Tokyo J. Math.
Vol. 16, No. 2, 1993

Positive Characteristic Finite Generation of Symbolic Rees Algebras and Roberts' Counterexamples to the Fourteenth Problem of Hilbert

Kazuhiko KURANO

Tokyo Metropolitan University

1. Introduction.

In this paper we shall study the problem whether the symbolic Rees algebras occurring in Roberts' new counterexamples [10] to the 14th problem of Hilbert are Noetherian rings or not, in the case where the characteristic of the ground field is positive.

For each prime ideal Q in a commutative Noetherian ring R we put $R_s(Q)=\sum_{n\geq 0}Q^{(n)}\xi^n$ and call it the *symbolic Rees algebra* of Q, where $Q^{(n)}$ denotes the n^{th} symbolic power of Q and ξ is an indeterminate over R. The determination of finite generation in $R_s(Q)$ is one of the central problems in both commutative algebra and algebraic geometry (cf. [8], [7], [9], [10], [3], and [4]). It is generally a quite hard problem but, according to the recent research [4], in the positive characteristic case there might be more chances for $R_s(Q)$ to be a Noetherian ring than in the case where the characteristic is zero.

Originally this kind of question was raised in 1985 by Cowsik [3], asking if $R_s(Q)$ are always Noetherian especially when the base ring R is regular (and local). However, as is now well known, this is not true in general. Three counterexamples [9], [10], and [4] are already known. In this paper we are particularly interested in the second example [10] due to Roberts, so we would like to cite here his examples explicitly.

Let F be a field and $R_0=F[x, y, z]$ be a polynomial ring with three indeterminates over F. $R=R_0[S, T, U, V]$ and $R_0[W]$ denote polynomial rings over R_0. For each positive integer t let $\varphi: R\to R_0[W]$ be the homomorphism of R_0-algebras defined by $\varphi(S)=x^{t+1}W$, $\varphi(T)=y^{t+1}W$, $\varphi(U)=z^{t+1}W$ and $\varphi(V)=(xyz)^tW$. We put $Q=\text{Ker}(\varphi)$. Let $R_1=R_0\cdot S+R_0\cdot T+R_0\cdot U+R_0\cdot V$ be a free R_0-module and let $\phi: R_1\to R_0$ be an R_0-linear map such that $\phi(S)=x^{t+1}$, $\phi(T)=y^{t+1}$, $\phi(U)=z^{t+1}$ and $\phi(V)=(xyz)^t$. We denote by M the kernel of ϕ and put $S(M)=R_0[M]$ $(\subseteq R)$. Let $\overline{S(M)}$ be the ideal-transform of

Received September 14, 1992
Revised February 26, 1993
Partially supported by Grant-in-Aid for Co-operation Research

$S(M)$ with respect to the ideal $(x, y, z)R_0$, that is $\overline{S(M)}=\{f \in R \mid (x, y, z)^m \cdot f \subseteq S(M)$ for some $m \geq 0\}$.

Then it is not hard to check that $\overline{S(M)}$ is a homomorphic image of $R_s(Q)$ and that $\overline{S(M)}=R \cap K$, where K is the quotient field of $S(M)$. Roberts proved the following epochmaking theorem:

THEOREM 1.1 ([10]). *Suppose* $\mathrm{ch}(F)=0$ *and* $t \geq 2$. *Then* $\overline{S(M)}$ *is not a Noetherian ring so that* $R_s(Q)$ *can not be a finitely generated* R*-algebra.*

Hence $\overline{S(M)}$ and $R_s(Q)$ are counterexamples to the 14[th] problem of Hilbert and Cowsik's question, respectively.

In our paper we want to know what happens on Roberts' examples $R_s(Q)$ if $\mathrm{ch}(F)>0$, for there are still no counterexample to Cowsik's question in the positive characteristic case. But, as we noted before, if $\mathrm{ch}(F)>0$ there are more possibilities for $R_s(Q)$ to be Noetherian rings (see, e.g., exploration in [4]). In this context it is quite challenging to explore Roberts' example in the case where $\mathrm{ch}(F)>0$, asking whether $R_s(Q)$ are Noetherian rings or not. The whole paper is devoted to studying this question and our results will actually claim that $R_s(Q)$ are finitely generated R-algebras if $\mathrm{ch}(F)>0$ and $t \geq 2$ (see Theorem 2.1). Therefore $\overline{S(M)}$ is a Noetherian ring too in that case. We shall show also that $\overline{S(M)}$ is Noetherian for $t=1$, whatever $\mathrm{ch}(F)$ is. Thus we conclude that Roberts' counterexamples are in this sense the best possible ones, and to the author's slight disappointment, Cowsik's question remains open in the positive characteristic case.

Let us now turn to the organization of this paper. Our method in studying the finite generation of symbolic Rees algebras is based on the following result, whose proof shall be given in different paper [5].

LEMMA 1.2 (Theorem 3.3 in [5]). *Let* R_0 *be a Noetherian integral domain containing a field. Let* t_1, t_2, t_3 *and* y *be elements in* R_0 *and assume that* t_1, t_2, t_3 *forms an* R_0*-regular sequence. Let* $R=R_0[S, T, U, V]$ *and* $R_0[W]$ *be polynomial rings over* R_0 *and* $\varphi : R \to R_0[W]$ *denotes the homomorphism of* R_0*-algebras defined by* $\varphi(S)=t_1 W$, $\varphi(T)=t_2 W$, $\varphi(U)=t_3 W$ *and* $\varphi(V)=yW$. *Put* $Q=\mathrm{Ker}(\varphi)$. *Then* $R_s(Q)$ *is a finitely generated* R*-algebra if for some* $l \geq 1$ *the symbolic power* $Q^{(l)}$ *contains a polynomial which is monic in* V *and of degree* l.

Let $\alpha, \beta \geq 1$ be integers and, in the same situation as in Roberts' theorem [10], let $\varphi : R \to R_0[W]$ be the homomorphism of R_0-algebras such that $\varphi(S)=x^\alpha W$, $\varphi(T)=y^\alpha W$, $\varphi(U)=z^\alpha W$ and $\varphi(V)=(xyz)^\beta W$. We put $Q=\mathrm{Ker}(\varphi)$. Then x^α, y^α, z^α obviously forms an R_0-regular sequence. In section 2, assuming $\mathrm{ch}(F)=p>0$, we shall give (in terms of α, β and p) a necessary and sufficient condition (Theorem 2.1) for the existence of integers $l \geq 1$ for which $Q^{(l)}$ contains a polynomial satisfying the requirements stated in Lemma 1.2. Then it will directly follow from the criterion that Roberts' examples $R_s(Q)$ in Theorem 1.1 are always finitely generated R-algebras if $\mathrm{ch}(F)>0$ and $t \geq 2$, that is the case $\beta=\alpha+1 \geq 3$ in our notation. Hence $\overline{S(M)}$ is a Noetherian ring too in that case.

However, the above method does not work if $\alpha=1$ and $\beta=2$ (except for the case of $\mathrm{ch}(F)=2$). Therefore, to see the Noetherian property of $\overline{S(M)}$ for $t=1$, we need a series of direct computation, which we shall perform in section 3.

Throughout this paper, we shall maintain the same notation as is given in this section unless otherwise specified. We assume that all rings are commutative Noetherian and with 1. The author thanks the referee for several advices.

2. When does $\overline{S^l(M)}$ contain a monic polynomial?

The aim of this section is to give a necessary and sufficient condition for the existence of a monic polynomial satisfying the requirements stated in Lemma 1.2.

Let F be a field. Now we think $R=\sum_{l\geq 0}R_l=F[x, y, z, S, T, U, V]$ the graded ring with $\deg x=\deg y=\deg z=0$, $\deg S=\deg T=\deg U=\deg V=1$. As usual we denote by L_i the homogeneous component of L in degree i for a graded R-module L. Let α and β be positive integers and let $\varphi: R=F[x, y, z, S, T, U, V]\to F[x, y, z, W]$ be the ring homomorphism of polynomial rings over F defined by $\varphi(S)=x^\alpha W$, $\varphi(T)=y^\alpha W$, $\varphi(U)=z^\alpha W$ and $\varphi(V)=(xyz)^\beta W$. Then $Q=\mathrm{Ker}(\varphi)$ is a homogeneous prime ideal of R of height 3. Consider the symbolic Rees algebra $R_s(Q)=\sum_{n\geq 0}Q^{(n)}\xi^n\subseteq R[\xi]$, where ξ is an indeterminate.

Our goal in this section is the following theorem:

THEOREM 2.1. *Suppose that the characteristic of F is $p>0$. Then there exists a positive integer l for which $Q^{(l)}$ contains a monic polynomial in V of degree l if and only if $\beta/\alpha\geq(2p-1)/3p$ (resp. $\beta/\alpha\geq 2/3$) is satisfied in the case of $p\equiv 2$ (3) (resp. $p\not\equiv 2$ (3)).*

Therefore, when this is the case, $R_s(Q)$ is Noetherian.

First of all, we note that

REMARK 2.2. If $\beta\geq\alpha$, then Q contains $V-x^{\beta-\alpha}y^\alpha z^\alpha S$ over any field F. Conversely suppose $\beta<\alpha$. Then it is easily checked that Q contains no monic polynomial in V of degree 1 over an arbitrary field F.

Next assume that the characteristic of F is zero. If there exists a positive integer l for which $Q^{(l)}$ contains a monic polynomial in V of degree l, then it is well known that Q itself contains a monic polynomial in V of degree 1 (see, e.g., Remark 2.7 in [5]). Therefore, in the case of $\mathrm{ch}(F)=0$, there exists l for which $Q^{(l)}$ contains a monic polynomial in V of degree l if and only if $\beta\geq\alpha$.

Hence we assume $\beta<\alpha$ in the rest of this section.

Let $\phi: R_1\to R_0$ be the R_0-linear map of R_0-free modules defined by $S\mapsto x^\alpha$, $T\mapsto y^\alpha$, $U\mapsto z^\alpha$, $V\mapsto(xyz)^\beta$. Then $M=\mathrm{Ker}(\phi)$ is an R_0-submodule of R_1 and the following set minimally generates M as an R_0-module:

$$\{y^\alpha S-x^\alpha T,\ z^\alpha T-y^\alpha U,\ x^\alpha U-z^\alpha S,\ x^{\alpha-\beta}V-y^\beta z^\beta S,\ y^{\alpha-\beta}V-x^\beta z^\beta T,\ z^{\alpha-\beta}V-x^\beta y^\beta U\}\,.$$

We denote by $S(M)$ the graded subring of R generated by R_0 and M. $S^l(M)$ denotes the homogeneous component of $S(M)$ in degree l. Furthermore we put

$$\overline{S^l(M)}=\{f\in R_l \mid x^m f, y^m f, z^m f\in S^l(M) \text{ for a sufficiently large } m\}.$$

Then $\overline{S(M)}=\bigoplus_{l\geq 0}\overline{S^l(M)}$ is also a graded subring of R.

Before proving Theorem 2.1, we make two remarks (which are not used in this paper).

REMARK 2.3. Put $\mathfrak{m}=(x, y, z)R_0$.

1. $S^l(M)=\mathrm{Sym}^l(M)/\mathrm{H}^0_{\mathfrak{m}}(\mathrm{Sym}^l(M))$ for any $l\geq 0$.
2. $\mathrm{Sym}(M)\not\simeq S(M)$. (In the case of $\beta\geq\alpha$, we have $\mathrm{Sym}(M)\simeq S(M)$.)
3. $\overline{S^l(M)}=(S^l(M))^{**}=(\mathrm{Sym}^l(M))^{**}$ for any $l\geq 0$.

($\mathrm{Sym}(M)$ (resp. $\mathrm{Sym}^l(M)$) is the symmetric algebra (resp. the l-th symmetric module) of M. $\mathrm{H}^0_{\mathfrak{m}}(\mathrm{Sym}^l(M))$ is the 0-th local cohomology group of $\mathrm{Sym}^l(M)$. $(-)^*$ means R_0-dual, i.e., $(-)^*=\mathrm{Hom}_{R_0}(-, R_0)$.)

REMARK 2.4. In the same way as in [10], we can prove that $\overline{S(M)}$ is not Noetherian if F is a field of characteristic 0 and $1>\beta/\alpha\geq 2/3$ is satisfied.

Let us return to the proof of Theorem 2.1.

Put $P=Q\cap F[x, y, z, S, T, U]$. Since $x^\alpha, y^\alpha, z^\alpha$ forms an R_0-regular sequence, we have

$$P=I_2\begin{pmatrix} S & T & U \\ x^\alpha & y^\alpha & z^\alpha \end{pmatrix},$$

where $I_2(*)$ means the ideal generated by all the 2×2-minors. Then it is well known that $P^l=P^{(l)}$ for any positive integer l (e.g., see [1]).

We think $F[x, y, z, S, T, U]$ a graded subring of R and put $N=P_1=R_0(y^\alpha S-x^\alpha T)+R_0(z^\alpha T-y^\alpha U)+R_0(x^\alpha U-z^\alpha S)$. We define $S(N)$ and $\overline{S^l(N)}$ similarly to $S(M)$ and $\overline{S(M)}$, respectively. Precisely speaking, $S(N)=\bigoplus_l S^l(N)$ is the graded subring of $F[x, y, z, S, T, U]$ generated by R_0 and N, and we define

$$\overline{S^l(N)}=\{f\in(F[x, y, z, S, T, U])_l \mid x^m f, y^m f, z^m f\in S^l(N) \text{ for a sufficiently large } m\}.$$

REMARK 2.5. Since $P^l=P^{(l)}$, $[P^{(l)}]_l=[P^l]_l=S^l(N)=\overline{S^l(N)}$ holds.

Similarly we have $[Q^l]_l=S^l(M)$ and $[Q^{(l)}]_l=\overline{S^l(M)}$ for any l because QR_x, QR_y and QR_z are complete intersections.

When there exists l for which $Q^{(l)}$ contains a monic polynomial in V of degree l, we may assume that it is homogeneous of degree l because $Q^{(l)}$ is a homogeneous ideal. Hence $Q^{(l)}$ contains a monic polynomial in V of degree l if and only if $\overline{S^l(M)}$ contains a monic polynomial in V.

By virtue of the previous remark we have only to explore when $\overline{S^l(M)}$ contains a monic polynomial in V for each l.

Let

$$\delta : F[x, y, z, S, T, U] \to F[x, y, z, S', T', U']$$

and

$$\gamma : F[x, y, z, S, T, U, V] \to F[x, y, z, S', T', U', V']$$

be the ring homomorphisms of polynomial rings over F defined by $S \mapsto x^\alpha S'$, $T \mapsto T' + y^\alpha S'$, $U \mapsto U' + z^\alpha S'$, $V \mapsto V' + (xyz)^\beta S'$.

The next lemma characterizes the symbolic powers $P^{(l)}$ and $Q^{(l)}$ as kernels of ring homomorphisms.

LEMMA 2.6. *For any positive integer l, we have*

$$P^{(l)} = \{f \in F[x, y, z, S, T, U] \mid \delta(f) \in (T', U')^l F[x, y, z, S', T', U']\},$$

$$Q^{(l)} = \{f \in R \mid \gamma(f) \in (T', U', V')^l F[x, y, z, S', T', U', V']\}.$$

In particular, since $[Q^{(l)}]_l = \overline{S^l(M)}$ *and* $[P^{(l)}]_l = \overline{S^l(N)} = S^l(N)$, *we get*

$$\begin{aligned} \overline{S^l(N)} &= S^l(N) \\ &= \{f \in (F[x, y, z, S, T, U])_l \mid S' \text{ does not appear in } \delta(f)\}, \\ \overline{S^l(M)} &= \{f \in R_l \mid S' \text{ does not appear in } \gamma(f)\}, \\ S^l(N) &= \overline{S^l(M)} \cap F[x, y, z, S, T, U] \end{aligned}$$

for any l.

PROOF. Let $\theta : F[x, y, z, S', T', U'] \to F[x, x^{-1}, y, z, S', T', U']$ be the ring homomorphism defined by localization. Then the composite map $\theta \circ \gamma$ is also the map given by localization. Since $\theta \circ \gamma(Q)$ coincides with $(T', U', V')F[x, y, z, S', T', U', V']$, we have

$$Q^{(l)} = (T', U', V')^l F[x, y, z, S', T', U', V'] \cap F[x, y, z, S, T, U, V].$$

Other statements will be easily proved in the same way. Q.E.D.

To prove Theorem 2.1, *we assume* $\mathrm{ch}(F) = p > 0$ *and* $\alpha > \beta$ *in the rest of this section.*

It is necessary for us to prove several lemmas before proving Theorem 2.1.

LEMMA 2.7. *If there exists $l > 0$ for which $\overline{S^l(M)}$ contains a monic polynomial in V, then there exists $n > 0$ such that so does $\overline{S^{p^n}(M)}$.*

PROOF. Let $h = V^l + g_1 V^{l-1} + \cdots + g_l$ ($g_1, \cdots, g_l \in F[x, y, z, S, T, U]$) be a monic polynomial contained in $\overline{S^l(M)}$. Then S' does not appear in

$$\begin{aligned} \gamma(h) &= \gamma(V)^l + \delta(g_1)\gamma(V)^{l-1} + \cdots + \delta(g_l) \\ &= (V' + (xyz)^\beta S')^l + \delta(g_1)(V' + (xyz)^\beta S')^{l-1} + \cdots + \delta(g_l) \end{aligned}$$

by Lemma 2.6. Looking at the coefficient of $S'V'^{l-1}$, we see that $l(xyz)^{\beta}$ is contained in $(x^{\alpha}, y^{\alpha}, z^{\alpha})R_0$. Therefore l is divisible by p.

We put $l=p^n t$, where p and t are relatively prime. Note that, as we have already seen, n must be a positive integer. When $t=1$, we have nothing to prove.

When $t\geq 2$, put

$$h'=tV^{p^n}+g_1V^{p^n-1}+\cdots+g_{p^n}\in R_{p^n}.$$

Then it is easy to check that S' does not appear in $\gamma(h')$ since it does not in $\gamma(h)$. Therefore we get $h'\in\overline{S^{p^n}(M)}$ as required. Q.E.D.

We denote the Koszul relations on x^{α}, y^{α}, z^{α} by

$$A=y^{\alpha}S-x^{\alpha}T,$$

$$B=z^{\alpha}S-x^{\alpha}U,$$

$$C=z^{\alpha}T-y^{\alpha}U.$$

Then we have $S(N)=\overline{S(N)}=F[x, y, z, A, B, C]\subset F[x, y, z, S, T, U]$.

LEMMA 2.8. *If $\overline{S^{p^n}(M)}$ contains an element of the form*

$$h=V^{p^n}+g_iV^{p^n-i}+(\textit{lower degree terms in } V)$$

such that $0<i<p^n$ and $g_i\in F[x, y, z, S, T, U]$, then g_i is contained in $S^i(N)$.

PROOF. Since S' does not appear in

$$\gamma(h)=(V'+(xyz)^{\beta}S')^{p^n}+\delta(g_i)(V'+(xyz)^{\beta}S')^{p^n-i}+\cdots,$$

it does not in $\delta(g_i)$ too. As g_i is a homogeneous element of degree i, we obtain $g_i\in S^i(N)$. Q.E.D.

We think R_0 and R the Z^3-graded ring with $\deg_{Z^3}x=(1, 0, 0)$, $\deg_{Z^3}y=(0, 1, 0)$, $\deg_{Z^3}z=(0, 0, 1)$, $\deg_{Z^3}S=(\alpha, 0, 0)$, $\deg_{Z^3}T=(0, \alpha, 0)$, $\deg_{Z^3}U=(0, 0, \alpha)$ and $\deg_{Z^3}V=(\beta, \beta, \beta)$. Then R_l is a Z^3-graded R_0-module for each $l\geq 0$. Furthermore $S^l(M)$ and $\overline{S^l(M)}$ are Z^3-graded R_0-submodules of R_l. For a Z^3-homogeneous element f with $\deg_{Z^3}f=(a, b, c)$, we denote the total degree of f by $\mathrm{t.deg}_{Z^3}f=a+b+c$.

LEMMA 2.9. *Suppose $\beta/\alpha<2/3$. Let $h\in\overline{S^{p^n}(M)}$ be a Z^3-homogeneous monic polynomial in V. Then $h-V^{p^n}$ is contained in $F[x, y, z, S, T, U]$.*

PROOF. By assumption we have $\deg_{Z^3}h=(\beta p^n, \beta p^n, \beta p^n)$. Put

$$h=V^{p^n}+g_iV^{p^n-i}+(\text{lower degree terms in } V)$$

with $0\neq g_i\in F[x, y, z, S, T, U]$ and $0<i<p^n$. Then g_i is contained in $S^i(N)$ by Lemma 2.8. Since $\mathrm{t.deg}_{Z^3}A=\mathrm{t.deg}_{Z^3}B=\mathrm{t.deg}_{Z^3}C=2\alpha$, we have $\mathrm{t.deg}_{Z^3}g_i\geq 2i\alpha$. Therefore we obtain

$$3\beta p^n = \text{t.deg}_{Z^3} g_i V^{p^n - i} \geq 2i\alpha + 3\beta(p^n - i) .$$

But it contradicts $\beta/\alpha < 2/3$. Q.E.D.

LEMMA 2.10. *Suppose $\beta/\alpha < 1/2$. Then, for any positive integer l, $\overline{S^l(M)}$ contains no monic polynomial in V.*

PROOF. Assume the contrary, i.e., there exists a positive integer l for which $\overline{S^l(M)}$ contains a monic polynomial h in V. We may put $l = p^n$ by virtue of Lemma 2.7 and may assume that h is Z^3-homogeneous. Then n must be a positive integer by Remark 2.2 and $g = h - V^{p^n}$ is contained in $F[x, y, z, S, T, U]$ by Lemma 2.9.

Since $x^{\alpha-\beta}V - y^\beta z^\beta S$ is contained in M, $x^{(\alpha-\beta)p^n}V^{p^n} - y^{\beta p^n} z^{\beta p^n} S^{p^n}$ is in $S^{p^n}(M)$. Furthermore, since $x^{(\alpha-\beta)p^n}V^{p^n} + x^{(\alpha-\beta)p^n} g^{p^n}$ is contained in $\overline{S^{p^n}(M)}$, we have $x^{(\alpha-\beta)p^n} g^{p^n} + y^{\beta p^n} z^{\beta p^n} S^{p^n} \in \overline{S^{p^n}(M)} \cap F[x, y, z, S, T, U] = S^{p^n}(N)$ (see Lemma 2.6).

That is to say, there exists a Z^3-homogeneous element $f \in S^{p^n}(N)$ such that $\deg_{Z^3} f$ is equal to $(\alpha p^n, \beta p^n, \beta p^n)$ and $f - y^{\beta p^n} z^{\beta p^n} S^{p^n}$ is divisible by $x^{(\alpha-\beta)p^n}$. Since $f \in S^{p^n}(N)$ and $\deg_{Z^3} f = (\alpha p^n, \beta p^n, \beta p^n)$, we may suppose

$$f = \sum_{\substack{i+j+k=p^n \\ \alpha(i+k) \leq \beta p^n \\ \alpha(j+k) \leq \beta p^n}} d_{ijk} x^{\alpha p^n - \alpha(i+j)} y^{\beta p^n - \alpha(i+k)} z^{\beta p^n - \alpha(j+k)} A^i B^j C^k ,$$

where d_{ijk}'s are elements of F. Since the term $y^{\beta p^n} z^{\beta p^n} S^{p^n}$ actually appears in f, we can find i, j satisfying $i + j = p^n$ and $d_{ij0} \neq 0$. Then we obtain $\alpha i \leq \beta p^n$ and $\alpha(p^n - i) \leq \beta p^n$. It contradicts $\beta/\alpha < 1/2$. Q.E.D.

LEMMA 2.11. *Let l be a positive integer. Then, for a fixed base field F, whether $\overline{S^l(M)}$ contains a monic polynomial in V or not depends only on the rational number β/α.*

PROOF. Let $R'_0 = F[x', y', z']$ be a polynomial ring over F. Regard R'_0 as an R_0-algebra by means of the ring homomorphism $R_0 \to R'_0$ defined by $x \mapsto x'^t$, $y \mapsto y'^t$ and $z \mapsto z'^t$. Put $R' = R \otimes_{R_0} R'_0 = F[x', y', z', S, T, U, V]$ and $R'_1 = R_1 \otimes_{R_0} R'_0 = R'_0 \cdot S + R'_0 \cdot T + R'_0 \cdot U + R'_0 \cdot V$. Let M' be the kernel of the R'_0-linear map $R'_1 \to R'_0$ defined by $S \mapsto x'^{t\alpha}$, $T \mapsto y'^{t\alpha}$, $U \mapsto z'^{t\alpha}$ and $V \mapsto (x'y'z')^{t\beta}$. Then we have $M' = M \otimes_{R_0} R'_0$. Denote by $S(M') = \bigoplus_{l \geq 0} S^l(M')$ the graded subring of R' generated by R'_0 and M'. Furthermore define $\overline{S^l(M')}$ in the same way as in the case of $\overline{S^l(M)}$, i.e.,

$$\overline{S^l(M')} = \{f \in R'_l = R_l \otimes_{R_0} R'_0 \mid x'^m f,\ y'^m f,\ z'^m f \in S^l(M') \text{ for a sufficiently large } m\} .$$

Then we have $S^l(M') = S^l(M) \otimes_{R_0} R'_0$ and $S^l(M') \subseteq \overline{S^l(M)} \otimes_{R_0} R'_0 \subseteq R'_l$. By definition, $\overline{S^l(M)} \otimes_{R_0} R'_0 \subseteq \overline{S^l(M')}$. On the other hand, since x is a non zero divisor of $R_l/\overline{S^l(M)}$, x' is that of $R'_l/\overline{S^l(M')}$. Therefore we get $\overline{S^l(M')} = \overline{S^l(M)} \otimes_{R_0} R'_0$. Hence $\overline{S^l(M)}$ contains a monic polynomial in V if and only if so does $\overline{S^l(M')}$.

Thus, for a fixed base field F, whether $\overline{S^l(M)}$ contains a monic polynomial in V

or not depends only on the rational number β/α. Q.E.D.

LEMMA 2.12. *Suppose $\beta'/\alpha'<\beta/\alpha\leq 1$ and fix a base field F. If $\overline{S^l(M)}$ contains a monic polynomial in V in the case of α'/β', then so does $\overline{S^l(M)}$ in the case of α/β.*

PROOF. We may assume $\alpha'=\alpha+1$, $\beta'=\beta$ by Lemma 2.11. We denote by M' the kernel of the R_0-linear map $R_1\to R_0$ defined by $S\mapsto x^{\alpha+1}$, $T\mapsto y^{\alpha+1}$, $U\mapsto z^{\alpha+1}$ and $V\mapsto(xyz)^\beta$. Ψ denotes the graded ring homomorphism $R\to R$ defined by $S\mapsto xS$, $T\mapsto yT$, $U\mapsto zU$ and $V\mapsto V$. Then we have

$$\Psi(y^{\alpha+1}S-x^{\alpha+1}T)=xy^{\alpha+1}S-x^{\alpha+1}yT=xy(y^\alpha S-x^\alpha T)\in M\,,$$

$$\Psi(x^{\alpha+1-\beta}V-y^\beta z^\beta S)=x^{\alpha+1-\beta}V-xy^\beta z^\beta S=x(x^{\alpha-\beta}V-y^\beta z^\beta S)\in M\,.$$

Therefore we obtain $\Psi(M')\subseteq M$ and, hence, $\Psi(S^l(M'))\subseteq S^l(M)$. Then it is easy to see $\Psi(\overline{S^l(M')})\subseteq\overline{S^l(M)}$. The assertion immediately follows from the fact that Ψ maps monic polynomials in V to monic polynomials. Q.E.D.

REMARK 2.13. With the same situation as in the proof of Lemma 2.12, if $\overline{S^l(M')}$ contains V^l+g for some $g\in F[x, y, z, S, T, U]$, $\Psi(V^l+g)=V^l+\Psi(g)$ is in $\overline{S^l(M)}$ with $\Psi(g)\in F[x, y, z, S, T, U]$.

At first we investigate the case of $\mathrm{ch}(F)=2$.

PROPOSITION 2.14. *Suppose $\mathrm{ch}(F)=2$. Then there exists a positive integer l for which $\overline{S^l(M)}$ contains a monic polynomial in V if and only if $\beta/\alpha\geq 1/2$.*

PROOF. If $\beta/\alpha<1/2$, then Lemma 2.10 implies that $\overline{S^l(M)}$ contains no monic polynomial in V for any $l>0$.

Therefore we have only to prove that $\overline{S^2(M)}$ contains a monic polynomial in V when $\alpha=2$ and $\beta=1$ (cf. Lemma 2.11 and Lemma 2.12).

Since $xV-yzS\in M$, $x^2V^2-y^2z^2S^2$ is contained in $S^2(M)$. Furthermore we have

$$AB=(y^2S-x^2T)(z^2S-x^2U)=y^2z^2S^2-x^2y^2SU-x^2z^2ST+x^4TU\in S^2(M)$$

and hence,

$$(xV-yzS)^2+AB=x^2V^2-x^2y^2SU-x^2z^2ST+x^4TU\in S^2(M)\,.$$

Thus we obtain $V^2-y^2SU-z^2ST+x^2TU\in\overline{S^2(M)}$. Q.E.D.

Next we would like to prove Theorem 2.1 in the case of $p\equiv 1\,(6)$.

PROPOSITION 2.15. *Suppose $\mathrm{ch}(F)=p\equiv 1\,(6)$. Then there exists a positive integer l for which $\overline{S^l(M)}$ contains a monic polynomial in V if and only if $\beta/\alpha\geq 2/3$.*

In order to prove the previous proposition, by virtue of Lemma 2.7, Lemma 2.9, Lemma 2.11, Lemma 2.12 and Ramark 2.13, it is sufficient to show the following two lemmas:

LEMMA 2.16. *Suppose* $\mathrm{ch}(F)=p\equiv 1\,(6)$, $\alpha=3$ *and* $\beta=2$. *Then* $\overline{S^p(M)}$ *contains a monic polynomial in* V.

LEMMA 2.17. *Suppose* $\mathrm{ch}(F)=p\equiv 1\,(6)$ *and* $\beta/\alpha=2/3$. *Then* $V^{p^n}+g$ *is not contained in* $\overline{S^{p^n}(M)}$ *for any* $n>0$ *and for any* $g\in F[x, y, z, S, T, U]$.

PROOF OF LEMMA 2.16. At first we would like to prove that $S^p(N)$ contains a Z^3-homogeneous element of the form

$$y^{2p}z^{2p}S^p+f_jS^{p-j}+(\text{lower degree terms in } S)$$

which satisfies $f_j\in F[x, y, z, T, U]$ and $j\geq(p-1)/3$. Since

$$A^{(p+1)/2}B^{(p-1)/2}=y^{3(p+1)/2}z^{3(p-1)/2}S^p+(\text{lower degree terms in } S)\in S^p(N)$$

and $2p>3(p+1)/2$, $S^p(N)$ actually contains a Z^3-homogeneous element of the form $y^{2p}z^{2p}S^p+f_iS^{p-i}+(\text{lower degree terms in } S)$. Choose the largest i such that $S^p(N)$ contains a Z^3-homogeneous element of the form $y^{2p}z^{2p}S^p+f_iS^{p-i}+(\text{lower degree terms in } S)$. Assume $i<(p-1)/3$. (Since $(p-1)/3<p$, we have $0\neq f_i\in F[x, y, z, T, U]$ and note that $(p-1)/3$ is even.) Let h be a Z^3-homogeneous element of the form

$$h=y^{2p}z^{2p}S^p+f_iS^{p-i}+(\text{lower degree terms in } S)\in S^p(N)\,.$$

In the same way as in the proof of Lemma 2.8, we recognize that there exists $d\in F[x, y, z]$ such that $f_i=dC^i$. Since $\deg_{Z^3}C^iS^{p-i}=(3p-3i, 3i, 3i)$ and $\deg_{Z^3}y^{2p}z^{2p}S^p=(3p, 2p, 2p)$, there exists $q\in F^\times$ such that $f_i=qx^{3i}y^{2p-3i}z^{2p-3i}C^i$. (Note $2p-3i>0$ since $i<(p-1)/3$). Therefore we have

$$h=y^{2p}z^{2p}S^p+qx^{3i}y^{2p-3i}z^{2p-3i}C^iS^{p-i}+(\text{lower degree terms in } S)\in S^p(N)\,.$$

Here assume that i is odd. Then $p-i$ is even and we have $2p-3i\geq 3(p-i)/2$. When this is the case, we obtain

$$C^iA^{(p-i)/2}B^{(p-i)/2}=y^{3(p-i)/2}z^{3(p-i)/2}C^iS^{p-i}+(\text{lower degree terms in } S)\in S^p(N)\,.$$

Then we can find a larger integer than i satisfying the requirement. It contradicts the maximality of i.

Next assume that i is even. Then $p-i$ is odd and we have $2p-3i\geq 3(p-i+1)/2$. Similarly to the case where t is even, since

$$C^iA^{(p-i+1)/2}B^{(p-i-1)/2}$$
$$=y^{3(p-i+1)/2}z^{3(p-i-1)/2}C^iS^{p-i}+(\text{lower degree terms in } S)\in S^p(N)\,,$$

we can also find a larger integer than i satisfying the requirement. It is a contradiction.

Therefore, for $t=(p-1)/3$, $S^p(N)$ contains a Z^3-homogeneous element of the form

$$h=y^{2p}z^{2p}S^p+qx^{3t}y^{2p-3t}z^{2p-3t}C^tS^{p-t}+(\text{lower degree terms in } S)$$

such that $q\in F$. Since $h-y^{2p}z^{2p}S^p-qx^{3t}y^{2p-3t}z^{2p-3t}C^tS^{p-t}$ is a polynomial in S of degree at most $p-t-1$, it is divisible by $x^{3(t+1)}=x^{p+2}$. (We will know $q\neq 0$ in Lemma 2.17. But we do not need this fact here.)

Put

$$\begin{aligned}k&=C^tA^tB^t(xV-y^2z^2S)\\&=C^t(y^3S-x^3T)^t(z^3S-x^3U)^t(xV-y^2z^2S)\\&=xy^{3t}z^{3t}C^tS^{2t}V-y^{3t+2}z^{3t+2}C^tS^{2t+1}+x^3(\text{terms of degree at most 1 in } V)\,.\end{aligned}$$

By definition, k is contained in $S^p(M)$. Note $3t+1=p$ and $3t+2=2p-3t$. Then we have

$$\begin{aligned}qx^{p-1}k=&\,qx^py^{3t}z^{3t}C^tS^{2t}V-qx^{3t}y^{2p-3t}z^{2p-3t}C^tS^{p-t}\\&+x^{p+2}(\text{terms of degree at most 1 in } V)\end{aligned}$$

and thus,

$$\begin{aligned}&h+qx^{p-1}k+(xV-y^2z^2S)^p\\&\qquad=x^pV^p+qx^py^{3t}z^{3t}C^tS^{2t}V+x^{p+2}(\text{terms of degree at most 1 in } V)\end{aligned}$$

is contained in $S^p(M)$. Therefore $\overline{S^p(M)}$ contains a monic polynomial in V. Q.E.D.

Proof of Lemma 2.17. Put $t=(p^n-1)/3$. Note that t is an even integer. We shall prove this lemma in two steps.

Step 1. In this part, we show that $S^{p^n}(N)$ contains a Z^3-homogeneous element of the form

$$y^{\beta p^n}z^{\beta p^n}S^{p^n}+qx^{\alpha t}y^{\beta p^n-\alpha t}z^{\beta p^n-\alpha t}C^tS^{p^n-t}+(\text{terms of degree less than } p^n-t \text{ in } S)$$

such that $q\in F^\times$. Here note $\beta p^n-\alpha t>0$.

For $i=0, 1, \cdots, t-1$, we put

$$H(i)=x^{\alpha i}y^{\beta p^n-\alpha(d_i+i)}z^{\beta p^n-\alpha(e_i+i)}A^{d_i}B^{e_i}C^i$$

such that $d_i=(p^n-i+1)/2$, $e_i=d_i-1$ (resp. $d_i=e_i=(p^n-i)/2$) when i is even (resp. odd). Note $\beta p^n-\alpha(d_i+i)\geq 0$ and $\beta p^n-\alpha(e_i+i)\geq 0$ for $i=0, 1, \cdots, t-1$. Obviously $H(i)\in S^{p^n}(N)$ is Z^3-homogeneous with $\deg_{Z^3}H(i)=(\alpha p^n, \beta p^n, \beta p^n)$. We put

$$[a, b]=x^{\alpha(p^n-a)}y^{\beta p^n-\alpha b}z^{\beta p^n}S^aT^b\,.$$

Then we have

$$\begin{aligned}H(i)&=x^{\alpha i}y^{\beta p^n-\alpha(d_i+i)}z^{\beta p^n-\alpha(e_i+i)}(y^\alpha S-x^\alpha T)^{d_i}(z^\alpha S-x^\alpha U)^{e_i}(z^\alpha T-y^\alpha U)^i\\&=\sum_{j=0}^{d_i}(-1)^j\binom{d_i}{j}[d_i-j+e_i, i+j]+U(\cdots)\,.\end{aligned}$$

Here note $d_i - j + e_i = p^n - i - j$.

Once we can find $c_0, \cdots, c_{t-1} \in F$ such that

$$H = \sum_{i=0}^{t-1} c_i H(i) = [p^n, 0] + q[p^n - t, t] + \sum_{l>t} q_l [p^n - l, l] + U(\cdots)$$

with $q \in F^\times$ and $q_{t+1}, \cdots \in F$, then we have

$$H = y^{\beta p^n} z^{\beta p^n} S^{p^n} + q x^{\alpha t} y^{\beta p^n - \alpha t} z^{\beta p^n - \alpha t} C^t S^{p^n - t} + (\text{lower degree terms in } S).$$

Therefore we have only to find $c_0, \cdots, c_{t-1} \in F$ as above.

Put

$$H'(i) = \sum_{j=0}^{d_i} (-1)^j \binom{d_i}{j} [p^n - i - j, i + j].$$

We would like to find $c_0, \cdots, c_{t-1} \in F$ satisfying

$$\sum_{i=0}^{t-1} c_i H'(i) = [p^n, 0] + q[p^n - t, t] + \sum_{l>t} q_l [p^n - l, l]$$

with $q \in F^\times$ and $q_{t+1}, \cdots \in F$.

If we put

$$\sum_{i=0}^{t-1} c_i H'(i) = \sum_{l=0}^{p^n} k_l [p^n - l, l],$$

then we have the following equations:

$$\begin{pmatrix} k_0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ k_t \end{pmatrix} = \begin{pmatrix} \binom{d_0}{0} & 0 & & & \mathbf{0} \\ -\binom{d_0}{1} & \binom{d_1}{0} & 0 & & \\ \binom{d_0}{2} & -\binom{d_1}{1} & \binom{d_2}{0} & \ddots & \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \vdots & \vdots & \vdots & & \binom{d_{t-1}}{0} \\ (-1)^t \binom{d_0}{t} & (-1)^{t-1} \binom{d_1}{t-1} & (-1)^{t-2} \binom{d_2}{t-2} & \cdots & -\binom{d_{t-1}}{1} \end{pmatrix} \begin{pmatrix} c_0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ c_{t-1} \end{pmatrix}.$$

Therefore it is sufficient to show that the $t \times t$-submatrix of the above $(t+1) \times t$-matrix consisting of the 2nd row, the 3rd row, $\cdots$, the $(t+1)$th row is regular. By multiplying -1 to all the even rows and all the odd columns of this $t \times t$-matrix, we have

$$L_1=\begin{pmatrix} \binom{d_0}{1} & \binom{d_1}{0} & 0 & & \mathbf{0} \\ \binom{d_0}{2} & \binom{d_1}{1} & \binom{d_2}{0} & \ddots & \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \vdots & \vdots & \vdots & & \binom{d_{t-1}}{0} \\ \binom{d_0}{t} & \binom{d_1}{t-1} & \binom{d_2}{t-2} & \cdots & \binom{d_{t-1}}{1} \end{pmatrix}.$$

Note that

$$d_{t-1}=\frac{p^n+2}{3}>\frac{p^n-1}{3}=t\,.$$

By definition, d_i-d_{i+1} is equal to 0 or 1. Therefore we get $\det(L_1)=\det(L_2)$ by a succession of elementary transforms, where

$$L_2=\begin{pmatrix} \binom{d_{t-1}}{1} & \binom{d_{t-1}}{0} & 0 & & \mathbf{0} \\ \binom{d_{t-1}}{2} & \binom{d_{t-1}}{1} & \binom{d_{t-1}}{0} & \ddots & \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \vdots & \vdots & \vdots & & \binom{d_{t-1}}{0} \\ \binom{d_{t-1}}{t} & \binom{d_{t-1}}{t-1} & \binom{d_{t-1}}{t-2} & \cdots & \binom{d_{t-1}}{1} \end{pmatrix}.$$

Then Giambelli's identity ([6]) implies that $\det(L_2)=\dim_{\mathbf{C}}\mathrm{Sym}^t G=\binom{d_{t-1}+t-1}{t}$, where $\mathbf{C}$ is the field of complex numbers, G is a $\mathbf{C}$-vector space of dimension d_{t-1} and $\mathrm{Sym}^t G$ is the t-th symmetric module of G.

Since $d_{t-1}+t-1=2t$, we have only to prove that $\binom{2t}{t}$ is not divisible by p.

In order to prove it, since

$$\binom{2t}{t}=\frac{(t+1)(t+2)\cdots 2t}{1\cdot 2\cdots t}$$

and $2t\le p^n$, it is sufficient to show that, for $l=1, 2, \cdots, n-1$, the number of multiples of p^l in $\{1, 2, \cdots, t\}$ coincides with the number of that in $\{t+1, t+2, \cdots, 2t\}$. But it is easy to calculate. (In fact, since $t=(p^n-1)/3\not\equiv 0\ (p)$, we can find an integer s such that

$sp^l < t < (s+1)p^l$. Then we have $t - sp^l = (p^l - 1)/3$, because t is equal to $(p^n - p^l)/3 + (p^l - 1)/3$. Then we obtain $2sp^l < 2t < (2s+1)p^l$.)

Step 2. We return to the proof of Lemma 2.17. Suppose that there exists $g \in F[x, y, z, S, T, U]$ such that $V^{p^n} + g$ is $\boldsymbol{Z}^3$-homogeneous and contained in $\overline{S^{p^n}(M)}$.

Since $x^{\alpha-\beta}V - y^\beta z^\beta S \in M$, we get $x^{(\alpha-\beta)p^n}V^{p^n} - y^{\beta p^n}z^{\beta p^n}S^{p^n} \in S^{p^n}(M)$. Furthermore, since $x^{(\alpha-\beta)p^n}V^{p^n} + x^{(\alpha-\beta)p^n}g$ is contained in $\overline{S^{p^n}(M)}$, we have

$$y^{\beta p^n}z^{\beta p^n}S^{p^n} + x^{(\alpha-\beta)p^n}g \in \overline{S^{p^n}(M)} \cap F[x, y, z, S, T, U] = S^{p^n}(N)\,.$$

As we have already seen in Step 1, $S^{p^n}(N)$ contains a $\boldsymbol{Z}^3$-homogeneous element of the form

$$y^{\beta p^n}z^{\beta p^n}S^{p^n} + qx^{\alpha t}y^{\beta p^n - \alpha t}z^{\beta p^n - \alpha t}C^t S^{p^n - t} + (\text{terms of degree at most } p^n - t \text{ in } S)$$

with $q \in F^\times$ and therefore,

$$G = x^{(\alpha-\beta)p^n}g - qx^{\alpha t}y^{\beta p^n - \alpha t}z^{\beta p^n - \alpha t}C^t S^{p^n - t} + (\text{terms of degree at most } p^n - t \text{ in } S)$$

is contained in $S^{p^n}(N)$. Since $(\alpha-\beta)p^n > \alpha t$, the degree of g in S is less than $p^n - t$. Therefore $G \neq 0$ is a $\boldsymbol{Z}^3$-homogeneous element with $\deg_{\boldsymbol{Z}^3} G = (\alpha p^n, \beta p^n, \beta p^n)$. Hence we may suppose

$$G = \sum_{\substack{i+j+k=p^n \\ \alpha(i+k) \le \beta p^n \\ \alpha(j+k) \le \beta p^n}} d_{ijk} x^{\alpha p^n - \alpha(i+j)} y^{\beta p^n - \alpha(i+k)} z^{\beta p^n - \alpha(j+k)} A^i B^j C^k\,,$$

where $d_{ijk} \in F$.

Here we may assume that, for each k, the number of pairs (i, j) such that $d_{ijk} \neq 0$ is at most one. (We shall prove this by induction on k. For $k \ge 0$, suppose $d_{i_1 j_1 k} \neq 0$ and $d_{i_2 j_2 k} \neq 0$. Then $i_1 + j_1 + k = i_2 + j_2 + k = p^n$, $\alpha(i_1 + k) \le \beta p^n$, $\alpha(i_2 + k) \le \beta p^n$, $\alpha(j_1 + k) \le \beta p^n$ and $\alpha(j_2 + k) \le \beta p^n$ are satisfied. Suppose $0 \le i_1 < i_2$ $(j_1 > j_2 \ge 0)$ and set $i_3 = i_1 + 1$, $j_3 = j_1 - 1$. Note $i_3 + j_3 + k = p^n$, $\alpha(i_3 + k) \le \beta p^n$ and $\alpha(j_3 + k) \le \beta p^n$. Multiplying $A^{i_1}B^{j_3}C^k$ to $z^\alpha A - y^\alpha B + x^\alpha C = 0$, we obtain $z^\alpha A^{i_3}B^{j_3}C^k - y^\alpha A^{i_1}B^{j_1}C^k + x^\alpha A^{i_1}B^{j_3}C^{k+1} = 0$, here $z^\alpha A^{i_3}B^{j_3}C^k$, $y^\alpha A^{i_1}B^{j_1}C^k$ and $x^\alpha A^{i_1}B^{j_3}C^{k+1}$ are $\boldsymbol{Z}^3$-homogeneous elements of degree $(\alpha(i_3 + j_3), \alpha(i_3 + k), \alpha(j_1 + k))$. Since $\alpha(i_3 + j_3) \le \alpha p^n$, $\alpha(i_3 + k) \le \beta p^n$ and $\alpha(j_1 + k) \le \beta p^n$, we may replace G by

$$G + d_{i_1 j_1 k} x^{\alpha p^n - \alpha(i_3 + j_3)} y^{\beta p^n - \alpha(i_3 + k)} z^{\beta p^n - \alpha(j_1 + k)} (z^\alpha A^{i_3}B^{j_3}C^k - y^\alpha A^{i_1}B^{j_1}C^k + x^\alpha A^{i_1}B^{j_3}C^{k+1})\,.$$

Then we may assume $d_{i_1 j_1 k} = 0$. We have only to repeat this process.)

The above argument enables us to assume that the number of pairs (i, j) such that $d_{ijk} \neq 0$ is at most one for each k. Put

$$k_1 = \min\{k \ge 0 \mid \text{there exists } (i, j) \text{ such that } d_{ijk} \neq 0\}\,.$$

(Since $G \neq 0$, k_1 is well-defined.) Suppose $d_{i_1 j_1 k_1} \neq 0$. Then the degree of G in S is equal to $i_1 + j_1$. Therefore we have $i_1 + j_1 = p^n - t$ and, hence, $k_1 = t$. Since $p^n - t$ is an odd integer, either i_1 or j_1 is at least $(p^n-t+1)/2$. Suppose $i_1\geq(p^n-t+1)/2$. Then we have $i_1+t\geq(p^n+t+1)/2$ and, therefore,

$$\beta p^n-\alpha(i_1+t)<0$$

is satisfied. It contradicts the choice of k_1. Q.E.D.

We have completed the proof of Proposition 2.15.

Next consider the case of $p\equiv 5$ (6).

PROPOSITION 2.18. *Suppose* $\mathrm{ch}(F)=p\equiv 5$ (6). *Then there exists* $l>0$ *for which* $\overline{S^l(M)}$ *contains a monic polynomial in* V *if and only if* $\beta/\alpha\geq(2p-1)/3p$.

PROOF. At first we prove that $\overline{S^p(M)}$ contains a monic polynomial in V when $1>\beta/\alpha\geq(2p-1)/3p$. It is sufficient to show that $\overline{S^p(M)}$ contains a monic polynomial in V if $\beta/\alpha=(2p-1)/3p$ by Lemma 2.11 and Lemma 2.12.

Put $t=(p+1)/3$. Note that t is even.

In the same way as in the proof of Lemma 2.16, we can prove that $S^p(N)$ contains a $\boldsymbol{Z}^3$-homogeneous element of the form

$$h=y^{\beta p}z^{\beta p}S^p+qx^{\alpha t}y^{\beta p-\alpha t}z^{\beta p-\alpha t}C^tS^{p-t}+(\text{lower degree terms in } S)$$

with $q\in F$.

Since $x^{\alpha-\beta}V-y^\beta z^\beta S\in M$,

$$(x^{\alpha-\beta}V-y^\beta z^\beta S)^p+h=x^{(\alpha-\beta)p}V^p+qx^{\alpha t}y^{\beta p-\alpha t}z^{\beta p-\alpha t}C^tS^{p-t}$$
$$+(\text{terms in } F[x, y, z, S, T, U] \text{ of degree at most } p-t \text{ in } S)$$

is contained in $S^p(M)$. Since it is $\boldsymbol{Z}^3$-homogeneous, $(x^{\alpha-\beta}V-y^\beta z^\beta S)^p+h-x^{(\alpha-\beta)p}V^p$ is divisible by $x^{\alpha t}$. Therefore $\overline{S^p(M)}$ contains a monic polynomial in V since $\alpha t=(\alpha-\beta)p$.

In the rest of this proof, we shall prove that there exists no $l>0$ for which $\overline{S^l(M)}$ contains a monic polynomial in V if $\beta/\alpha<(2p-1)/3p$. Assume the contrary, i.e., there exist $n\geq 2$, α and β such that β/α is less than $(2p-1)/3p$ and $\overline{S^{p^n}(M)}$ contains a $\boldsymbol{Z}^3$-homogeneous element $V^{p^n}+g$ with $g\in F[x, y, z, S, T, U]$. (By virtue of Lemma 2.7 and Lemma 2.9, for α and β which satisfy $\beta/\alpha<(2p-1)/3p$, once we can find a monic polynomial contained in $\overline{S^l(M)}$ for some l, we may assume that there exists $n\geq 2$ such that $\overline{S^{p^n}(M)}$ contains a monic polynomial of the form $V^{p^n}+g$ with $g\in F[x, y, z, S, T, U]$.) Since

$$\frac{2}{3}>\frac{2p-1}{3p}=\frac{2p^n-p^{n-1}}{3p^n}>\frac{2p^n-p^{n-1}-3}{3p^n},$$

we may suppose

$$\frac{2p-1}{3p}>\frac{\beta}{\alpha}>\frac{2p^n-p^{n-1}-3}{3p^n}$$

by Lemma 2.11 and Lemma 2.12 and Remark 2.13. Put $t'=(p^n-2p^{n-1}-3)/3$. Note that t' is even and we have $0<t'<p^n$ since $n\geq 2$.

Next, in the same way as in Step 1 in Lemma 2.17, we prove that $S^{p^n}(N)$ contains a $\boldsymbol{Z}^3$-homogeneous element of the form

$$y^{\beta p^n}z^{\beta p^n}S^{p^n}+qx^{\alpha t'}y^{\beta p^n-\alpha t'}z^{\beta p^n-\alpha t'}C^{t'}S^{p^n-t'}+(\text{lower degree terms in } S)$$

with $q\in F^{\times}$. (By definition of α, β and t', $\beta p^n-\alpha t'>0$ is satisfied.)

For $i=0, 1, \cdots, t'-1$, put

$$H(i)=x^{\alpha i}y^{\beta p^n-\alpha(d_i+i)}z^{\beta p^n-\alpha(e_i+i)}A^{d_i}B^{e_i}C^i\,,$$

where $d_i=(p^n-i+1)/2$ and $e_i=d_i-1$ (resp. $d_i=e_i=(p^n-i)/2$) when i is even (resp. odd). We repeat the same argument as in Step 1 of Lemma 2.17. Note that, for $i=0, 1, \cdots, t'-1$, $\beta p^n-\alpha(d_i+i)\geq 0$ and $\beta p^n-\alpha(e_i+i)\geq 0$ are satisfied. Since

$$d_{t'-1}=\frac{p^n+p^{n-1}+3}{3}>t'\,,$$

we have only to prove that the $t\times t$-matrix

$$L=\begin{pmatrix} \binom{d_{t'-1}}{1} & \binom{d_{t'-1}}{0} & 0 & & \mathbf{0} \\ \binom{d_{t'-1}}{2} & \binom{d_{t'-1}}{1} & \binom{d_{t'-1}}{0} & \ddots & \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \vdots & \vdots & \vdots & & \binom{d_{t'-1}}{0} \\ \binom{d_{t'-1}}{t'} & \binom{d_{t'-1}}{t'-1} & \binom{d_{t'-1}}{t'-2} & \cdots & \binom{d_{t'-1}}{1} \end{pmatrix}$$

is regular as in Step 1 in Lemma 2.17. Then $\det(L)=\binom{d_{t'-1}+t'-1}{t'}$ holds by Giambelli's identity. Put

$$e=d_{t'-1}+t'-1=\frac{2p^n-p^{n-1}-3}{3}\,.$$

We have only to show that $\binom{e}{t'}$ is not divisible by p. Since $e=(2p^n-p^{n-1}-3)/3<p^n$, it is sufficient to prove that the number of multiples of p^l in $\{1, 2, \cdots, t'\}$ coincides with the number of that in $\{e-t'+1, e-t'+2, \cdots, e\}$ for $l=1, 2, \cdots, n-1$. It is obvious because

$$e-t'=\frac{p^n+p^{n-1}}{3}\equiv 0\ \ (p^{n-1})\,.$$

Now we return to the proof of Proposition 2.18.

Since $x^{\alpha-\beta}V-y^{\beta}z^{\beta}S\in M$, we have $x^{(\alpha-\beta)p^n}V^{p^n}-y^{\beta p^n}z^{\beta p^n}S^{p^n}\in S^{p^n}(M)$. Therefore $y^{\beta p^n}z^{\beta p^n}S^{p^n}+x^{(\alpha-\beta)p^n}g\in S^{p^n}(N)$ holds. As we have already seen, $S^{p^n}(N)$ contains a $\boldsymbol{Z}^3$-homogeneous element of the form

$$y^{\beta p^n}z^{\beta p^n}S^{p^n}+qx^{\alpha t'}y^{\beta p^n-\alpha t'}z^{\beta p^n-\alpha t'}C^{t'}S^{p^n-t'}+(\text{lower degree terms in } S)$$

with $q\in F^{\times}$. Hence, there exists

$$G=x^{(\alpha-\beta)p^n}g-qx^{\alpha t'}y^{\beta p^n-\alpha t'}z^{\beta p^n-\alpha t'}C^{t'}S^{p^n-t'}+(\text{terms of degree at most } p^n-t' \text{ in } S)$$

contained in $S^{p^n}(N)$. Since $(\alpha-\beta)p^n>\alpha t'$, the degree of g in S is less than p^n-t' and, therefore, we have $G\neq 0$. As G is a $\boldsymbol{Z}^3$-homogeneous element with $\deg_{\boldsymbol{Z}^3}G=(\alpha p^n, \beta p^n, \beta p^n)$, we may suppose

$$G=\sum_{\substack{i+j+k=p^n\\ \alpha(i+k)\leq\beta p^n\\ \alpha(j+k)\leq\beta p^n}} d_{ijk}x^{\alpha p^n-\alpha(i+j)}y^{\beta p^n-\alpha(i+k)}z^{\beta p^n-\alpha(j+k)}A^iB^jC^k\,,$$

with $d_{ijk}\in F$. In the same way as in Step 2 in Lemma 2.17, we may assume that the number of pairs (i,j) such that $d_{ijk}\neq 0$ is at most one for each k. Put

$$k_1=\min\{k\geq 0 \mid \text{there exists } (i,j) \text{ such that } d_{ijk}\neq 0\}\,.$$

(Since $G\neq 0$, k_1 is well-defined.) Suppose $d_{i_1j_1k_1}\neq 0$. Then the degree of G in S is equal to i_1+j_1. Therefore $i_1+j_1=p^n-t'$ and $k_1=t'$ hold. Since p^n-t' is odd, either i_1 or j_1 is at least $(p^n-t'+1)/2$. But, if we suppose $i_1\geq(p^n-t'+1)/2$, it is easy to see

$$\beta p^n-\alpha(i_1+t')<0\,.$$

It is a contradiction. Q.E.D.

Next consider the case of $p=3$.

Proposition 2.19. *Suppose* $\mathrm{ch}(F)=3$. *Then there exists* $l>0$ *for which* $\overline{S^l(M)}$ *contains a monic polynomial in* V *if and only if* $\beta/\alpha\geq 2/3$.

Proof. At first, we would like to prove that $\overline{S^l(M)}$ contains a monic polynomial in V for some $l>0$ when $\beta/\alpha\geq 2/3$. By virtue of Lemma 2.11 and Lemma 2.12, we have only to prove it when $\alpha=3$ and $\beta=2$. When this is the case, we have

$$(xV-y^2z^2S)^3+z^3A^2B=x^3V^3+x^3(\text{terms in } F[x, y, z, S, T, U])\,.$$

Therefore $\overline{S^3(M)}$ surely contains a monic polynomial in V.

Next we shall prove the opposite implication.

Suppose that there exist $n\geq 2$, α and β satisfying $\beta/\alpha<2/3$ such that $\overline{S^{3^n}(M)}$ contains a $\boldsymbol{Z}^3$-homogeneous element $V^{3^n}+g$ with $g\in F[x, y, z, S, T, U]$. By means of Lemma 2.11 and Lemma 2.12, we may assume

$$\frac{2}{3} > \frac{\beta}{\alpha} > \frac{2 \cdot 3^{n-1} - 1}{3^n}.$$

Put $t = 3^{n-1} - 1$ and note that t is even. (Since $n \geq 2$, we have $t > 0$.)

In the same way as in Step 1 of Lemma 2.17, we want to show that $S^{3^n}(N)$ contains a $\boldsymbol{Z}^3$-homogeneous element of the form

$$y^{\beta 3^n} z^{\beta 3^n} S^{3^n} + q x^{\alpha t} y^{\beta 3^n - \alpha t} z^{\beta 3^n - \alpha t} C^t S^{3^n - t} + (\text{lower degree terms in } S)$$

with $q \in F^{\times}$. (Note that, by definition of α, β and t, we have $\beta 3^n - \alpha t > 0$.)

For $i = 0, 1, \cdots, t-1$, we put

$$H(i) = x^{\alpha i} y^{\beta 3^n - \alpha(d_i + i)} z^{\beta 3^n - \alpha(e_i + i)} A^{d_i} B^{e_i} C^i ,$$

where $d_i = (3^n - i + 1)/2$ and $e_i = d_i - 1$ (resp. $d_i = e_i = (3^n - i)/2$) when i is even (resp. odd). We repeat the same argument as in Step 1 in Lemma 2.17. Note $\beta 3^n - \alpha(d_i + i) \geq 0$ and $d_{t-1} = 3^{n-1} + 1 > t$. We have only to show that the following $t \times t$-matrix is regular:

$$L = \begin{pmatrix} \binom{d_{t-1}}{1} & \binom{d_{t-1}}{0} & 0 & & \mathbf{0} \\ \binom{d_{t-1}}{2} & \binom{d_{t-1}}{1} & \binom{d_{t-1}}{0} & \ddots & \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \vdots & \vdots & \vdots & & \binom{d_{t-1}}{0} \\ \binom{d_{t-1}}{t} & \binom{d_{t-1}}{t-1} & \binom{d_{t-1}}{t-2} & \cdots & \binom{d_{t-1}}{1} \end{pmatrix}$$

We get $\det(L) = \binom{d_{t-1} + t - 1}{t}$ by Giambelli's identity. Put

$$e = d_{t-1} + t - 1 = 2 \cdot 3^{n-1} - 1 .$$

It is sufficient to show that $\binom{e}{t}$ is not divisible by 3.

Since $e < 3^n$, it suffices to show that the number of multiples of 3^l in $\{1, 2, \cdots, t\}$ is equal to the number of that in $\{e - t + 1, e - t + 2, \cdots, e\}$ for $l = 1, 2, \cdots, n-1$. But it follows from

$$e - t = 3^{n-1}$$

immediately.

Now we return to the proof of Proposition 2.19.

Since $x^{\alpha - \beta} V - y^{\beta} z^{\beta} S \in M$, $x^{(\alpha - \beta) 3^n} V^{3^n} - y^{\beta 3^n} z^{\beta 3^n} S^{3^n}$ is contained in $S^{3^n}(M)$. Furthermore, since the Z^3-homogeneous element $V^{3^n}+g$ is contained in $\overline{S^{3^n}(M)}$, $y^{\beta 3^n}z^{\beta 3^n}S^{3^n}+x^{(\alpha-\beta)3^n}g\in S^{3^n}(N)$ is satisfied. As we have already seen, $S^{3^n}(N)$ contains a Z^3-homogeneous element of the form

$$y^{\beta 3^n}z^{\beta 3^n}S^{3^n}+qx^{\alpha t}y^{\beta 3^n-\alpha t}z^{\beta 3^n-\alpha t}C^tS^{3^n-t}+(\text{terms of degree at most } 3^n-t \text{ in } S)$$

with $q\in F^\times$. Hence $S^{3^n}(N)$ contains

$$G=x^{(\alpha-\beta)3^n}g-qx^{\alpha t}y^{\beta 3^n-\alpha t}z^{\beta 3^n-\alpha t}C^tS^{3^n-t}+(\text{terms of degree at most } 3^n-t \text{ in } S)\,.$$

Here, the degree of g in S is less than 3^n-t since $(\alpha-\beta)3^n>\alpha t$. Therefore we have $G\neq 0$. Furthermore since G is a Z^3-homogeneous element with $\deg_{Z^3}G=(\alpha 3^n, \beta 3^n, \beta 3^n)$, we may suppose

$$G=\sum_{\substack{i+j+k=3^n\\ \alpha(i+k)\leq\beta 3^n\\ \alpha(j+k)\leq\beta 3^n}} d_{ijk}x^{\alpha 3^n-\alpha(i+j)}y^{\beta 3^n-\alpha(i+k)}z^{\beta 3^n-\alpha(j+k)}A^iB^jC^k$$

with $d_{ijk}\in F$. In the same way as in Step 1 in Lemma 2.17, we may suppose that the number of pairs (i,j) such that $d_{ijk}\neq 0$ is at most one for each k. Put

$$k_1=\min\{k\geq 0 \mid \text{there exists } (i,j) \text{ such that } d_{ijk}\neq 0\}\,.$$

(k is well-defined since $G\neq 0$.) Suppose $d_{i_1j_1k_1}\neq 0$. Then the degree of G in S is equal to i_1+j_1. Hence we have $i_1+j_1=3^n-t$ and, therefore, $k_1=t$ is satisfied. Since 3^n-t is odd, either i_1 or j_1 is at least $(3^n-t+1)/2$. But, if we suppose $i_1\geq(3^n-t+1)/2$, then it is easy to see

$$\beta 3^n-\alpha(i_1+t)<0\,.$$

It is a contradiction. Q.E.D.

Now Theorem 2.1 is an immediate consequence of Proposition 2.14, Proposition 2.15, Proposition 2.18 and Proposition 2.19.

3. The case of $\alpha=2$ and $\beta=1$.

In the case of $\alpha=2$ and $\beta=1$ (i.e., the case of $t=1$ in Roberts' notation [10]), we have already seen that $\overline{S(M)}$ is Noetherian in Propositions 2.14 when $\mathrm{ch}(F)=2$, for $\overline{S(M)}$ is a homomorphic image of $R_s(Q)$ as we noted in section 1. On the other hand, when $\mathrm{ch}(F)\neq 2$, we can not apply results in [5] to this case and Roberts' method in [10] does not work even if $\mathrm{ch}(F)=0$.

In this section, we would like to show that $\overline{S(M)}$ is Noetherian for $\alpha=2$ and $\beta=1$ whatever $\mathrm{ch}(F)$ is, i.e.,

PROPOSITION 3.1. *Suppose $\alpha=2$ and $\beta=1$. Then $\overline{S(M)}$ is Noetherian over any field F.*

We prove this proposition in the rest of this paper.

We have already proved in Proposition 2.14 that $\overline{S(M)}$ is Noetherian when $\mathrm{ch}(F)=2$ and, therefore, *assume* $\mathrm{ch}(F)\neq 2$ *in the rest of this section.* We put

$$\begin{aligned}
A &= y^2S - x^2T \\
B &= z^2S - x^2U \\
C &= z^2T - y^2U \\
D &= xV - yzS \\
E &= yV - xzT \\
F &= zV - xyU \\
G &= xV^2 - 2yzSV + xz^2ST + xy^2SU - x^3TU \\
H &= yV^2 - 2xzTV + yz^2ST - y^3SU + x^2yTU \\
I &= zV^2 - 2xyUV - z^3ST + y^2zSU + x^2zTU\,.
\end{aligned}$$

In order to prove the Noetherian property of $\overline{S(M)}$, we claim that

$$\overline{S(M)} = R_0[A, B, C, D, E, F, G, H, I]\,.$$

First of all, note that, since

$$\begin{aligned}
xG &= D^2 - AB \\
yH &= E^2 + AC \\
zI &= F^2 - BC\,,
\end{aligned}$$

we get $G, H, I \in \overline{S^2(M)}$. Therefore we have $\overline{S(M)} \supseteq R_0[A, B, C, D, E, F, G, H, I]$.

We would like to prove the opposite containment.

The next lemma will play an essential role in proving the finite generation of $\overline{S(M)}$.

LEMMA 3.2. *Let q and t be integers such that $q \geq t \geq 0$. If $K_0V^t + K_1V^{t-1} + \cdots + K_t \in \overline{S^q(M)}$ is a $\boldsymbol{Z}^3$-homogeneous element such that $K_0, \cdots, K_t \in F[x, y, z, S, T, U]$, then we have*

$$K_0 = \sum_{i+j+k=q-t} d_{ijk} A^i B^j C^k$$

with $d_{ijk} \in R_0$ such that $\mathrm{t.deg}_{\boldsymbol{Z}^3} d_{ijk} \geq [(t+1)/2]$ *for any i, j, k.*

PROOF. Suppose that $K_0V^t + K_1V^{t-1} + \cdots + K_t$ is a $\boldsymbol{Z}^3$-homogeneous element contained in $\overline{S^q(M)}$ such that $q \geq t \geq 0$ and $K_0, \cdots, K_t \in F[x, y, z, S, T, U]$. Then S' does not appear in

$$\gamma(K_0V^t + K_1V^{t-1} + \cdots + K_t) = \delta(K_0)\gamma(V)^t + \delta(K_1)\gamma(V)^{t-1} + \cdots + \delta(K_t)$$

by Lemma 2.6 and, therefore, it does not in $\delta(K_0)$ too. Thus we get $K_0 \in S^{q-t}(N)$ and, hence, we may put $K_0 = \sum_{i+j+k=q-t} d_{ijk}A^iB^jC^k$ with $d_{ijk} \in R_0$.

When $t=0$, the assertion is obvious because $[(t+1)/2]=0$.

Assume the contrary and choose the minimal $t>0$ such that, for some $q \geq t$, there exists a $\boldsymbol{Z}^3$-homogeneous element

$$K_0V^t + K_1V^{t-1} + \cdots + K_t \in \overline{S^q(M)}$$

with

$$K_0 = \sum_{i+j+k=q-t} d_{ijk}A^iB^jC^k$$

satisfying $\text{t.deg}_{\boldsymbol{Z}^3} d_{ijk} < [(t+1)/2]$ for any i, j, k.

At first, suppose that t is even and put $t=2l$ $(l \geq 1)$. In this case, $[(t+1)/2]$ is equal to l. By our assumption, for some $q \geq 2l$, there exists a $\boldsymbol{Z}^3$-homogeneous element

$$L_1 := K_0V^{2l} + \sum_{i=1}^{2l} K_iV^{2l-i} \in \overline{S^q(M)}$$

with $K_0, \cdots, K_{2l} \in F[x, y, z, S, T, U]$ and $K_0 = \sum_{i+j+k=q-2l} d_{ijk}A^iB^jC^k \neq 0$ such that $d_{ijk} \in R_0 = F[x, y, z]$ satisfies $\text{t.deg}_{\boldsymbol{Z}^3} d_{ijk} = l-1$ for each i, j, k. Since

$$\sum_{i+j+k=q-2l} d_{ijk}(G, H, I)A^iB^jC^k \in \overline{S^{q-2}(M)},$$

we obtain

$$L_2 := \left(\sum_{i+j+k=q-2l} d_{ijk}(G, H, I)A^iB^jC^k\right)G = xK_0V^{2l} + \cdots \in \overline{S^q(M)}.$$

Put

$$L_3 := xL_1 = xK_0V^{2l} + \sum_{i=1}^{2l} xK_iV^{2l-i} \in \overline{S^q(M)}.$$

Suppose $L_2 \neq L_3$ and let a be the degree of $L_3 - L_2$ in V. Here note $0 \leq a < 2l$. On the other hand, $\text{t.deg}_{\boldsymbol{Z}^3}(L_3 - L_2)$ is equal to $4q - l$. Furthermore the coefficient of V^a in $L_3 - L_1$ is contained in $S^{q-a}(N)$. Then we get

$$4q - l \geq [(a+1)/2] + 4(q-a) + 3a$$

by the minimality of t and, therefore, we have $a > 2l-1$. It is a contradiction. Hence we get $L_2 = L_3$. In particular, L_2 is divisible by x. Therefore $\sum_{i+j+k=q-2l} d_{ijk}(G, H, I)A^iB^jC^k$ is divisible by x. As we have already seen in Step 2 of Lemma 2.17, we may suppose

$$K_0 = x^{\xi}(mA^{i_0}B^{j_0}C^{k_0} + x(\cdots)),$$

where $i_0+j_0+k_0=q-2l$ and $m=by^{m_1}z^{m_2}$ ($b\in F^\times$, $\xi+m_1+m_2=l-1$). Since G^ξ is not divisible by x,

$$bH^{m_1}I^{m_2}A^{i_0}B^{j_0}C^{k_0}+G(\cdots)$$

have to be divisible by x. But the constant term (in V) of the above polynomial is equal to

$$b(yz^2ST-y^3SU)^{m_1}(-z^3ST+y^2zSU)^{m_2}(y^2S)^{i_0}(z^2S)^{j_0}(z^2T-y^2U)^{k_0} \qquad \mathrm{mod}(x)$$

and it is not divisible by x. It is a contradiction.

Next suppose t is odd and put $t=2l+1$ ($l\geq 0$). In this case, we have $[(t+1)/2]=l+1$. By our assumption, for some integer $q\geq 2l+1$, there exists a $\boldsymbol{Z}^3$-homogeneous element

$$L_4:=K_0V^{2l+1}+\sum_{i=1}^{2l+1}K_iV^{2l+1-i}\in\overline{S^q(M)}$$

with $K_0,\cdots,K_{2l+1}\in F[x,y,z,S,T,U]$ and $K_0=\sum_{i+j+k=q-2l-1}d_{ijk}A^iB^jC^k\neq 0$ such that $d_{ijk}\in R_0=F[x,y,z]$ satisfies $\mathrm{t.deg}_{\boldsymbol{Z}^3}d_{ijk}=l$. Put

$$\begin{aligned}L_5:&=\sum_{i+j+k=q-2l-1}d_{ijk}(G,H,I)A^iB^jC^k\\&=K_0V^{2l}+\sum_{i=1}^{2l}K_i'V^{2l-i}\in\overline{S^{q-1}(M)},\end{aligned}$$

where $K_i'\in F[x,y,z,S,T,U]$, and

$$\begin{aligned}L_6:&=(xV-yzS)L_5\\&=xK_0V^{2l+1}+\sum_{i=1}^{2l}xK_i'V^{2l+1-i}-yzS\Bigg(\sum_{i+j+k=q-2l-1}d_{ijk}(G,H,I)A^iB^jC^k\Bigg)\in\overline{S^q(M)}.\end{aligned}$$

Furthermore put

$$L_7:=xL_4=xK_0V^{2l+1}+\sum_{i=1}^{2l+1}xK_iV^{2l+1-i}\in\overline{S^q(M)}.$$

Then we have

$$L_7-L_6=\sum_{i=1}^{2l+1}xK_iV^{2l+1-i}-\sum_{i=1}^{2l}xK_i'V^{2l+1-i}+yzS\Bigg(\sum_{i+j+k=q-2l-1}d_{ijk}(G,H,I)A^iB^jC^k\Bigg).$$

Here the degree of L_7-L_6 in V is at most $2l$.

If L_6 is equal to L_7, L_5 is divisible by x. As in the case where t is even, we may suppose

$$K_0=x^\xi(mA^{i_0}B^{j_0}C^{k_0}+x(\cdots)),$$

where $i_0+j_0+k_0=q-2l-1$ and $m=by^{m_1}z^{m_2}$ ($b\in F^\times$, $\xi+m_1+m_2=l$). Since G^ξ is not divisible by x,

$$bH^{m_1}I^{m_2}A^{i_0}B^{j_0}C^{k_0}+G(\cdots)$$

have to be divisible by x. But the constant term (in V) of the above polynomial is equal to

$$b(yz^2ST-y^3SU)^{m_1}(-z^3ST+y^2zSU)^{m_2}(y^2S)^{i_0}(z^2S)^{j_0}(z^2T-y^2U)^{k_0} \qquad \mathrm{mod}(x)$$

and it is not divisible by x. It is a contradiction.

Hence, suppose $L_7-L_6\neq 0$ and let a_1 be the degree of L_7-L_6 in V. Note $0\leq a_1\leq 2l$, $\mathrm{t.deg}_{Z^3}(L_7-L_6)=4q-l$ and the coefficient of V^{a_1} in L_7-L_6 is an element of $S^{q-a_1}(N)$. Then we obtain

$$4q-l\geq[(a_1+1)/2]+4(q-a_1)+3a_1$$

by the minimality of t. Then a_1 is equal to $2l$ since $a_1\leq 2l$. Put

$$L_7-L_6=\sum_{i=0}^{2l} J_iV^{2l-i}\in\overline{S^q(M)}\,,$$

where $J_i\in F[x, y, z, S, T, U]$. Since $J_0\neq 0$, we may suppose

$$J_0=\sum_{i+j+k=q-2l} e_{ijk}A^iB^jC^k\in S^{q-2l}(N)\,,$$

where $e_{ijk}\in R_0=F[x, y, z]$ and $\mathrm{t.deg}_{Z^3}e_{ijk}=l$. Furthermore, put

$$\begin{aligned} L_8 :&= \sum_{i+j+k=q-2l} e_{ijk}(G, H, I)A^iB^jC^k \\ &= J_0V^{2l}+(\text{lower degree terms in } V)\in\overline{S^q(M)}\,. \end{aligned}$$

Then the degree of $L_7-L_6-L_8$ in V is less than $2l$.

Suppose $L_7-L_6-L_8\neq 0$ and let a_2 be the degree of $L_7-L_6-L_8$ in V. Then $0\leq a_2<2l$ is satisfied. On the other hand, we have

$$4q-l\geq[(a_2+1)/2]+4(q-a_2)+3a_2$$

by the minimality of t and, therefore, $a_2>2l-1$ is satisfied. It is a contradiction. Hence we obtain $L_7-L_6=L_8$. In particular, if we denote

$$L_9 := yzS\left(\sum_{i+j+k=q-2l-1} d_{ijk}(G, H, I)A^iB^jC^k\right),$$

we have $L_8\equiv L_9 \ \mathrm{mod}(x)$.

As in the case where t is even, we may put

$$K_0=x^{\xi}(mA^{i_0}B^{j_0}C^{k_0}+x(\cdots))\,,$$

where $i_0+j_0+k_0=q-2l-1$ and $m=by^{m_1}z^{m_2}$ ($b\in F^{\times}$, $\xi+m_1+m_2=l$). Furthermore we may suppose

$$J_0 = x^\zeta(nA^{i_1}B^{j_1}C^{k_1} + x(\cdots)),$$

where $i_1 + j_1 + k_1 = q - 2l$ and $n = cy^{n_1}z^{n_2}$ $(c \in F^\times, \zeta + n_1 + n_2 = l)$.

If $r < \xi$ or $r > 2l - \xi$, the coefficient of V^r in L_9 is divisible by x. In the case of $\xi < l$, it is easy to see that the coefficients of V^ξ and $V^{2l-\xi}$ are never divisible by x.

If $r < \zeta$ or $r > 2l - \zeta$, the coefficient of V^r in L_8 is divisible by x. In the case of $\zeta < l$, it is easy to see that the coefficients of V^ζ and $V^{2l-\zeta}$ are never divisible by x.

Since $L_8 \not\equiv L_9$ mod(x), it is easy to see $\xi = \zeta$.

Suppose $\xi = l$. Then we have $K_0 = bx^l A^{i_0}B^{j_0}C^{k_0}$ and $J_0 = cx^l A^{i_1}B^{j_1}C^{k_1}$, where $b, c \in F^\times$, $i_0 + j_0 + k_0 = q - 2l - 1$ and $i_1 + j_1 + k_1 = q - 2l$. Then we obtain

$$\begin{aligned} L_9 &= byzSG^l A^{i_0}B^{j_0}C^{k_0} \\ &\equiv byzS(-2yzSV)^l(y^2S)^{i_0}(z^2S)^{j_0}(z^2T - y^2U)^{k_0} \qquad \text{mod}(x), \\ L_8 &= cG^l A^{i_1}B^{j_1}C^{k_1} \\ &\equiv c(-2yzSV)^l(y^2S)^{i_1}(z^2S)^{j_1}(z^2T - y^2U)^{k_1} \qquad \text{mod}(x). \end{aligned}$$

Hence $L_8 \not\equiv L_9$ mod(x). It is a contradiction.

Next suppose $\xi < l$. The coefficient of $V^{2l-\xi}$ in L_9 is

$$(yzS)(-2yzS)^\xi by^{m_1}z^{m_2}(y^2S)^{i_0}(z^2S)^{j_0}(z^2T - y^2U)^{k_0} \qquad \text{mod}(x)$$

and that in L_8 is

$$(-2yzS)^\xi cy^{n_1}z^{n_2}(y^2S)^{i_1}(z^2S)^{j_1}(z^2T - y^2U)^{k_1} \qquad \text{mod}(x).$$

On the other hand, the coefficient of V^ξ in L_9 is

$$(yzS)(-2yzS)^\xi by^{m_1}(-z)^{m_2}(z^2ST - y^2SU)^{m_1+m_2}(y^2S)^{i_0}(z^2S)^{j_0}(z^2T - y^2U)^{k_0} \qquad \text{mod}(x)$$

and that in L_8 is

$$(-2yzS)^\xi cy^{n_1}(-z)^{n_2}(z^2ST - y^2SU)^{n_1+n_2}(y^2S)^{i_1}(z^2S)^{j_1}(z^2T - y^2U)^{k_1} \qquad \text{mod}(x).$$

(Note ch$(F) \neq 2$.) Hence we have $k_0 = k_1$. Since $m_1 + m_2 = n_1 + n_2$, we have

$$by^{1+m_1+2i_0}z^{1+m_2+2j_0} = cy^{n_1+2i_1}z^{n_2+2j_1}$$

and

$$(-1)^{m_2}by^{1+m_1+2i_0}z^{1+m_2+2j_0} = (-1)^{n_2}cy^{n_1+2i_1}z^{n_2+2j_1}.$$

It is easy to see that two equations as above are not satisfied at the same time.

Q.E.D.

We now return to the proof of Proposition 3.1.

We have already seen $\overline{S(M)} \supseteq R_0[A, B, C, D, E, F, G, H, I]$.

Let q and t be non-negative integers such that $q \geq t \geq 0$. For any integers i, j and

k such that $i+j+k=q-t$ and for any monomial m of degree $[(t+1)/2]$ in x, y and z, $R_0[A, B, C, D, E, F, G, H, I]$ contains an element of the form

$$mA^iB^jC^kV^t + (\text{lower degree terms in } V).$$

Then

$$\overline{S(M)} = R_0[A, B, C, D, E, F, G, H, I]$$

is an immediate consequence of Lemma 3.2.

We have completed the proof of Proposition 3.1.

References

[1] W. Bruns and U. Vetter, *Determinantal Rings*, Lecture Notes in Math., **1327** (1988), Springer.
[2] R. Cowsik, Symbolic powers and the number of defining equations, *Algebra and its Applications*, Lecture Notes in Pure and Appl. Math., **91** (1985), 13–14, Dekker.
[3] S. D. Cutokosky, Symbolic algebras of monomial primes, J. Reine Angew. Math., **416** (1991), 71–89.
[4] S. Goto, K. Nishida and K.-i. Watanabe, Non-Cohen-Macaulay symbolic blow-ups for space monomial curves and counterexamples to Cowsik's question, to appear in Proc. Amer. Math. Soc.
[5] K. Kurano, On finite generation of Rees rings defined by filtrations of ideals, to appear in J. Math. Kyoto Univ.
[6] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Clarendon Press (1979).
[7] M. Nagata, On the fourteenth problem of Hilbert, *Proc. Internat. Congress Math. (1958)*, Cambridge Univ. Press (1960), 459–462.
[8] D. Rees, On a problem of Zariski, Illinois J. Math., **2** (1958), 145–149.
[9] P. Roberts, A prime ideal in a polynomial ring whose symbolic blow-up is not Noetherian, Proc. Amer. Math. Soc., **94** (1985), 589–592.
[10] P. Roberts, An infinitely generated symbolic blow-ups in a power series ring and a new counterexample to Hilbert's fourteenth problem, J. Alg., **132** (1990), 461–473.

Present Address:
Department of Mathematics, Tokyo Metropolitan University
Minami-Ohsawa, Hachioji-shi, Tokyo 192–03, Japan
e-mail: kurano@math.metro-u.ac.jp