Then we get*

$$\textit{Index}(M) \leq 2d-1\ .$$

Moreover, for generic M, Index$(M)=2d-1$ and Nullity$(M)=3$.

PROOF. We have proved that $\mathfrak{G}_{F,k,l}$ is contained in some algebraic subvarieties in $\mathfrak{G}_{k,l}$ which does not cover the whole parameter space. Therefore it has real codimension greater than 1. Hence $\mathfrak{G}_{k,l} \setminus \mathfrak{G}_{F,k,l}$ is arcwise-connected. For any $g \in \mathfrak{G}_{k,l} \setminus \mathfrak{G}_{F,k,l}$, there is a path which connects g to $g^0_{k,l}$ and does not intersect $\mathfrak{G}_{F,k,l}$. Since the eigenvalues of the corresponding operator vary continuously along this path ([K-S]), if Index$(g) \neq 2d-1$, at least one element of the negative eigenspace must move to the nullity space. This means that the path intersects $\mathfrak{G}_F$, which contradicts the choice of the path. Therefore Index(g) must be $2d-1$ and Nullity$(g)=3$ for any $g \in \mathfrak{G} \setminus \mathfrak{G}_F$. To prove that Index$(M)$ is not greater than $2d-1$ for any M, we will see what happens if there exists g such that Index(g) is greater than $2d-1$. Then there exists a neighborhood U of g in $\mathfrak{G}_{k,l}$, any of whose element has index greater than $2d-1$. That is a contradiction to the fact that Index$(g)=2d-1$ for generic g.

COROLLARY 4.3. *For any complete orientable finitely branched minimal surface of genus zero with finite total curvature 8π, we have*

$$\textit{Index}(M)=3 \quad \textit{and} \quad \textit{Nullity}(M)=3\ .$$

PROOF. If Nullity(M) is greater than three for a minimal surface M, then its Gauss map belongs to $\mathfrak{G}_F$. From lemma in §2 Φ_z has at least 4 distinct zeros. Hence by Riemann-Hurwitz' theorem we get $2d-2 \geq 4$, which says that $d=(1/4\pi)\int |K| \geq 3$.

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