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The Witten Laplacian on Negatively Curved Simply Connected Manifolds

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1. Introduction.

The Hopf conjecture states that the Euler characteristic of a compact Riemannian $2n$-manifold $\bar{M}$ of negative sectional curvature satisfies $(-1)^n\chi(\bar{M})>0$ [6]. Applying the Chern-Gauss-Bonnet theorem gives the conjecture for $n=1, 2$, for spaces of constant curvature, and for spaces of sufficiently pinched curvature [5]. Singer's idea of instead using the L^2 index theorem to establish the Hopf conjecture has been successfully carried out for Kähler manifolds by Gromov [18] (cf. [11]). It is worth noting that the first examples of negatively curved manifolds not admitting metrics of constant negative curvature are rather recent [20], [19].

Singer's method depends on the vanishing of L^2 harmonic forms (except in the middle dimension) on the universal cover of a compact negatively curved manifold, as explained in §4. This raised the question of such vanishing for arbitrary simply connected negatively curved manifolds. Anderson's paper [1] shows that such vanishing results are not possible without a pinching condition; however, his examples admit no compact quotient, so Singer's approach is not ruled out. One of our main results (Corollary 4.4) is that for one-forms vanishing occurs except in the pinching region ruled out by Anderson's examples. In general, we obtain vanishing results and hence $(-1)^n\chi(\bar{M})\geq 0$ (Theorem 4.5) for manifolds of pinched negative curvature, where the pinching constant is more relaxed than in previous work, e.g. [5].

The vanishing theorems depend upon Witten's deformation $\Box_\tau$ of the Laplacian on forms on M [21]. In contrast to Witten's work, in which the Morse inequalities are recovered by letting the deformation parameter τ go to infinity, the vanishing theorems arise through the study of small deformations. Moreover, instead of deforming the Laplacian by a Morse function as in [21], we use the distance function to a point. The

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pinching condition comes in through the study of the Hessian of the distance function. The use of the distance function and the Hessian comparison theorem for vanishing theorems appeared previously in [12] in another form.

In §2 the Witten Laplacian and Bismut's variant are introduced, and in §3 some general vanishing results for the L^2 kernel of $\Box_\tau$ are obtained. In §4 the main result is proven. In §5, it is shown that in all dimensions an L^2 harmonic form on M which decays exponentially in distance from a point must vanish, if the decay constant is sufficiently large. As explained in §5, this is another indication of the difficulty of proving the Hopf conjecture by Singer's approach.

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2. The Witten Laplacian.

Let M be a complete n-dimensional Riemannian manifold. Suppose $h: M \to \boldsymbol{R}$ is C^2. Following Witten [21], define the modifications d_τ, $\tau \in \boldsymbol{R}$, to exterior differentiation

$$d_\tau = e^{\tau h} d e^{-\tau h} \tag{2.1}$$

with formal adjoint d_τ^* and corresponding deformed Laplacian

$$\Box_\tau = d_\tau d_\tau^* + d_\tau^* d_\tau . \tag{2.2}$$

In general, when E is a vector space and A an endomorphism of E there are induced linear maps $\bigwedge^q A : \bigwedge^q E \to \bigwedge^q E$ and $(d\bigwedge)^q A : \bigwedge^q E \to \bigwedge^q E$ defined by

$$\bigwedge^q A(v^1 \wedge \cdots \wedge v^q) = Av^1 \wedge \cdots \wedge Av^q$$

and

$$(d\bigwedge)^q A(v^1 \wedge \cdots \wedge v^q) = \sum_{j=1}^{q} v_1 \wedge \cdots \wedge v_{j-1} \wedge Av_j \wedge v_{j+1} \wedge \cdots \wedge v_q .$$

Write $\bigwedge A$ and $(d\bigwedge)A$ for the corresponding direct sums acting on $\bigwedge E$.

From [8, Proposition 11.13, equation (11.35), Theorem 12.10],

$$\Box_\tau = \Delta + \tau^2 \|dh\|^2 - \tau(\Delta h + 2(d\bigwedge)(\nabla^2 h)^*) \tag{2.3}$$

where Δ is the Laplace-Beltrami operator (i.e. $\Box_0$) and $(\nabla^2 h)_x^* : T_x^*M \to T_x^*M$ is the adjoint of the Hessian $(\nabla^2 h)_x : T_xM \to T_xM$.

Equations (2.1), (2.2), (2.3) are satisfactory when considering d_τ and $\Box_\tau$ as operators on smooth forms. To consider the L^2 theory, we will need to define $\Box_\tau$ as an essentially self-adjoint operator on the space $L^2\Omega^*$ of L^2 forms. A convenient way for us to do this is via Bismut's version [4] of the deformed Laplacian:

Let μ_τ be the measure on M given in terms of the Riemannian measure dx by

$$\mu_\tau(dx) = e^{2\tau h(x)} dx$$

with $L^2\Omega^q(\mu_\tau)$ the corresponding Hilbert space of q-forms on M. Let δ_τ be the formal adjoint of d acting on C^∞ forms with compact support in $L^2\Omega^q(\mu_\tau)$. Then

$$\delta_\tau = \delta - 2\tau i_{\nabla h} \tag{2.4}$$

where $\delta = d_0^* = \delta_0$ and i_Z is interior multiplication by a vector field Z. Define, on smooth forms with compact support

$$\Delta_\tau = (d + \delta_\tau)^2 .$$

Then

$$\Delta_\tau = d\delta_\tau + \delta_\tau d = \Delta - 2\tau L_{\nabla h} \tag{2.5}$$

where $L_Z = \{i_Z, d\}$ (the anticommutator) is Lie differentiation in the direction of Z. The operators Δ_τ, $\Box_\tau$ are related by the formula $\Delta_\tau = e^{\tau h}\Box_\tau e^{-\tau h}$.

As in [7], $d+\delta_\tau$ is essentially self adjoint on $L^2\Omega^*(\mu_\tau)$ with core the space $C_0^\infty\Omega^*$ of C^∞ forms with compact support, and so are all its powers. It therefore has a unique self adjoint extension, which is its closure $\overline{d+\delta_\tau}$, as does the Laplacian, whose extension we denote $\bar{\Delta}_\tau$ (cf. [3] for the case of functions). Moreover, it is easy to see that $\overline{d+\delta_\tau} = \bar{d} + \bar{\delta}_\tau$. Define $\bar{\Box}_\tau$ on the usual L^2 spaces by

$$\bar{\Box}_\tau = e^{-\tau h}\bar{\Delta}_\tau e^{\tau h} .$$

Then $\bar{\Box}_\tau$ is self adjoint and is isospectral to $\bar{\Delta}_\tau$.

Note also that if $*$ is the Hodge star operator then $*\bar{\Box}_\tau = \bar{\Box}_{-\tau}*$. Thus $\bar{\Box}_\tau$ on q-forms is isospectral to $\bar{\Box}_{-\tau}$ on $(n-q)$-forms, and in particular the L^2 kernels of $\bar{\Box}_\tau$ on q-forms and $\bar{\Box}_{-\tau}$ on $(n-q)$-forms are isomorphic.

The L^2 cohomology for the measure μ_τ is defined by

$$L_\tau^2 H^q(M) = \frac{(\text{Ker } \bar{d} \text{ on } L^2\Omega^q(\mu_\tau))}{(\text{Image } \bar{d} \text{ on } L^2\Omega^{q-1}(\mu_\tau))} .$$

Note that we do not take the closure of the image of $\bar{d}$. The ordinary L^2 cohomology groups $L_0^2 H^q(M)$ will be denoted $L^2 H^q(M)$.

Let $\mathscr{H}_\tau^q(M)$ be the kernel of $\bar{\Delta}_\tau$ in $L^2\Omega^q(\mu_\tau)$. As usual, $\phi \in \mathscr{H}_\tau^q(M)$ if and only if $\phi \in L^2\Omega^q(\mu_\tau)$ and $\bar{d}\phi = 0$, $\bar{\delta}_\tau\phi = 0$. All such ϕ are C^∞. Moreover, if $\phi \in \mathscr{H}_\tau^q(M)$ and $\phi = \bar{d}\theta$, then $\langle \bar{\delta}_\tau \bar{d}\theta, \psi \rangle_\tau = 0$ for all $\psi \in L^2\Omega^q(\mu_\tau)$, so that $\phi = 0$ (by setting $\psi = \theta$). Thus there is the injective map

$$j_q : \mathscr{H}_\tau^q(M) \to L_\tau^2 H^q(M) .$$

As usual, this is bijective if and only if $\bar{d}$ has closed range on $L^2\Omega^{q-1}(\mu_\tau)$.

The following is essentially taken from [10], where the case $\tau = 0$ is considered. We let $\bar{d}^q$ denote $\bar{d}$ on q-forms, and similarly for $\bar{\Delta}_\tau^q$.

LEMMA 2.6. *$\bar{d}^q$ and $\bar{d}^{q-1}$ have closed range if and only if* $\inf\{\operatorname{spec}(\bar{\Delta}_\tau^q)-\{0\}\}>0$. *If so, both j_q and j_{q-1} are isomorphisms.*

PROOF. In [10, Prop. 6.2], Donnelly proves the abstract result that for any self adjoint operator A the existence of a spectral gap at zero (i.e. $\inf\{\operatorname{spec}(A)-\{0\}\}>0$) is equivalent to having closed range. We have $\bar{\Delta}_\tau^q=\bar{d}^{q-1}(d^{q-1})^*+(d^q)^*\bar{d}^q$ as in [16, §2]; the proof there for $\tau=0$ generalizes immediately for arbitrary τ. Since this is a μ_τ-orthogonal splitting, $\bar{\Delta}_\tau^q$ has a spectral gap if and only if $\bar{d}^{q-1}(d^{q-1})^*$ and $(d^q)^*\bar{d}^q$ have closed range. This is easily seen to be equivalent to $(d^{q-1})^*$ and $\bar{d}^q$ having closed range. By Banach's theorem, $(d^{q-1})^*$ has closed range if and only if $\bar{d}^{q-1}$ does, since $d^*=\bar{d}^*$ automatically. □

Let λ_τ^q be the infimum of the spectrum of $\bar{\Delta}_\tau$ on $L^2\Omega^q(\mu_\tau)$. The following proposition enables us to obtain vanishing theorems for harmonic forms (at $\tau=0$) from the behavior of λ_τ^q as $\tau\to 0$.

PROPOSITION 2.7. *For fixed q with $1\le q\le n$ suppose*
(i) *$h(x)\le 0$ for all x in M,*
(ii) *$\tau^2/\lambda_\tau^q\to 0$ as $\tau\downarrow 0$.*
Then there is no non-zero L^2 harmonic q-form w on M with $i_{\nabla h}w\in L^2$.

PROOF. Suppose $w\in L^2\Omega^q$ and $\Delta w=0$. There exists $\varepsilon>0$ such that $\lambda_\tau^q>0$ for $0<\tau\le\varepsilon$ and therefore $L_\tau^2H^q=0$ for such τ. Since $h\le 0$, $w\in L^2\Omega^q(\mu_\tau)$ and so, since $dw=0$, $w=\bar{d}\beta_\tau$ for some $\beta_\tau\in L^2\Omega^{q-1}(\mu_\tau)$, each $0\le\tau\le\varepsilon$. We can choose $\beta_\tau\in(\ker d)^\perp=\overline{\delta_\tau C_0^\infty\Omega^q}$. However, since the closure $\bar{d}$ of d in $L^2\Omega^{q-1}(\mu_\tau)$ has closed range, because $\lambda_\tau^q>0$ and using Lemma 2.6, so has $\bar{\delta}$. Thus we can write $\beta_\tau=\bar{\delta}_\tau\theta_\tau$ for some $\theta_\tau\in\operatorname{Dom}\bar{\delta}_\tau$. If $\beta_\tau\ne 0$

$$
\begin{aligned}
|\bar{d}\beta_\tau|_\tau^2/|\beta_\tau|_\tau^2 &= |\bar{d}\bar{\delta}_\tau\theta_\tau|_\tau^2/|\bar{\delta}_\tau\theta_\tau|_\tau^2\\
&\ge \inf\{|\overline{d\delta_\tau}\theta|_\tau^2/|\bar{\delta}_\tau\theta|_\tau^2 : \theta\in\operatorname{Dom}\overline{d\delta_\tau}\subset L^2\Omega^q(\mu_\tau)\}\\
&= \inf\{|\overline{d\delta_\tau}\theta|_\tau^2/|\bar{\delta}_\tau\theta|_\tau^2 : \theta\in(\ker\bar{\delta}_\tau)^\perp\cap\operatorname{Dom}\overline{d\delta_\tau}\}\\
&= \inf\{|\overline{d\delta_\tau}\theta|_\tau^2/|\bar{\delta}_\tau\theta|_\tau^2 : \theta\in(\ker d)\cap\operatorname{Dom}\overline{d\delta_\tau}\}
\end{aligned}
$$

since $L_\tau^2H^q=0$. Thus

$$
\begin{aligned}
|\bar{d}\beta_\tau|_\tau^2/|\beta_\tau|_\tau^2 &\ge \inf\{|\bar{\Delta}_\tau\theta|_\tau^2/\langle\bar{\Delta}_\tau\theta,\theta\rangle_\tau : \theta\in(\ker d)\cap\operatorname{Dom}\overline{d\delta_\tau}\}\\
&\ge \inf\{|\bar{\Delta}_\tau\theta|_\tau^2/\langle\bar{\Delta}_\tau\theta,\theta\rangle_\tau : \theta\in\operatorname{Dom}\bar{\Delta}_\tau\}\\
&= \inf\{|\Delta_\tau\theta|_\tau^2/\langle\Delta_\tau\theta,\theta\rangle_\tau : \theta\in C_0^\infty\Omega^q\}\\
&= \inf\{\langle\Delta_\tau(\Delta_\tau)^{1/2}\theta,(\Delta_\tau)^{1/2}\theta\rangle_\tau/\langle(\Delta_\tau)^{1/2}\theta,(\Delta_\tau)^{1/2}\theta\rangle_\tau : \theta\in C_0^\infty\Omega^q\}\\
&\ge \lambda_\tau^q.
\end{aligned}
$$

This gives $|w|_\tau^2=|\bar{d}\beta_\tau|_\tau^2\ge\lambda_\tau^q|\beta_\tau|_\tau^2$.

Also $|\bar{d}\beta_\tau|_\tau^2=\langle w,\bar{d}\beta_\tau\rangle_\tau=\langle\bar{\delta}_\tau w,\beta_\tau\rangle_\tau=-2\tau\langle i_{\nabla h}w,\beta_\tau\rangle_\tau$ by (1.4) since $\delta w=0$. Thus

$\lambda_\tau^q|\beta_\tau|_\tau^2 \le 2\tau|i_{\nabla h}w|_\tau|\beta_\tau|_\tau$, giving

$$|\beta_\tau|_\tau \le 2\tau(\lambda_\tau^q)^{-1}|i_{\nabla h}w|_\tau .$$

Putting these together,

$$|w|_\tau^2 \le 2\tau|i_{\nabla h}w|_\tau|\beta_\tau|_\tau \le 4\tau^2(\lambda_\tau^q)^{-1}|i_{\nabla h}w|_\tau^2 .$$

By dominated convergence $|i_{\nabla h}w|_\tau^2 \to |i_{\nabla h}w|_0^2$ as $\tau\downarrow 0$ and so also, by (ii)

$$|w|_0^2 = \lim_{\tau\to 0}|w|_\tau^2 = 0 . \qquad \square$$

To finish this section we gather together the results in a form which will be easy to use.

PROPOSITION 2.8. *Suppose $h(x)\ge 0$ for all x in M and that $\lambda_{\tau_0}^q>0$ for some $\tau_0>0$. Then,*

(i) *For $r=q$ or $n-q$ if $\phi\in L^2\Omega^r$ has $e^{\tau_0 h}\phi\in L^2\Omega^r$ and $\Delta\phi=0$, then $\phi=0$.*

(ii) *If $e^{\tau_0 h}\phi\in L^2\Omega^q$ and $d\phi=0$, then $\phi=d\theta$ where $e^{\tau_0 h}\theta\in L^2\Omega^{q-1}$.*

(iii) *If $e^{-\tau_0 h}\phi\in L^2\Omega^{n-q}$ and $d\phi=0$, then $\phi=d\theta$ where $e^{-\tau_0 h}\theta\in L^2\Omega^{n-q-1}$.*

(iv) *If also $\lambda_\tau^q\ge\tau\delta$, for $0<\tau\le\tau_0$ and $r=q$ or $n-q$, then there is no non-zero harmonic $\phi\in L^2\Omega^r$ with $|\nabla h|\phi\in L^2\Omega^r$.*

PROOF. For (i), with $r=q$ note that $e^{\tau_0 h}\phi\in L^2$ implies $\phi\in L^2$, so that if $\Delta\phi=0$ then $d\phi=0$. Our hypothesis together with Lemma 2.6 gives $\phi=\bar{d}\theta$ where $\theta\in L^2\Omega(\mu_{\tau_0})$. But then $\theta\in L^2$, giving $\phi=0$. From this the case $r=n-q$ follows by duality for the $L^2\Omega^*$ complex. Part (ii) is direct from Lemma 2.6 using the vanishing of $L_\tau^2H^q$. Since $0<\lambda_{\tau_0}^q=\lambda_{-\tau_0}^{n-q}$, we have

$$0=L_{\tau_0}^2H^q\simeq\mathscr{H}_{\tau_0}^q\simeq\mathscr{H}_{-\tau_0}^{n-q}\simeq L_{-\tau_0}^2H^{n-q}$$

from which (iii) follows. For (iv) with $r=n-q$ use Proposition 2.7 above with h replaced by $-h$ and q by $n-q$. This shows that there is no non-zero harmonic $\phi\in L^2\Omega^{n-q}$ with $i_{\nabla h}\phi\in L^2\Omega^{n-q-1}$, and so in particular with $|\nabla h|\phi\in L^2\Omega^{n-q}$ (since $|i_{\nabla h}w|_{L^2,x}\le|\nabla h|_x|w|_{L^2,x}$), giving the result. The case $r=q$ follows by duality. $\square$

3. A criterion for positivity.

Let $\nu_0(x)\le\nu_1(x)\le\cdots\le\nu_{n-1}(x)$ be the eigenvalues of the Hessian ∇^2h of h at x. For $\delta>0$ and $q=1,\cdots,n-1$ consider the following condition on h:

CONDITION $A(q)^\delta$. $\nu_0(x)+\cdots+\nu_{n-q-1}(x)-\nu_{n-q}(x)-\cdots-\nu_{n-1}(x)\ge\delta$ for all $x\in M$.

PROPOSITION 3.1. *Suppose h satisfies $A(q)^\delta$ for some $\delta>0$. Then, for $\tau\ge 0$*

$$\lambda_\tau^q\ge\lambda_0^q+\tau\delta\ge\tau\delta \tag{3.2}$$

and

$$\lambda_{-\tau}^{n-q} \geq \lambda_0^{n-q} + \tau\delta \geq \tau\delta . \tag{3.3}$$

PROOF. Let $e_0, \cdots, e_{n-1} \in T_x^*M$ be a complete orthonormal set of eigenvectors of $(\nabla^2 h)_x$ with eigenvalues $\nu_0(x), \cdots, \nu_{n-1}(x)$. For a q-form w we can write

$$w_x = \sum_{\substack{I=(i_1,\cdots,i_q)\\ i_1<\cdots<i_q}} a_{i_1\cdots i_q} e_{i_1} \wedge \cdots \wedge e_{i_q} .$$

Then

$$\begin{aligned}
&\langle -\tau(\Delta h + 2(d\wedge)^q(\nabla^2 h)^*)w, w\rangle_x \\
&= \tau \sum_I \left\langle \left(\sum_{i=0}^{n-1} \nu_i(x)\right)(a_{i_1\cdots i_q} e_{i_1} \wedge \cdots \wedge e_{i_q}) - 2\left(\sum_{j=1}^{q} \nu_{i_j}\right) a_{i_1\cdots i_q} e_{i_1} \wedge \cdots \wedge e_{i_q}, w\right\rangle_x \\
&= \tau\left(\left(\sum_{i=0}^{n-1} \nu_i(x)\right)|w|_x^2 - 2\left\{\frac{\sum_I (\sum_{j=1}^q \nu_{i_j})|a_{i_1\cdots i_q}|^2}{\sum_I |a_{i_1\cdots i_q}|^2}\right\}|w|_x^2\right)
\end{aligned}$$

(since $|w|_x^2 = \sum_I |a_{i_1\cdots i_q}|^2$)

$$\begin{aligned}
&\geq \tau\left(\sum_{i=0}^{n-1} \nu_i(x) - 2\sum_{j=n-q}^{n-1} \nu_j(x)\right)|w|_x^2 \\
&= \tau\left(\sum_{i=0}^{n-q-1} \nu_i(x) - \sum_{j=n-q}^{n-1} \nu_j(x)\right)|w|_x^2 \\
&\geq \tau\delta |w|_x^2 ,
\end{aligned}$$

by hypothesis. The inequality (3.2) follows immediately by (2.3).

For $\tau < 0$, note that we can equally well replace h by $-h$ and keep $\tau > 0$. The eigenvalues are then $-\nu_{n-1}(x) \leq \cdots \leq -\nu_0(x)$ so that condition $A(q)^\delta$ for h is condition $A(n-q)^\delta$ for $-h$. Alternatively this follows by the duality formula $*\Box_\tau = \Box_{-\tau}*$. □

NOTE. $A(q)^\delta$ holds if $\nu_0(x) \geq \delta$ and

$$\nu_{n-1}(x) \leq \left(\frac{n-q-1}{q}\right)\nu_0(x)$$

for all $x \in M$. It can never hold in the middle dimension, for even n, i.e. for $n = 2k$ and $q = k$.

4. A special case.

We are particularly interested in $h(x) = \pm r(x)$ where $r(x)$ is the distance of x from some fixed point p of a simply connected negatively curved manifold. The eigenvalues $\nu_i(x)$ occurring in condition $A(q)^\delta$ can be estimated using the Hessian comparison theorem [17]. For a simply connected manifold of constant curvature $-c^2$ the Hessian is:

$$\nabla^2 r = (c \coth cr)H + 0 \qquad (r>0)$$

[17, p. 34, 35] when decomposed into radial and transverse components, with H the identity transformation on the transverse component (i.e. the orthogonal complement of ∇r). Thus the comparison theorem gives us that if $\pi_1(M)=0$ and $-c_1^2 \le \mathrm{Sect}_M \le -c_2^2 < 0$ then for $h(x)=r(x)$:

$$v_0(x)=0\,,$$

$$c_2 \coth c_2 r \le v_j(x) \le c_1 \coth c_1 r\,, \qquad 1 \le j \le n-1\,,$$

if $r(x) \ne 0$. Near $r(x)=0$, we need to do some smoothing. For this define a C^∞ map $f\colon [0,\infty) \to [0,\infty)$ with $f(0)=0$, $f'(0)=0$ and

$$f''(r) = \begin{cases} 1 & 0 \le r \le a \\ 0 & b \le r < \infty \end{cases}$$

for numbers $0<a<b<\infty$ to be chosen later. Thus

$$f(r) = \begin{cases} \frac{1}{2} r^2 & 0 \le r \le a \\ \alpha r + \beta & b \le r < \infty \end{cases}$$

with $\alpha>0$, $\beta>0$. Note that if $ac \coth cb \ge 1$ then

(4.1) $$f''(r) \le f'(r) c \coth cr\,.$$

Indeed for $0<r<a$ we know $f'(r) c \coth cr = rc \coth cr \ge 1 = f''(r)$ and for $a<cr<cb$

$$f'(r) \coth cr > ac \coth cr > ac \coth cb \ge 1 \ge f''(r)$$

while for $b<r$, $f''=0$.

Lemma 4.2. *Suppose $\pi_1(M)=0$ and $-1 \le \mathrm{Sect}_M < -1+\varepsilon$ where*

$$\varepsilon = \frac{(n-1)(n-2q-1)}{(n-q-1)^2}$$

for $2q<n-1$. Set $h(x)=f(r(x))$, for f as above with $a=(1-\varepsilon)^{-1/2}$ and $b>a$. Then h satisfies $A(q)^\delta$ for some $\delta>0$.

Proof. Since $\nabla h = f'(r) \nabla r$ and

$$\nabla^2 h = f''(r) dr \otimes dr + f'(r) \nabla^2 r$$

each v_j satisfies

$$v_j(x) \ge \min\{f''(r), \sqrt{1-\varepsilon} f'(r) \coth(\sqrt{1-\varepsilon} r)\} \ge f''(r)$$

by (4.1), since $a\sqrt{1-\varepsilon} \coth(\sqrt{1-\varepsilon} b) \ge \coth((\sqrt{1-\varepsilon}) b) \ge 1$. Thus $v_0(x)=f''(r)$ and

$$f'(r)a\sqrt{1-\varepsilon}\coth a\sqrt{1-\varepsilon}r \le \nu_j(x) \le f'(r)\coth r\,, \qquad j=1,\cdots,n-1\,.$$

This gives

$$\begin{aligned}\nu_0+\cdots+\nu_{n-q-1}-\nu_{n-1}-\cdots-\nu_{n-q} &\\ \ge f''(r)-qf'(r)\coth r+(n-q-1)f'(r)\sqrt{1-\varepsilon}\coth\sqrt{1-\varepsilon}r &\\ \ge f''(r)+f'(r)\coth r((n-q-1)\sqrt{1-\varepsilon}-q) &\\ \stackrel{\text{def}}{=} D(r)\,. &\end{aligned}$$

If $(n-q-1)\sqrt{1-\varepsilon}>q$ then clearly $\delta\stackrel{\text{def}}{=}\inf_{r>0}D(r)>0$. □

THEOREM 4.3. *Suppose* $\pi_1(M)=0$ *and* $-1\le \mathrm{Sect}_M\le -1+\varepsilon$, *where* $\varepsilon<1-q^2/(n-q-1)^2$. *Take* $h(x)=f(r(x))$ *as above. Then if* $\tau>0$ *and* $2q<n-1$,

(i) $\lambda_\tau^q>0$,

(ii) *if* $e^{\tau r}\phi\in L^2\Omega^q$ *and* $d\phi=0$, *then* $\phi=d\theta$ *where* $e^{\tau r}\theta\in L^2\Omega^{q-1}$. *In particular any class* $[\phi]$ *in* L^2H^q *or in* L^2H^{n-q} *with* $e^{\tau r}\phi\in L^2$ *is trivial,*

(iii) *if* $e^{-\tau r}\phi\in L^2\Omega^{n-q}$ *and* $d\phi=0$, *then* $\phi=d\theta$ *where* $e^{-\tau r}\theta\in L^2\Omega^{n-q-1}$,

(iv) *for* $p=q$ *or* $n-q$ *there is no non-zero harmonic* $\phi\in L^2\Omega^p$.

PROOF. By Proposition 3.1 and Lemma 4.2 we have $\lambda_\tau^q(f)\ge\tau\delta$ for some $\delta>0$, for all $\tau>0$. Parts (i), (ii), (iii) are immediate from Proposition 2.8. Also, by duality $\lambda_\tau^{n-q}(-f)=\lambda_{-\tau}^{n-q}(f)\ge\tau\delta$. Thus $\tau^2/\lambda_\tau^{n-q}(-f)\le\tau\delta\to 0$ as $\tau\downarrow 0$, and Proposition 2.7 shows that there is no non-zero L^2 harmonic $(n-q)$-form with $i_{\nabla h}w\in L^2$. Since $|i_{\nabla h}w|_{L^2}\le|\nabla h|_{L^\infty}|w|_{L^2}\le C|w|_{L^2}$ for some constant C, there is no non-zero L^2 harmonic $(n-q)$-form, and so by duality no non-zero L^2 harmonic q form. □

In contrast to (iv) above, Anderson [1] has shown that L^2 hamonic forms can exist in dimension q if the pinching condition is relaxed past $-1\le\mathrm{Sect}_M<(-1)/(n-2q)^2$. In particular, for $q=1$ both upper bounds for the curvature equal $(-1)/(n-2)$, showing that in this dimension Anderson's and our work give optimal results.

COROLLARY 4.4. *For* $n=2k$, $\pi_1(M)=0$, *and*

$$-1\le\mathrm{Sect}_M<-\frac{(k-1)^2}{k^2}\,,$$

there are no non-zero L^2 *harmonic q-forms for* $q\ne k$.

THEOREM 4.5. *For* $n=2k$ *and* $\bar{M}$ *a compact smooth n-manifold with*

$$-1\le\mathrm{Sect}_{\bar{M}}<-\frac{(k-1)}{k^2}\,,$$

we have

$$(-1)^k\chi(\bar{M})\ge 0$$

where $\chi(\bar{M})$ is the Euler characteristic of $\bar{M}$.

PROOF. As in [9], this follows immediately from Atiyah's L^2 index theorem [2] and the last corollary. Namely, define the L^2 Betti numbers on the universal cover M of $\bar{M}$ by

$$\beta_2^q = \int_F p^q(x, x)dx$$

where F is a fundamental domain for $\bar{M}$ in M, and $p^q(x, y)$ is the kernel of the projection of $L^2\Omega^q$ onto the kernel of Δ^q. Set $\chi_2(M) = \sum(-1)^q \beta_2^q$. Atiyah's theorem implies $\chi_2(M) = \chi(\bar{M})$. Under the hypothesis of the corollary, we have $\chi(\bar{M}) = (-1)^k \beta_2^k$. □

In $n = 4l$ and $\bar{M}$ has non-zero signature, then the L^2 index theorem applied to the signature operator gives $\beta_2^{2l} \neq 0$, and so $\chi(\bar{M}) > 0$ in this case.

In [12] Donnelly and Xavier proved the following:

THEOREM 4.6 (Donnelly and Xavier). *Suppose M is simply connected with pinched sectional curvatures $-1 \leq \mathrm{Sect}_M \leq -1 + \varepsilon$ for $0 \leq \varepsilon < 1$. If $q < \frac{1}{2}(n-1)$ then*

$$\lambda_0^q \geq \frac{1}{4}[(n-1)\sqrt{(1-\varepsilon)} - 2q]^2$$

for $\varepsilon < 1 - 4q^2/(n-1)^2$.

The methods they used depend on an interesting integral inequality and, like ours, on estimates of the eigenvalues of the Hessian of the distance function r. Their results give vanishing of L^2 harmonic forms except on the middle dimension when n is even and also vanishing of L^2H^q for $q < \frac{1}{2}(n-1)$.

Our pinching is given by $-1 \leq \mathrm{Sect}_M \leq -1 + \varepsilon$ where $\varepsilon < 1 - q^2/(n-q-1)^2$. This is more relaxed than Donnelly and Xavier's for all $0 < q < \frac{1}{2}(n-1)$. We get vanishing of L^2 harmonic forms away from the middle dimension and growth conditions on non-zero L^2 cohomology classes: however we do not obtain vanishing of L^2H^q.

Donnelly and Xavier give our Theorem 4.5 with their tighter pinching. They point out that the stronger result $(-1)^k \chi(\bar{M}) > 0$ has been proved by Bourguignon and Karcher [5] under the pinching $\varepsilon < 3/(n+1)$, using the classical Chern-Gauss-Bonnet theorem, and they observe that this is a more relaxed pinching than theirs. However it is tighter than ours.

Essentially our pinching is more relaxed but we are getting somewhat weaker conclusions than [5] or [12]. The growth conditions in the next section require no pinching.

5. Growth conditions via semigroup domination.

Let $R^q \in \mathrm{End}(\Omega^q)$ be the curvature term in the Weitzenböck formula $\Delta = \nabla^*\nabla + R^q$ for the Laplacian on q-forms. In particular $R^1 = \mathrm{Ric}$. Consider the lower bound $\gamma_\tau^q: M \to \boldsymbol{R}$:

$$\gamma_\tau^q(x) = \inf\{\langle R^q(V), V\rangle_x - 2\tau\langle (d\bigwedge)^q(\nabla^2 h)V, V\rangle_x : V \in \bigwedge^q T_xM \text{ and } |V|_x = 1\}$$

and let $C_\tau^q = \inf\{\gamma_\tau^q(x) : x \in M\} \geq -\infty$. Assume $C_\tau^1 > -\infty$. According to Bakry [3], this ensures that the Brownian motion with drift $\tau\nabla h$ on M exists for all time or equivalently that the probabilistically defined semigroup (the minimal semigroup) determined by $-\frac{1}{2}\Delta + \tau\nabla h$ maps the constant function 1 to itself.

From [13, §5 Remark (2)] there is the semigroup domination

$$|Q_t^q(\phi)(V)| \leq e^{-(1/2)C_\tau^q t} Q_t^0(|\phi|)(x)|V|_x$$

for $\phi \in L^2\Omega^q(\mu_\tau)$, $V \in \bigwedge^q T_xM$, where $\{Q_t^q : t \geq 0\}$ is the semigroup $e^{t\bar{\Delta}_\tau}$ on $L^2\Omega^q(\mu_\tau)$. In fact in [14] this is given for Q_t^q and Q_t^0 the probabilistically defined semigroups on the spaces of L^∞ forms. However these semigroups agree on the intersection of their domains given our assumption on C_τ^1, which ensures that the relevant stochastic process exists for all time, and the assumption that $C_\tau^\infty > -\infty$ (cf. [14, Ch. IV, Prop. 1A]; the uniform boundedness of Q_t is not needed there, as pointed out to us by X.-M. Li). As in [15] for $\tau = 0$, we have

PROPOSITION 5.1. *If $C_\tau^1 > -\infty$ then*

$$\lambda_\tau^q \geq C_\tau^q + \lambda_\tau^0 .$$

Take $h(x) = -f(r(x))$ for f as in §4 so that $f(r) = \frac{1}{2}r^2$ for $0 \leq r \leq a$ and $f(r) = \alpha r + \beta$ if $r \geq b$. From the proof of Lemma 4.2 we see

$$\gamma_\tau^q(x) \geq C_0^q + 2\tau q f''(r(x)), \qquad x \in M .$$

Thus

$$\gamma_\tau^q(x) \geq C_0^q + 2\tau q, \qquad 0 \leq r(x) \leq a .$$

Also if $\mathrm{Sect}_M \leq -k^2 < 0$, as in §4 we see

$$\begin{aligned}\gamma_\tau^q(x) &\geq C_0^q + 2\tau f''(r(x)) + 2\tau f'(r(x))(q-1)k\coth kr(x)\\ &\geq C_0^q + 2\tau(q-1)ak\coth ak\end{aligned}$$

if $r(x) \geq a$. Thus, for this h,

$$\lambda_\tau^q \geq C_0^q + \lambda_\tau^0 + \tau\min\{2q, 2(q-1)ak\coth ak\} . \tag{5.2}$$

Using this we can get growth conditions on harmonic forms without pinching:

THEOREM 5.3. *Suppose* $\pi_1(M)=0$, $\mathrm{Sect}_M \le -k^2<0$ *for some k and* Ric_M *is bounded below. If* $q\neq n,\ n-1$ *and* $\tau> -\inf R^{n-q}/2(n-q)$ *then*

(i) *For* $p=q$ *or* $n-q$ *if* $\phi\in L^2\Omega^p$ *has* $\Delta\phi=0$ *and* $e^{\tau r}\phi\in L^2$, *then* $\phi=0$.

(ii) *If* $e^{\tau r}\phi\in L^2\Omega^q$ *and* $d\phi=0$, *then* $\phi=d\theta$ *where* $e^{\tau r}\theta\in L^2\Omega^{q-1}$.

(iii) *If* $e^{-\tau r}\phi\in L^2\Omega^{n-q}$ *and* $d\phi=0$, *then* $\phi=d\theta$ *where* $e^{-\tau r}\theta\in L^2\Omega^{n-q-1}$.

PROOF. The case $q=0$ is trivial. Equation (5.2) plus duality gives

$$\lambda^{n-q}_{-\tau}\ge C^q_0+\lambda^0_\tau+\tau\min\{2q,\,2(q-1)ak\coth ak\}=C^q_0+\lambda^0_\tau+\tau 2q$$

if $q\neq 1$ and a is sufficiently large. Thus if we change q to $n-q$ and change the sign of h to take $h(x)=f(r(x))$ we get $\lambda^q_\tau\ge C^{n-q}_0+\lambda^0_\tau+\tau 2(n-q)$, $q\neq n-1$, whence $\lambda^q_\tau>0$. Now apply Proposition 2.8. □

COROLLARY 5.4. *If* $\pi_1(M)=0$, $\mathrm{Sect}_M\le -k^2<0$ *for some k and* R^q *is bounded below for each q, there exists* $\tau\in[0,\infty)$ *such that any harmonic form* ϕ *with* $e^{\tau r}\phi\in L^2$ *vanishes identically.*

As pointed out in [9], harmonic $(n/2)$-forms on $\boldsymbol{R}^n$ restrict to give L^2 harmonic forms on hyperbolic n-space, thought of as the unit ball. It is easy to check that these forms can decay exponentially in the distance, but not fast enough to violate Theorem 5.3. Anderson's examples of L^2 harmonic forms outside his pinching range involve similar forms of slow exponential decay. This indicates that existence proofs for L^2 harmonic forms in the middle dimension (which is necessary to show that the Euler characteristic of $\bar{M}$ is non-zero) may involve delicate estimates.

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