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An Example of Nonsymmetric Dipolarizations in a Lie Algebra

Shaoqiang DENG and Soji KANEYUKI

Nankai University and Sophia University

Introduction.

Let $\mathfrak{g}$ be a real Lie algebra, and $\mathfrak{g}^{\pm}$ be two subalgebras of $\mathfrak{g}$ and f be a linear form on $\mathfrak{g}$. We say that $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ is a *dipolarization* in $\mathfrak{g}$ if the following conditions are satisfied:

(D1) $\mathfrak{g}=\mathfrak{g}^+ + \mathfrak{g}^-$,

(D2) Let $\mathfrak{h}=\mathfrak{g}^+ \cap \mathfrak{g}^-$. Then $f([X, \mathfrak{g}])=0$ iff $X \in \mathfrak{h}$.

(D3) $f([\mathfrak{g}^+, \mathfrak{g}^+])=f([\mathfrak{g}^-, \mathfrak{g}^-])=0$.

The notion of a dipolarization was introduced by S. Kaneyuki [1], [2], in order to describe, algebraically, a homogeneous symplectic structure with two Lagrangian foliations—a homogeneous parakähler structure—on a manifold. A dipolarization $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$is called *symmetric*, if the two subalgebras $\mathfrak{g}^+$ and $\mathfrak{g}^-$ are isomorphic. Otherwise it is called *nonsymmetric*. In [1], [2], S. Kaneyuki has constructed a large class of symmetric dipolarizations in a semisimple Lie algebra, in terms of its gradations. The purpose of this note is to construct an example of nonsymmetric dipolarizations in the Lie algebra of upper triangular matrices. One can pose the problems: Are there nonsymmetric dipolarizations in semisimple Lie algebras? If so, classify them.

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Let $\mathfrak{g}$ denote the Lie algebra $\mathfrak{t}(n, \boldsymbol{R})$ of $n \times n$ real upper triangular matrices. Let E_{ij} be the $n \times n$ matrix whose (k, l) entry is $\delta_{ik}\delta_{jl}$, and let $\mathfrak{g}_{ij}=\boldsymbol{R}E_{ij}$ $(1 \leq i \leq j \leq n)$. Consider the following subspaces of $\mathfrak{g}$

(1) $$\mathfrak{g}_i=\sum_{k=1}^{n-i} \mathfrak{g}_{k,k+i}, \qquad 0 \leq i \leq n-1 .$$

Then $\mathfrak{g}$ can be written as a graded Lie algebra:

(2) $$\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1+\cdots+\mathfrak{g}_{n-1} .$$

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(3) $$\mathfrak{g}^+=\mathfrak{g}_0+\mathfrak{g}_2+\cdots+\mathfrak{g}_{n-1}\,,$$
$$\mathfrak{g}^-=\boldsymbol{R}E+\mathfrak{g}_1+\cdots+\mathfrak{g}_{n-1}\,,$$

where E denotes the unit matrix of degree n. Obviously $\mathfrak{g}^\pm$ satisfy (D1). Let $\langle\ ,\ \rangle$ be an inner product on $\mathfrak{g}$ with respect to which $\{E_{ij}: 1\leq i\leq j\leq n\}$ is an orthonormal basis of $\mathfrak{g}$. We define a linear form on $\mathfrak{g}$ by putting

(4) $$f(X)=\sum_{i=1}^{n}\langle X, E_{ii}\rangle+\sum_{i=1}^{n-1}\langle X, E_{i,i+1}\rangle,\qquad X\in\mathfrak{g}\,.$$

THEOREM. *Let $\mathfrak{g}=\mathfrak{t}(n,\boldsymbol{R})$ be the Lie algebra of $n\times n$ real upper triangular matrices, and let $\mathfrak{g}^\pm$ and f be the ones given in* (3) *and* (4). *Then $\{\mathfrak{g}^+,\mathfrak{g}^-,f\}$ is a dipolarization in $\mathfrak{g}$. Furthermore, if $n\geq 4$, then this dipolarization is nonsymmetric.*

We put $\mathfrak{g}^{(2)}=\sum_{k=2}^{n-1}\mathfrak{g}_k$. Then we have $[\mathfrak{g}^\varepsilon,\mathfrak{g}^\varepsilon]\subset\mathfrak{g}^{(2)}$, where ε denotes $+$ or $-$. Therefore, in view of (4), we see that f satisfies (D3). Let $\mathfrak{h}=\mathfrak{g}^+\cap\mathfrak{g}^-$. Then $\mathfrak{h}=\boldsymbol{R}E+\mathfrak{g}_2+\cdots+\mathfrak{g}_{n-1}$, and so we have $[\mathfrak{h},\mathfrak{g}]\subset\mathfrak{g}^{(2)}$ and hence

(5) $$f([\mathfrak{h},\mathfrak{g}])=0\,.$$

LEMMA 1. *Let $X\in\mathfrak{g}$. Suppose that $f([X,\mathfrak{g}])=0$. Then $X\in\mathfrak{h}$.*

PROOF. We write $X=\sum_{i=0}^{n-1}X_i$, *where* $X_i\in\mathfrak{g}_i$ $(0\leq i\leq n-1)$. By the assumption we have

(6) $$f([X,\mathfrak{g}_0])=f([X,\mathfrak{g}_1])=0\,.$$

We also have $[X,\mathfrak{g}_0]\equiv[X_1,\mathfrak{g}_0]\bmod\mathfrak{g}^{(2)}$ and $[X,\mathfrak{g}_1]\equiv[X_0,\mathfrak{g}_1]\bmod\mathfrak{g}^{(2)}$. Therefore (6) implies that

(7) $$f([X_1,\mathfrak{g}_0])=f([X_0,\mathfrak{g}_1])=0\,.$$

Writing $X_1=\sum_{k=1}^{n-1}x_{k,k+1}E_{k,k+1}$, we get the equalities $[X_1,E_{11}]=-x_{12}E_{12}$ and $[X_1,E_{ii}]=x_{i-1,i}E_{i-1,i}-x_{i,i+1}E_{i,i+1}$ $(2\leq i\leq n-1)$. Applying f to both equalities, we have from (7) that $x_{12}=x_{23}=\cdots=x_{n-1,n}=0$, or equivalently $X_1=0$. Now let $X_0=\sum_{k=1}^{n}x_{kk}E_{kk}$. Then, for $1\leq i\leq n-1$, one has $[X_0,E_{i,i+1}]=(x_{ii}-x_{i+1,i+1})E_{i,i+1}$. Applying f to this equality and considering (7), we have $x_{11}=x_{22}=\cdots=x_{nn}$. We have thus proved $X\in\mathfrak{h}$. q.e.d.

Let $\mathfrak{z}(\mathfrak{g}^\pm)$ denote the center of $\mathfrak{g}^\pm$, respectively.

LEMMA 2. *If $n\geq 4$, then $\mathfrak{z}(\mathfrak{g}^+)=\boldsymbol{R}E$.*

PROOF. Note that $\mathfrak{g}^+=\mathfrak{g}_0+\mathfrak{g}^{(2)}$. Choose an arbitrary element $X\in\mathfrak{z}(\mathfrak{g}^+)$ and write it in the form $X=X_0+X_1$, where $X_0\in\mathfrak{g}_0$ and $X_1\in\mathfrak{g}^{(2)}$. We then have $[X_1,\mathfrak{g}_0]=[X,\mathfrak{g}_0]=0$ and consequently $[X_1,E_{ii}]=0$ for $1\leq i\leq n$. We write $X_1=\sum_{k+2\leq h}x_{kh}E_{kh}$. Substituting this into the above equality, we have

(8) $$0=[X_1, E_{ii}]=\sum_{k\le i-2} x_{ki}E_{ki}-\sum_{i+2\le k} x_{ik}E_{ik}\,, \qquad 1\le i\le n;$$

here the first term (resp. the second term) of the third member of (8) does not appear provided that $i=1, 2$ (resp. $i=n-1, n$). It follows from (8) that $x_{kh}=0$ for $h\ge k+2$, $1\le k\le n-2$, that is, $X_1=0$. Therefore $X=X_0\in\mathfrak{g}_0$. We then have $[X_0, \mathfrak{g}^{(2)}]=[X_0, \mathfrak{g}^+]=0$, and consequently $[X_0, E_{ij}]=0$ for $j\ge i+2$, $1\le i\le n-2$. If we write $X_0=\sum_{k=1}^{n} x_{kk}E_{kk}$, then the above equality implies that

(9) $$0=[X_0, E_{ij}]=(x_{ii}-x_{jj})E_{ij}\,, \qquad 1\le i\le n-2,\quad j\ge i+2\,.$$

Therefore under the assumption $n\ge 4$, we have that all x_{ii}'s are identical, and hence $X=X_0=x_{11}E$. q.e.d.

PROOF OF THE THEOREM. We have already seen that $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ satisfies (D1) and (D3); (D2) is also satisfied by Lemma 1 and (5). Since $\mathfrak{g}_{n-1}$ is in the center of the nilpotent graded Lie algebra $\mathfrak{g}_1+\cdots+\mathfrak{g}_{n-1}$, we have $\boldsymbol{R}E+\mathfrak{g}_{n-1}\subset\mathfrak{z}(\mathfrak{g}^-)$. And hence $\dim\mathfrak{z}(\mathfrak{g}^-)\ge 2$. Therefore, taking Lemma 2 into account, we see that $\mathfrak{g}^+$ is not isomorphic to $\mathfrak{g}^-$. q.e.d.

References

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Present Addresses:
SHAOQIANG DENG
DEPARTMENT OF MATHEMATICS, NANKAI UNIVERSITY, TIANJING, P.R. CHINA

SOJI KANEYUKI
DEPARTMENT OF MATHEMATICS, SOPHIA UNIVERSITY, TOKYO 102, JAPAN