TOKYO J. MATH.
VOL. 16, NO. 2, 1993

Modified Jacobi-Perron Algorithm and Generating Markov Partitions for Special Hyperbolic Toral Automorphisms

Shunji ITO and Makoto OHTSUKI

Tsuda College

0. Introduction.

The fact that the boundaries of Markov partitions of hyperbolic toral automorphism on T^3 are not smooth was pointed out by Bowen [2]. Using the generating method of fractal curves by Dekking [3], T. Bedford gave Markov partitions with fractal boundaries on a suitable subclass of hyperbolic toral automorphism on T^3.

THEOREM (T. Bedford). *Let us assume that the* 3×3 *integral matrix* B *satisfies the following properties*:
(1) B *is non-negative and* $\det B=1$,
(2) *the maximum eigenvalue* λ_0 *of* B *is a Pisot number, that is,* B *has a single real expanding eigen value* $\lambda_0>1$ *and double contracting eigenvalues* λ_1, λ_2 $(1>|\lambda_i|>0)$.
Then there exists a Markov partition of toral automorphism T_B *on* T^3 *with structure matrix* ${}^tB$.

His main idea is to construct a bounded domain X and a partition $\{X_i: i=1, 2, 3\}$ on the expanding invariant plane $\mathbb{P}$ with respect to a linear map L_B^{-1} which induces the Markov endomorphism on X with structure matrix B by using the generating method of fractal curves. The purpose of this paper is to study more precisely the generating method of these domains. For this purpose, we introduce the concept of tilings of $\mathbb{P}$ by three kinds of parallelograms and the concept of the substitution Σ on the configurations of parallelograms. (See Figure 1.)

Using this idea, on the special class for B such that $B=A_1A_2\cdots A_k$, where

$$A_i\in\left\{\begin{pmatrix} a & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}, \begin{pmatrix} a & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix} ;\quad a\in N\right\},$$

Received August 29, 1992
Revised February 1, 1993

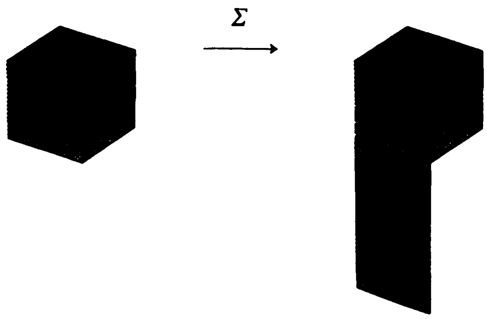

FIGURE 1

we are able to study the generating method of the domain X and partition $\{X_i: i=1, 2, 3\}$ which are not only the domains inducing the Markov endomorphism on $\mathbb{P}$ with structure matrix B, but also inducing the domain exchange transformation W on X such that $Wx = x - \pi e_i$ if $x \in X_i$, where π is the projection to $\mathbb{P}$ along the contracting eigenvector. And we see the domain exchange has a self-similarity in the following sense:

$$\begin{array}{ccc} X & \xrightarrow{W} & X \\ {\scriptstyle L_B}\downarrow & & \downarrow{\scriptstyle L_B} \\ L_B(X) & \xrightarrow{W_{L_B}} & L_B(X) \end{array}$$

where W_{L_B} means the induced automorphism on the set $L_B(X)$ induced from W. You will find the assertion is very close to Rauzy's result [7], which is the existence of the self-similar domain exchange trnasformation for the special matrix; $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. It is the main purpose to integrate two works [1] and [7] by using the substitution Σ. Finally, we claim that the multidimensional continued fraction algorithm called modified Jacobi-Perron algorithm in this paper and its natural extension work well behind all our results.

1. Definition of modified Jacobi-Perron algorithm.

Let us define an algorithm called modified Jacobi-Perron algorithm, which is introduced by Podsypanin [6], as follows. Let X be the domain given by $X = [0, 1] \times [0, 1)$ and let us define the transformation T on X by

$$T(\alpha, \beta) = \begin{cases} \left(\dfrac{\beta}{\alpha}, \dfrac{1}{\alpha} - \left[\dfrac{1}{\alpha}\right]\right) & \text{if } (\alpha, \beta) \in X_0 - \{(0, 0)\} \\ \left(\dfrac{1}{\beta} - \left[\dfrac{1}{\beta}\right], \dfrac{\alpha}{\beta}\right) & \text{if } (\alpha, \beta) \in X_1 \\ (0, 0) & \text{if } (\alpha, \beta) = (0, 0)\,, \end{cases} \tag{1-1}$$

where $X_0=\{(\alpha, \beta) \mid \alpha \geq \beta\}$ and $X_1=\{(\alpha, \beta) \mid \alpha<\beta\}$. By using the following integer valued functions

$$a(\alpha, \beta)=\begin{cases}\left[\dfrac{1}{\alpha}\right] & \text{if } (\alpha, \beta)\in X_0-\{(0,0)\}\\ \left[\dfrac{1}{\beta}\right] & \text{if } (\alpha, \beta)\in X_1\,,\end{cases}$$

$$\varepsilon(\alpha, \beta)=\begin{cases}0 & \text{if } (\alpha, \beta)\in X_0\\ 1 & \text{if } (\alpha, \beta)\in X_1\end{cases}$$

on $X-\{(0,0)\}$, we define for each $(\alpha, \beta)\in X-\{(0,0)\}$ a sequence of digits ${}^t(a_n, \varepsilon_n)$ by

$${}^t(a_n, \varepsilon_n):=(a(T^{n-1}(\alpha, \beta))\,,\, \varepsilon(T^{n-1}(\alpha, \beta))) \qquad \text{if} \quad T^{n-1}(\alpha, \beta)\neq(0,0)\,.$$

The triple $(X, T, (a(\alpha, \beta), \varepsilon(\alpha, \beta)))$ is called *modified Jacobi-Perron algorithm*. And we denote $(\alpha_n, \beta_n):=T^n(\alpha, \beta)$. For the modified Jacobi-Perron algorithm, we introduce a transformation $(\bar{X}, \bar{T})$, which is called a natural extension of modified Jacobi-Perron algorithm, as follows: let $\bar{X}=X\times X$ and let us define the transformation $\bar{T}$ on $\bar{X}$ by

$$\text{(1-2)}\qquad \bar{T}(\alpha, \beta, \gamma, \delta)=\begin{cases}\left(\dfrac{\beta}{\alpha}, \dfrac{1}{\alpha}-a_1, \dfrac{\delta}{a_1+\gamma}, \dfrac{1}{a_1+\gamma}\right) & \text{if } (\alpha, \beta)\in X_0-\{(0,0)\}\\ \left(\dfrac{1}{\beta}-a_1, \dfrac{\alpha}{\beta}, \dfrac{1}{a_1+\delta}, \dfrac{\gamma}{a_1+\delta}\right) & \text{if } (\alpha, \beta)\in X_1\\ (0,0,\gamma,\delta) & \text{if } (\alpha, \beta)=(0,0)\,.\end{cases}$$

Then we know that the transformation $\bar{T}$ is bijective from $(X-\{(0,0)\})\times X$ to $X\times(X-\{(0,0)\})$. We denote

$$(\alpha_n, \beta_n, \gamma_n, \delta_n)=\bar{T}^n(\alpha, \beta, \gamma, \delta)\,.$$

Let us introduce the family of matrices as follows: for each integral vector ${}^t(a, \varepsilon)$, $a\in \boldsymbol{N}$, $\varepsilon\in\{0,1\}$,

$$\text{(1-3)}\qquad A_{\binom{a}{0}}=\begin{pmatrix} a & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix} \quad\text{and}\quad A_{\binom{a}{1}}=\begin{pmatrix} a & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}.$$

Then we see

$$\text{(1-4)}\qquad A^{-1}_{\binom{a}{0}}=\begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 1 & -a & 0\end{pmatrix} \quad\text{and}\quad A^{-1}_{\binom{a}{1}}=\begin{pmatrix} 0 & 0 & 1\\ 1 & 0 & -a\\ 0 & 1 & 0\end{pmatrix}.$$

Using these matrices, we have the following proposition.

PROPOSITION 1.1. *Assume that* $T^k(\alpha, \beta)\neq(0,0)$, $0\leq k\leq n-1$. *Then we have*

$$\begin{pmatrix}1\\ \alpha_n\\ \beta_n\end{pmatrix}=\frac{1}{\theta\theta_1\cdots\theta_{n-1}}A^{-1}_{\binom{a_n}{\varepsilon_n}}A^{-1}_{\binom{a_{n-1}}{\varepsilon_{n-1}}}\cdots A^{-1}_{\binom{a_1}{\varepsilon_1}}\begin{pmatrix}1\\ \alpha\\ \beta\end{pmatrix},$$

$$\begin{pmatrix}1\\ \gamma_n\\ \delta_n\end{pmatrix}=\frac{1}{\eta\eta_1\cdots\eta_{n-1}}{}^tA_{\binom{a_n}{\varepsilon_n}}{}^tA_{\binom{a_{n-1}}{\varepsilon_{n-1}}}\cdots {}^tA_{\binom{a_1}{\varepsilon_1}}\begin{pmatrix}1\\ \gamma\\ \delta\end{pmatrix},$$

where

$$\theta_k=\max(\alpha_k, \beta_k),$$

$$\eta_k=\begin{cases}a_k+\gamma_{k-1} & \textit{if}\quad (\alpha_{k-1}, \beta_{k-1})\in X_0\\ a_k+\delta_{k-1} & \textit{if}\quad (\alpha_{k-1}, \beta_{k-1})\in X_1 .\end{cases}$$

PROOF. From the definition of the algorithm (1-1), we have

$$\begin{pmatrix}1\\ \alpha_1\\ \beta_1\end{pmatrix}=\frac{1}{\alpha}A^{-1}_{\binom{a_1}{0}}\begin{pmatrix}1\\ \alpha\\ \beta\end{pmatrix}\qquad \text{if}\quad (\alpha, \beta)\in X_0-\{(0, 0)\},$$

$$\begin{pmatrix}1\\ \alpha_1\\ \beta_1\end{pmatrix}=\frac{1}{\beta}A^{-1}_{\binom{a_1}{1}}\begin{pmatrix}1\\ \alpha\\ \beta\end{pmatrix}\qquad \text{if}\quad (\alpha, \beta)\in X_1 .$$

Therefore, we have

$$\begin{pmatrix}1\\ \alpha_1\\ \beta_1\end{pmatrix}=\frac{1}{\theta}A^{-1}_{\binom{a_1}{\varepsilon_1}}\begin{pmatrix}1\\ \alpha\\ \beta\end{pmatrix}\qquad \text{if}\quad (\alpha, \beta)\neq(0, 0),$$

where $\theta=\max\{\alpha, \beta\}$. On the other hand, from the natural extension (1-2) we know

$$\begin{pmatrix}1\\ \gamma_1\\ \delta_1\end{pmatrix}=\frac{1}{a_1+\gamma}{}^tA_{\binom{a_1}{0}}\begin{pmatrix}1\\ \gamma\\ \delta\end{pmatrix}\qquad \text{if}\quad (\alpha, \beta)\in X_0-\{(0, 0)\},$$

$$\begin{pmatrix}1\\ \gamma_1\\ \delta_1\end{pmatrix}=\frac{1}{a_1+\delta}{}^tA_{\binom{a_1}{1}}\begin{pmatrix}1\\ \gamma\\ \delta\end{pmatrix}\qquad \text{if}\quad (\alpha, \beta)\in X_1 .$$

Therefore we have

$$\begin{pmatrix}1\\ \gamma_1\\ \delta_1\end{pmatrix}=\frac{1}{\eta}{}^tA_{\binom{a_1}{\varepsilon_1}}\begin{pmatrix}1\\ \gamma\\ \delta\end{pmatrix}\qquad \text{if}\quad (\alpha, \beta)\neq(0, 0),$$

where

$$\eta=\begin{cases}a_1+\gamma & \text{if}\quad (\alpha, \beta)\in X_0\\ a_1+\delta & \text{if}\quad (\alpha, \beta)\in X_1 .\end{cases}$$

(q.e.d.)

For each n, let us introduce a transformation $\varphi_{\binom{a_n}{\varepsilon_n}}$ from $\boldsymbol{R}^3$ whose coordinate is denoted by ${}^t(x_n, y_n, z_n)$ to $\boldsymbol{R}^3$ whose coordinate is denoted by ${}^t(x_{n-1}, y_{n-1}, z_{n-1})$ as follows:

$$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = \varphi^{-1}_{\binom{a_n}{\varepsilon_n}} \begin{pmatrix} x_{n-1} \\ y_{n-1} \\ z_{n-1} \end{pmatrix} := A^{-1}_{\binom{a_n}{\varepsilon_n}} \begin{pmatrix} x_{n-1} \\ y_{n-1} \\ z_{n-1} \end{pmatrix}.$$

Then we have the following lemma.

LEMMA 1.2. *For each* $(\alpha, \beta, \gamma, \delta) \in \bar{X}$ *such that* $(\alpha, \beta) \neq (0, 0)$, *we have*

$$\left(\begin{pmatrix} 1 \\ \gamma_1 \\ \delta_1 \end{pmatrix}, \varphi^{-1}_{\binom{a_1}{\varepsilon_1}} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \right) = \eta \left(\begin{pmatrix} 1 \\ \gamma \\ \delta \end{pmatrix}, \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \right).$$

The proof is easy from the definition of $\varphi_{\binom{a_1}{\varepsilon_1}}$ and Proposition 1.1. Hereafter we denote also $\boldsymbol{\alpha}_n = {}^t(1, \alpha_n, \beta_n)$, $\boldsymbol{\gamma}_n = {}^t(1, \gamma_n, \delta_n)$. For each $(\alpha, \beta, \gamma, \delta) \in \bar{X}$ such that $(\alpha, \beta) \neq (0, 0)$, let us denote the plane which is orthogonal to $\boldsymbol{\gamma} = {}^t(1, \gamma, \delta)$ by

$$\mathbb{P}(\gamma, \delta) = \{\boldsymbol{x} \mid (\boldsymbol{x}, \boldsymbol{\gamma}) = 0\}$$

and let us define the domains $\boldsymbol{P}(\gamma, \delta)$ and $\boldsymbol{P}'(\gamma, \delta)$ by

$$\boldsymbol{P}(\gamma, \delta) = \{\boldsymbol{x} \mid (\boldsymbol{x}, \boldsymbol{\gamma}) > 0\}, \qquad \boldsymbol{P}'(\gamma, \delta) = \{\boldsymbol{x} \mid (\boldsymbol{x}, \boldsymbol{\gamma}) \geq 0\}.$$

Then from Lemma 1.2 we have

COROLLARY 1.3. *For each* $(\alpha_{n-1}, \beta_{n-1}, \gamma_{n-1}, \delta_{n-1}) \in \bar{X}$ *such that* $(\alpha_{n-1}, \beta_{n-1}) \neq (0, 0)$, *we have*

$$\varphi^{-1}_{\binom{a_n}{\varepsilon_n}}(\mathbb{P}(\gamma_{n-1}, \delta_{n-1})) = \mathbb{P}(\gamma_n, \delta_n),$$

$$\varphi^{-1}_{\binom{a_n}{\varepsilon_n}}(\boldsymbol{P}(\gamma_{n-1}, \delta_{n-1})) = \boldsymbol{P}(\gamma_n, \delta_n).$$

For each n let us define the projection π_n to $\mathbb{P}(\gamma_n, \delta_n)$ along $(1, \alpha_n, \beta_n)$. Then we have

COROLLARY 1.4. *The following commutative relation holds*:

$$\varphi_{\binom{a_n}{\varepsilon_n}} \circ \pi_n = \pi_{n-1} \circ \varphi_{\binom{a_n}{\varepsilon_n}}.$$

2. Substitutions on stepped surfaces.

Let E_1, E_2 and E_3 be unit squares spanned by $\{e_2, e_3\}$, $\{e_3, e_1\}$ and $\{e_1, e_2\}$, that is,

$$E_1 := \{\lambda e_2 + \mu e_3 \mid 0 \leq \lambda, \mu \leq 1\},$$

$$E_2 := \{\lambda e_3 + \mu e_1 \mid 0 \leq \lambda, \mu \leq 1\},$$

$$E_3 := \{\lambda e_1 + \mu e_2 \mid 0 \leq \lambda, \mu \leq 1\},$$

where

$$e_1=\begin{pmatrix}1\\0\\0\end{pmatrix},\quad e_2=\begin{pmatrix}0\\1\\0\end{pmatrix},\quad e_3=\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

For each $(\alpha, \beta, \gamma, \delta)\in \bar{X}$, let us define $\mathscr{S}(\gamma, \delta)$, which is a subset of $\boldsymbol{Z}^3\times\{E_1, E_2, E_3\}$, as follows:

$$\mathscr{S}(\gamma, \delta):=\left\{(\boldsymbol{x}, F)\;\middle|\;\begin{matrix}F\in\{E_1, E_2, E_3\}, \boldsymbol{x}\in\boldsymbol{Z}^3, \boldsymbol{x}+F\subset\boldsymbol{P}(\gamma, \delta)\\ \text{and } \boldsymbol{x}-\boldsymbol{e}_i\notin\boldsymbol{P}(\gamma, \delta) \text{ if } F=E_i\end{matrix}\right\},$$

and let us define $\mathscr{G}(\gamma, \delta)$ to be the family of all finite subsets of $\mathscr{S}(\gamma, \delta)$, that is,

$$\mathscr{G}(\gamma, \delta):=\left\{\sum_{\lambda\in\Lambda}(\boldsymbol{x}_\lambda, F_\lambda)\;\middle|\;\begin{matrix}\#\Lambda<\infty, (\boldsymbol{x}_\lambda, F_\lambda)\in\mathscr{S}(\gamma, \delta),\\ (\boldsymbol{x}_\lambda, F_\lambda)\neq(\boldsymbol{x}_{\lambda'}, F_{\lambda'}) \text{ if } \lambda\neq\lambda'\end{matrix}\right\},$$

where an element of $\mathscr{G}(\gamma, \delta)$ is denoted as a formal sum. We define $\mathscr{S}'$ and $\mathscr{G}'$ instead of $\mathscr{S}$ and $\mathscr{G}$ by using $\boldsymbol{P}'$ instead of $\boldsymbol{P}$, similarly. Then on the assumption that ${}^t(1, \gamma, \delta)$ is rationally independent, that is, if $l+m\gamma+n\delta=0$ for some $l, m, n\in\boldsymbol{Z}$ then $(l, m, n)=(0, 0, 0)$, we know that

$$\mathscr{S}\ni(\boldsymbol{e}_i, E_i),\ i=1, 2, 3,\qquad \mathscr{S}'\ni(0, E_i),\ i=1, 2, 3$$

and $\mathscr{G}'$ is obtained from $\mathscr{G}$ by changing only the elements $(\boldsymbol{e}_i, E_i)$ of $\Delta\in\mathscr{G}$ to $(0, E_i)$. For each $(\alpha, \beta, \gamma, \delta)\in\bar{X}$, let us define a map $\Sigma_{\binom{a_1}{\varepsilon_1}}$ from $\mathscr{G}(\gamma, \delta)$ to $\mathscr{G}(\gamma_1, \delta_1)$ as follows:

$$\Sigma_{\binom{a_1}{0}}:\quad\begin{matrix}(0, E_1) & \to & (0, E_3)+\sum_{1\le k\le a_1}(\boldsymbol{e}_1-k\boldsymbol{e}_3, E_1)\\ (0, E_2) & \to & (0, E_1)\\ (0, E_3) & \to & (0, E_2)\end{matrix}$$

and

$$\Sigma_{\binom{a_1}{1}}:\quad\begin{matrix}(0, E_1) & \to & (0, E_2)+\sum_{1\le k\le a_1}(\boldsymbol{e}_1-k\boldsymbol{e}_2, E_1)\\ (0, E_2) & \to & (0, E_3)\\ (0, E_3) & \to & (0, E_1)\end{matrix}$$

and for $(\boldsymbol{x}, F)\in\mathscr{S}(\gamma, \delta)$, we define

$$\Sigma_{\binom{a_1}{\varepsilon_1}}(\boldsymbol{x}, F):=\varphi^{-1}_{\binom{a_1}{\varepsilon_1}}(\boldsymbol{x})+\Sigma_{\binom{a_1}{\varepsilon_1}}(0, F)$$

and for $\sum_{\lambda\in\Lambda}(\boldsymbol{x}_\lambda, F_\lambda)\in\mathscr{G}(\gamma, \delta)$, we define

$$\Sigma_{\binom{a_1}{\varepsilon_1}}\left(\sum_{\lambda\in\Lambda}(\boldsymbol{x}_\lambda, F_\lambda)\right):=\sum_{\lambda\in\Lambda}(\Sigma_{\binom{a_1}{\varepsilon_1}}(\boldsymbol{x}_\lambda, F_\lambda)),$$

where $\boldsymbol{y}+(\boldsymbol{x}, F)$ means $(\boldsymbol{y}+\boldsymbol{x}, F)$. (See Figure 2.)

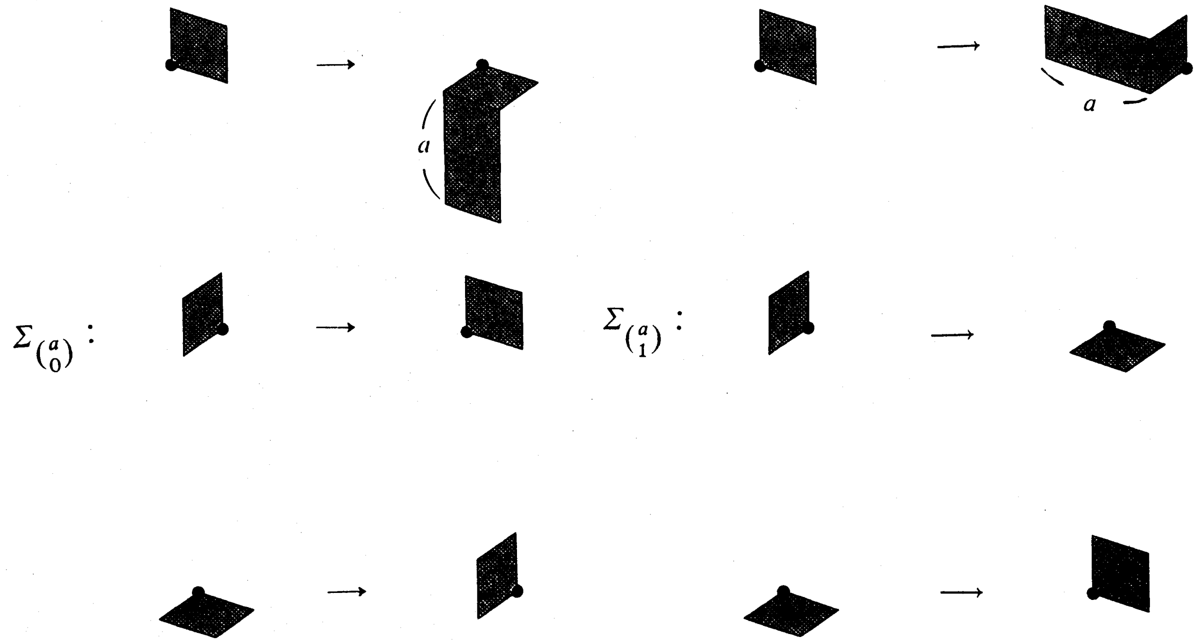

FIGURE 2. Figure of substitutions $\Sigma_{\binom{a_1}{\varepsilon_1}}(0, E_i)$, $i=1, 2, 3$

Then we can see by the following two lemmas that $\Sigma_{\binom{a_1}{\varepsilon_1}}$ is a map from $\mathcal{G}(\gamma, \delta)$ to $\mathcal{G}(\gamma_1, \delta_1)$.

LEMMA 2.1. *For each* $(\boldsymbol{x}, F) \in \mathcal{G}(\gamma, \delta)$, $\Sigma_{\binom{a_1}{\varepsilon_1}}(\boldsymbol{x}, F)$ *is an element of* $\mathcal{G}(\gamma_1, \delta_1)$.

PROOF. On the assumption that $\varepsilon_1 = 0$, we have the following 3 cases (i), (ii) and (iii).
(i) The case that $\boldsymbol{x} + E_1 \subset \boldsymbol{P}(\gamma, \delta)$ and $\boldsymbol{x} - \boldsymbol{e}_1 \notin \boldsymbol{P}(\gamma, \delta)$.
Remarking $\varphi^{-1}_{\binom{a_1}{0}}(\boldsymbol{x} + E_1) \subset \boldsymbol{P}(\gamma_1, \delta_1)$, we see that

$$\varphi^{-1}_{\binom{a_1}{0}}(\boldsymbol{x}) \in \boldsymbol{P}(\gamma_1, \delta_1)\,,$$

$$\varphi^{-1}_{\binom{a_1}{0}}(\boldsymbol{x} + \boldsymbol{e}_2) = \varphi^{-1}_{\binom{a_1}{0}}(\boldsymbol{x}) + \boldsymbol{e}_1 - a_1 \boldsymbol{e}_3 \in \boldsymbol{P}(\gamma_1, \delta_1)\,.$$

Therefore

$$\varphi^{-1}_{\binom{a_1}{0}}(\boldsymbol{x}) + \boldsymbol{e}_1 - j\boldsymbol{e}_3 \in \boldsymbol{P}(\gamma_1, \delta_1) \qquad (0 \le j \le a_1)\,.$$

Hence we have

$$\Sigma_{\binom{a_1}{0}}(\boldsymbol{x}, E_1) \subset \boldsymbol{P}(\gamma_1, \delta_1)\,.$$

On the other hand, we see that

$$\begin{aligned} (\gamma_1, \varphi^{-1}_{\binom{a_1}{0}}(\boldsymbol{x}) - \boldsymbol{e}_3) &= (\gamma_1, \varphi^{-1}_{\binom{a_1}{0}}(\boldsymbol{x} - \boldsymbol{e}_1)) && \text{(by (1-4))} \\ &= (\gamma, \boldsymbol{x} - \boldsymbol{e}_1) && \text{(by Lemma 1.2)} \\ &\le 0\,. \end{aligned}$$

Therefore, we see $\varphi^{-1}_{\binom{a_1}{0}}(x)-e_3 \notin P(\gamma_1, \delta_1)$ and so $\varphi^{-1}_{\binom{a_1}{0}}(x)-ke_3 \notin P(\gamma_1, \delta_1)$ for $1 \leq k$. This means

$$\Sigma_{\binom{a_1}{0}}(x, E_1) \subset \mathscr{S}(\gamma_1, \delta_1).$$

(ii) The case of $x+E_2 \subset P(\gamma, \delta)$ and $x-e_2 \notin P(\gamma, \delta)$.

It is sufficient to see that

$$(\gamma_1, \varphi^{-1}_{\binom{a_1}{0}}(x)-e_1) \leq 0.$$

We know from $(\gamma, x-e_2) \leq 0$, Lemma 1.2 and (1-3) that

$$\begin{aligned}(\gamma_1, \varphi^{-1}_{\binom{a_1}{0}}(x)-e_1) &= (\gamma_1, \varphi^{-1}_{\binom{a_1}{0}}(x)-\varphi^{-1}_{\binom{a_1}{0}}(e_2)-a_1\varphi^{-1}_{\binom{a_1}{0}}(e_1)) \\ &= \eta(\gamma, x-e_2-a_1e_1) \qquad \text{(by Lemma 1.2)} \\ &= \eta((\gamma, x-e_2)-a_1(\gamma, e_1)) \leq 0.\end{aligned}$$

(iii) The case of $x+E_3 \subset P(\gamma, \delta)$ and $x-e_3 \notin P(\gamma, \delta)$.

It is sufficient to see that $(\gamma_1, \varphi^{-1}_{\binom{a_1}{0}}(x)-e_2) \leq 0$. We know from $(\gamma, x-e_3) \leq 0$ and Lemma 1.2 and (1-5) that

$$\begin{aligned}(\gamma_1, \varphi^{-1}_{\binom{a_1}{0}}(x)-e_2) &= (\gamma_1, \varphi^{-1}_{\binom{a_1}{0}}(x)-\varphi^{-1}_{\binom{a_1}{0}}(e_3)) \\ &= \eta(\gamma, x-e_3) \leq 0.\end{aligned}$$

The case of $\varepsilon_1 = 1$ is obtained analogously. (q.e.d.)

LEMMA 2.2. *If $(x, F) \neq (x', F')$, then*

$$\Sigma_{\binom{a_1}{\varepsilon_1}}(x, F) \cap \Sigma_{\binom{a_1}{\varepsilon_1}}(x', F') = \varnothing.$$

PROOF. (i) Suppose that

$$\Sigma_{\binom{a_1}{0}}(x, E_1) \cap \Sigma_{\binom{a_1}{0}}(x', E_1) \neq \varnothing \qquad (x \neq x'),$$

that is, suppose that there exist k, j, $1 \leq k, j \leq a_1$, $k \neq j$ such that

$$\varphi^{-1}_{\binom{a_1}{0}}(x)+e_1-ke_3+E_1 = \varphi^{-1}_{\binom{a_1}{0}}(x')+e_1-je_3+E_1.$$

Then we have

$$\varphi^{-1}_{\binom{a_1}{0}}(x-x') = (k-j)e_3 = (k-j)\varphi^{-1}_{\binom{a_1}{0}}(e_1).$$

Therefore, we know

$$x-x' = (k-j)e_1,$$

that is,

$$\begin{aligned} & x-e_1 = x'+(k-j-1)e_1 \in P(\gamma, \delta) \\ \text{or} \quad & x'-e_1 = x+(j-k-1)e_1 \in P(\gamma, \delta).\end{aligned}$$

This contradicts

$$x-e_1,\ x'-e_1 \notin P(\gamma, \delta)\,.$$

(ii) Suppose that

$$\Sigma_{\binom{a_1}{0}}(x, E_1) \cap \Sigma_{\binom{a_1}{0}}(x', E_2) \neq \varnothing \qquad (x \neq x')\,,$$

that is, there exists j, $1 \leq j \leq a_1$, such that

$$\varphi^{-1}_{\binom{a_1}{0}}(x)+e_1-je_3+E_1=\varphi^{-1}_{\binom{a_1}{0}}(x')+E_1\,.$$

Then from that

$$\varphi^{-1}_{\binom{a_1}{0}}(x'-x)=e_1-je_3=\varphi^{-1}_{\binom{a_1}{0}}(e_2)+a_1\varphi^{-1}_{\binom{a_1}{0}}(e_1)-j\varphi^{-1}_{\binom{a_1}{0}}(e_1)\,,$$

we have

$$x'-x=e_2+(a_1-j)e_3\,.$$

That is $x'-e_2=x+(a_1-j)e_3 \in P(\gamma, \delta)$. This contradicts $x'-e_2 \notin P(\gamma, \delta)$. We see from the definition of $\Sigma_{\binom{a_1}{0}}$ that

$$\Sigma_{\binom{a_1}{0}}(x, E_1) \cap \Sigma_{\binom{a_1}{0}}(x', E_3)=\varnothing$$

and

$$\Sigma_{\binom{a_1}{0}}(x, E_i) \cap \Sigma_{\binom{a_1}{0}}(x', E_i)=\varnothing \qquad \text{if} \quad i=2, 3 \text{ and } x \neq x'\,.$$

Therefore we have the conclusion in the case of $\varepsilon_1=0$. The case of $\varepsilon_1=1$ can be discussed analogously. (q.e.d.)

By Lemma 2.1 and Lemma 2.2 we see that the map $\Sigma_{\binom{a_1}{\varepsilon_1}}$ is well defined as a map from $\mathscr{G}(\gamma, \delta)$ to $\mathscr{G}(\gamma_1, \delta_1)$. From now on, the map $\Sigma_{\binom{a_1}{\varepsilon_1}}$ is called the *substitution* associated with modified Jacobi-Perron algorithm.

LEMMA 2.3. *For any* $(x_1, F_1) \in \mathscr{S}(\gamma_1, \delta_1)$, *there exists* $(x, F) \in \mathscr{S}(\gamma, \delta)$ *such that*

$$(x_1, F_1) \in \Sigma_{\binom{a_1}{\varepsilon_1}}(x, F)\,.$$

PROOF. Let us assume that $\varepsilon_1=0$. (1) Assume that $(x_1, E_2) \in \mathscr{S}(\gamma_1, \delta_1)$, that is,

$$x_1+E_2 \subset P(\gamma_1, \delta_1) \quad \text{and} \quad x_1-e_2 \notin P(\gamma_1, \delta_1)\,.$$

Put $x=\varphi_{\binom{a_1}{0}}(x_1)$, then

$$(\gamma, x-e_3)=(\gamma, \varphi_{\binom{a_1}{0}}(x_1-e_2))=\eta^{-1}(\gamma_1, x_1-e_2) \leq 0\,.$$

This means

$$(x_1, E_2) \in \Sigma_{\binom{a_1}{0}}(x, E_3)\,.$$

(2) Assume that $(x_1, E_3) \in \mathscr{S}(\gamma_1, \delta_1)$, that is,

$$x_1+E_3\subset P(\gamma_1, \delta_1) \quad \text{and} \quad x_1-e_3\notin P(\gamma_1, \delta_1)\,.$$

Put $x=\varphi_{\binom{a_1}{0}}(x_1)$, then

$$(\gamma, x-e_1)=(\gamma, \varphi_{\binom{a_1}{0}}(x_1)-\varphi_{\binom{a_1}{0}}(e_3)) =\eta^{-1}(\gamma_1, x_1-e_3)\leq 0\,.$$

This means

$$(x_1, E_3)\in \Sigma_{\binom{a_1}{0}}(x, E_1)\,.$$

(3) Assume that $(x_1, E_1)\in \mathscr{S}(\gamma_1, \delta_1)$, that is,

$$x_1+E_1\subset P(\gamma_1, \delta_1) \quad \text{and} \quad x_1-e_1\notin P(\gamma_1, \delta_1)\,.$$

From $(\gamma_1, x_1-e_1)\leq 0$, there exists k such that $(\gamma_1, x_1-e_1+(k-1)e_3)<0$ and $(\gamma_1, x_1-e_1+ke_3)>0$ and k satisfies $1\leq k\leq a_1+1$. Because, from $-1=-(\gamma_1, e_1)$ and $(\gamma_1, x_1)>0$ we know that $0\geq(x_1-e_1, \gamma_1)>-1$. Therefore we see $1\leq k=[-(x_1-e_1, \gamma_1)/\delta_1]+1\leq [1/\delta_1]+1=a_1+1$. In the case of $1\leq k\leq a_1$, we take $(\varphi_{\binom{a_1}{0}}(x_1-e_1+ke_3), E_1)$, then we see that $(\varphi_{\binom{a_1}{0}}(x_1-e_1+ke_3), \gamma)=\eta^{-1}(x_1-e_1+ke_3, \gamma_1)>0$ and

$$\begin{aligned}(\varphi_{\binom{a_1}{0}}(x_1-e_1+ke_3)-e_1, \gamma)&=(\varphi_{\binom{a_1}{0}}(x_1-e_1+ke_3)-\varphi_{\binom{a_1}{0}}(e_3), \gamma)\\ &=(\varphi_{\binom{a_1}{0}}(x_1-e_1+(k-1)e_3), \gamma)\\ &=\eta^{-1}(x_1-e_1+(k-1)e_3, \gamma_1)\leq 0\,.\end{aligned}$$

This means

$$(\varphi_{\binom{a_1}{0}}(x_1-e_1+ke_3), E_1)\in \mathscr{G}(\gamma, \delta)\,.$$

Therefore, we see from the definition of $\Sigma_{\binom{a_1}{0}}$ that

$$\Sigma_{\binom{a_1}{0}}(\varphi_{\binom{a_1}{0}}(x_1-e_1+ke_3), E_1)\ni(x_1, E_1)\,.$$

In the case of $k=a_1+1$, we take

$$(\varphi_{\binom{a_1}{0}}(x_1-e_1+ke_3)-e_1+e_2, E_2)\,.$$

Then we see that

$$\begin{aligned}&(\varphi_{\binom{a_1}{0}}(x_1-e_1+(a_1+1)e_3)-e_1+e_2, \gamma)\\ &\quad=(\varphi_{\binom{a_1}{0}}(x_1-e_1+(a_1+1)e_3)-\varphi_{\binom{a_1}{0}}(e_3)+\varphi_{\binom{a_1}{0}}(e_1)-a_1\varphi_{\binom{a_1}{0}}(e_3), \gamma)\\ &\quad=(\varphi_{\binom{a_1}{0}}(x_1), \gamma)=\eta^{-1}(x_1, \gamma_1)>0\,.\end{aligned}$$

That is $(\varphi_{\binom{a_1}{0}}(x_1-e_1+(a_1+1)e_3)-e_1+e_2, E_2)\in \mathscr{S}(\gamma, \delta)$. Therefore, we see that

$$\Sigma_{\binom{a_1}{0}}(\varphi_{\binom{a_1}{0}}(x_1-e_1+(a_1+1)e_3)-e_1+e_2, E_2)=(x_1, E_1)\,.$$

Also we have

$$\varphi_{\binom{a_1}{0}}(x_1-e_1+(a_1+1)e_3)-e_1+e_2-e_2=\varphi_{\binom{a_1}{0}}(x_1-e_1+a_1e_3)\notin P(\gamma, \delta)\,.$$

The case of $\varepsilon_1=1$ can be discussed analogously. (q.e.d.)

Let us define a geometrical realization map $\mathscr{K}$ from $\mathscr{G}(\gamma,\delta)$ to the family of compact sets of $\boldsymbol{R}^3$ as follows:

$$\mathscr{K}((\boldsymbol{x},S)):=\boldsymbol{x}+S\,,$$

$$\mathscr{K}\left(\sum_{\lambda\in\Lambda}(\boldsymbol{x}_\lambda,S_\lambda)\right):=\bigcup_{\lambda\in\Lambda}(\boldsymbol{x}_\lambda+S_\lambda)\,.$$

Let us denote

$$\boldsymbol{S}(\gamma,\delta):=\bigcup_{(\boldsymbol{x},S)\in\mathscr{S}(\gamma,\delta)}\mathscr{K}((\boldsymbol{x},S))$$

and call it the *stepped surface* of the plane orthogonal to $\boldsymbol{\gamma}={}^t(1,\gamma,\delta)$. Then from Lemma 2.2 and Lemma 2.3 we have

PROPOSITION 2.4. *For any* $(\alpha,\beta,\gamma,\delta)\in\bar{X}$, $(\alpha,\beta)\neq(0,0)$, *the stepped surface* $\boldsymbol{S}(\gamma,\delta)$ *is invariant under the substitution* $\Sigma_{\binom{a_1}{\varepsilon_1}}$ *in the following sense*:

$$\boldsymbol{S}(\gamma_1,\delta_1)=\bigcup_{(\boldsymbol{x},S)\in\mathscr{S}(\gamma,\delta)}\mathscr{K}(\Sigma_{\binom{a_1}{\varepsilon_1}}(\boldsymbol{x},S))\,.$$

By the above discussion, it is easy to see that Lemmas 2.1, 2.2, 2.3 and Proposition 2.4 also hold on $\mathscr{S}'$ and $\mathscr{G}'$.

At the end of this section, we will show that for each $i=1,2,3$, $\mathscr{K}(\Sigma_{\binom{a_n}{\varepsilon_n}}\cdots\Sigma_{\binom{a_1}{\varepsilon_1}}(\boldsymbol{e}_i,E_i))$ is a topological cell, in other words $\pi_n\mathscr{K}(\Sigma_{\binom{a_n}{\varepsilon_n}}\cdots\Sigma_{\binom{a_1}{\varepsilon_1}}(\boldsymbol{e}_i,E_i))$ is a simply connected domain in $\mathbb{P}(\gamma_n,\delta_n)$. To prove this proposition, we will prepare several lemmas. Firstly let us introduce a set $\mathscr{C}_0$ as follows:

$$\begin{aligned}\mathscr{C}_0=\{&(0,E_1)+(-\boldsymbol{e}_1+\boldsymbol{e}_2,E_2),\ (0,E_1)+(-\boldsymbol{e}_1+\boldsymbol{e}_3,E_3),\ (0,E_1)+(\boldsymbol{e}_2,E_1),\\ &(0,E_1)+(\boldsymbol{e}_3,E_1),\ (0,E_1)+(0,E_2),\ (0,E_2)+(0,E_3),\ (0,E_3)+(0,E_1)\}\end{aligned}$$

and introduce a subset $\mathscr{C}$ of $\mathscr{G}(\gamma,\delta)$:

$$\mathscr{C}=\{T_{\boldsymbol{z}}\zeta\mid\zeta\in\mathscr{C}_0,\ \boldsymbol{z}\in\boldsymbol{Z}^3,\ T_{\boldsymbol{z}}\zeta\in\mathscr{G}(\gamma,\delta)\}\,,$$

where $T_{\boldsymbol{z}}$ is a translation map given by

$$T_{\boldsymbol{z}}\left(\sum_{\lambda\in\Lambda}(\boldsymbol{x}_\lambda,S_\lambda)\right)=\sum_{\lambda\in\Lambda}(\boldsymbol{x}_\lambda+\boldsymbol{z},S_\lambda)\,.$$

We say that a set Δ of $\mathscr{G}(\gamma,\delta)$ is $\mathscr{C}$-*covered* if there exists a finite subset $\{\zeta_\lambda\mid\lambda\in\Lambda\}$ of $\mathscr{C}$ satisfying the following properties:

(1) For any $\zeta_\lambda,\zeta_\mu\in\Delta$, there exist $\zeta_i\in\mathscr{C}$, $i=1,2,\cdots,n$, such that $\zeta_\lambda=\zeta_1$, $\zeta_i\cap\zeta_{i+1}\neq\varnothing$ $(i=1,2,\cdots,n-1)$ and $\zeta_\mu=\zeta_n$,

(2) $\bigcup_{\lambda\in\Lambda}\mathscr{K}(\zeta_\lambda)=\mathscr{K}(\Delta)$.

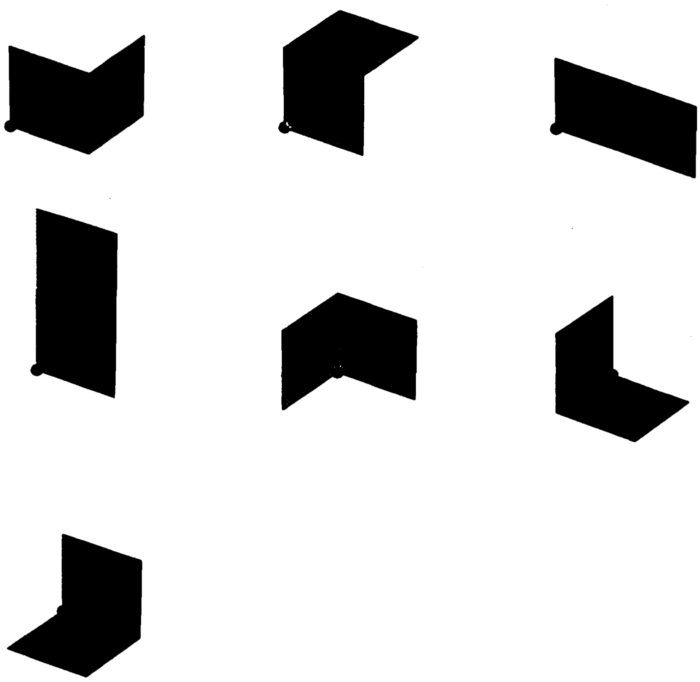

Figure of $\mathscr{C}_0$

LEMMA 2.5. *Assume that $\Delta \in \mathscr{G}(\gamma, \delta)$ is $\mathscr{C}$-covered. Then $\Sigma_{\binom{a_1}{\varepsilon_1}}(\Delta)$ is also $\mathscr{C}$-covered.*

PROOF. For each $\zeta \in \Delta$, $\zeta \in \mathscr{C}$, it is not difficult to see from the definition of $\Sigma_{\binom{a_1}{\varepsilon_1}}$ and Lemma 2.2 that $\Sigma_{\binom{a_1}{\varepsilon_1}}(\zeta)$ is $\mathscr{C}$-covered. (See Figure 3.) Therefore, we see that for any $\mathscr{C}$-covered set Δ, $\Sigma_{\binom{a_1}{\varepsilon_1}}(\Delta)$ is $\mathscr{C}$-covered. (q.e.d.)

We give the following definition.

DEFINITION 2.1. A $\mathscr{C}$-covered set Δ is called a *$\mathscr{C}$-covered cell* if its geometric realization $\mathscr{K}(\Delta)$ is a topological cell.

PROPOSITION 2.6. *If $\Delta \in \mathscr{G}(\gamma, \delta)$ is a $\mathscr{C}$-covered cell, then $\Sigma_{\binom{a_1}{\varepsilon_1}}(\Delta)$ is a $\mathscr{C}$-covered cell.*

PROOF. Suppose that $\mathscr{K}(\Sigma_{\binom{a_1}{\varepsilon_1}}(\Delta))$ is not a $\mathscr{C}$-covered cell, that is, the set $\mathscr{K}(S(\gamma_1, \delta_1)) - \mathscr{K}(\Sigma_{\binom{a_1}{\varepsilon_1}}(\Delta))$ has a bounded component D_1 and one unbounded component D_2. Then we are able to choose $(\boldsymbol{x}_1, S_1)$ and $(\boldsymbol{x}'_1, S'_1) \in \mathscr{G}(\gamma_1, \delta_1)$ such that $\mathscr{K}(\boldsymbol{x}_1, S_1) \subset D_1$ and $\mathscr{K}(\boldsymbol{x}'_1, S'_1) \subset D_2$. By Lemma 2.3, there exist $(\boldsymbol{x}, S)$ and $(\boldsymbol{x}', S') \in \mathscr{G}(\gamma, \delta)$ such that

$$\Sigma_{\binom{a_1}{\varepsilon_1}}(\boldsymbol{x}, S) \ni (\boldsymbol{x}_1, S_1), \qquad \Sigma_{\binom{a_1}{\varepsilon_1}}(\boldsymbol{x}', S') \ni (\boldsymbol{x}'_1, S'_1).$$

On the other hand, from the assumption that Δ is a topological cell, and from the fact that $(\boldsymbol{x}, S)$ and $(\boldsymbol{x}', S')$ do not belong to Δ, there exists a chain $\{\zeta_i \mid \zeta_i \in \mathscr{G}(\gamma, \delta)$ and $\zeta_i \in \mathscr{C}, i=1, 2, \cdots, m\}$ such that $\zeta_1 \cap (\boldsymbol{x}, S) \neq \emptyset$, $\zeta_i \cap \zeta_{i+1} \neq \emptyset$, $\zeta_m \cap (\boldsymbol{x}', S') \neq \emptyset$, $\zeta_i \cap \Delta = \emptyset$. Therefore the sequence $\Sigma_{\binom{a_1}{\varepsilon_1}}(\zeta_i)$, $i=1, 2, \cdots, m$, satisfy the following

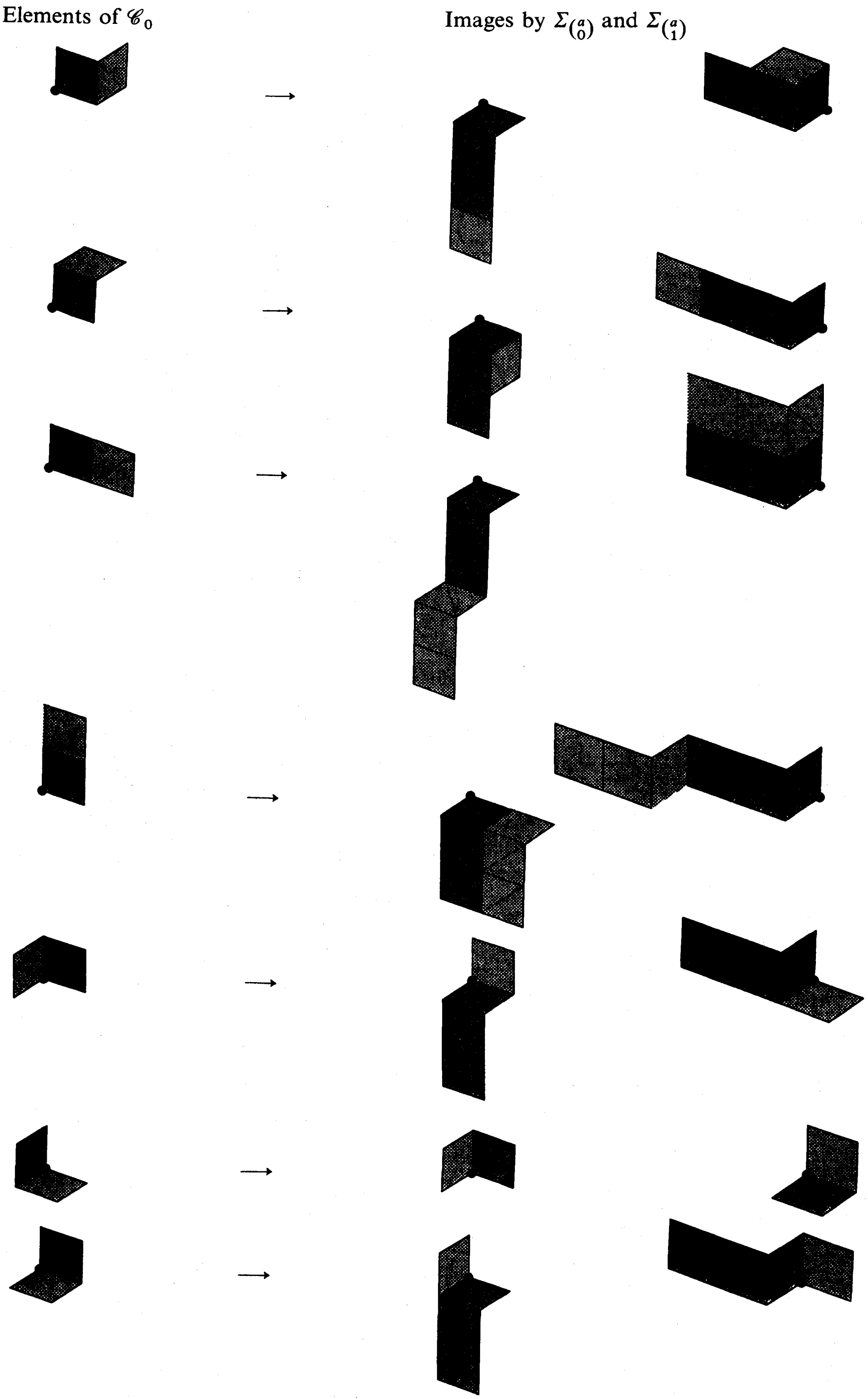

FIGURE 3. Figure of $\Sigma_{\binom{a}{\varepsilon}}(\zeta)$, $\zeta \in \mathscr{C}_0$

properties:

(1) $\Sigma_{\binom{a_1}{\varepsilon_1}}(\zeta_1) \ni (x_1, S_1)$,

(2) $\Sigma_{\binom{a_1}{\varepsilon_1}}(\zeta_i) \cap \Sigma_{\binom{a_1}{\varepsilon_1}}(\zeta_{i+1}) \neq \varnothing$,

(3) $\Sigma_{\binom{a_1}{\varepsilon_1}}(\zeta_m) \ni (x'_1, S'_1)$,

(4) $\Sigma_{\binom{a_1}{\varepsilon_1}}(\zeta_i) \cap \Sigma_{\binom{a_1}{\varepsilon_1}}(\Delta) = \varnothing$.

This means that D_1 and D_2 are connected by using pieces of $\mathscr{C}_0$ which are in the outside of $\mathscr{K}(\Sigma_{\binom{a_1}{\varepsilon_1}}(\Delta))$. This is a contradiction. (q.e.d.)

COROLLARY 2.7. (1) *Let us denote* $\mathscr{U} = \sum_{i=1,2,3}(e_i, E_i)$. *Then the sets*

$$\Sigma_{\binom{a_n}{\varepsilon_n}} \cdots \Sigma_{\binom{a_1}{\varepsilon_1}} \mathscr{U} \quad \text{and} \quad \Sigma_{\binom{a_n}{\varepsilon_n}} \cdots \Sigma_{\binom{a_1}{\varepsilon_1}} (e_i, E_i)\,, \quad i = 1, 2, 3\,,$$

are $\mathscr{C}$*-covered cells.*

(2) *Let us denote* $\mathscr{U}' = \sum_{i=1,2,3}(0, E_i)$. *Then the sets*

$$\Sigma_{\binom{a_n}{\varepsilon_n}} \cdots \Sigma_{\binom{a_1}{\varepsilon_1}} \mathscr{U}' \quad \text{and} \quad \Sigma_{\binom{a_n}{\varepsilon_n}} \cdots \Sigma_{\binom{a_1}{\varepsilon_1}} (0, E_i)\,, \quad i = 1, 2, 3\,,$$

are $\mathscr{C}$*-covered cells.*

3. Domain exchange transformations.

In this section, we introduce transformations on the plane $\mathbb{P}(\gamma, \delta)$ which we will call domain exchange transformations and discuss the properties as dynamical systems. From now on we assume that $(\alpha_n, \beta_n) \neq (0, 0)$ for all n. For each integers n, k such that $0 \leq k \leq n$, let us introduce the column vectors

$$(f_1^{(n,k)}, f_2^{(n,k)}, f_3^{(n,k)})$$

by

$$(f_1^{(n,k)}, f_2^{(n,k)} f_3^{(n,k)}) = A^{-1}_{\binom{a_n}{\varepsilon_n}} A^{-1}_{\binom{a_{n-1}}{\varepsilon_{n-1}}} \cdots A^{-1}_{\binom{a_{k+1}}{\varepsilon_{n+1}}}\,, \tag{3-1}$$

$$(f_1^{(n,n)}, f_2^{(n,n)}, f_3^{(n,n)}) = (e_1, e_2, e_3)$$

and denote the domains on $\mathbb{P}(\gamma_n, \delta_n)$ by

$$D^{(n,k)} := \pi_n \mathscr{K}(\Sigma_{\binom{a_n}{\varepsilon_n}} \cdots \Sigma_{\binom{a_{k+1}}{\varepsilon_{k+1}}} \mathscr{U})\,, \tag{3-2}$$

$$D_i^{(n,k)} := \pi_n \mathscr{K}(\Sigma_{\binom{a_n}{\varepsilon_n}} \cdots \Sigma_{\binom{a_{k+1}}{\varepsilon_{k+1}}} (e_i, E_i))\,,$$

$$D'^{(n,k)} := \pi_n \mathscr{K}(\Sigma_{\binom{a_n}{\varepsilon_n}} \cdots \Sigma_{\binom{a_{k+1}}{\varepsilon_{k+1}}} \mathscr{U}')\,,$$

$$D_i'^{(n,k)} := \pi_n \mathscr{K}(\Sigma_{\binom{a_n}{\varepsilon_n}} \cdots \Sigma_{\binom{a_{k+1}}{\varepsilon_{k+1}}} (0, E_i))\,.$$

Then we know from Lemmas 2.2 and 2.3 that

$$D^{(n,k)} = \bigcup_{i=1,2,3} D_i^{(n,k)}\,, \qquad \text{int.} D_i^{(n,k)} \cap \text{int.} D_j^{(n,k)} = \varnothing \ (i \neq j)\,,$$

$$D'^{(n,k)} = \bigcup_{i=1,2,3} D_i'^{(n,k)}, \qquad \text{int.}D_i'^{(n,k)} \cap \text{int.}D_j'^{(n,k)} = \varnothing \ (i \neq j).$$

LEMMA 3.1.

$$D^{(n,k)} = D'^{(n,k)}.$$

PROOF. The proof is obtained by induction for k. In the case of $k=n$, the relation $D^{(n,n)} = D'^{(n,n)}$ is trivial. We see $D^{(n,k-1)} = D'^{(n,k-1)}$ on the assumption $D^{(n,k)} = D'^{(n,k)}$. The sets $D^{(n,k-1)}$ and $D'^{(n,k-1)}$ are decomposed by

$$\begin{aligned} D^{(n,k-1)} &= \pi_n \mathscr{K}(\Sigma_{\binom{a_n}{\varepsilon_n}} \cdots \Sigma_{\binom{a_k}{\varepsilon_k}}(\mathscr{U})) \\ &= \pi_n \mathscr{K}(\Sigma_{\binom{a_n}{\varepsilon_n}} \cdots \Sigma_{\binom{a_{k+1}}{\varepsilon_{k+1}}}(\mathscr{U})) + \pi_n \mathscr{K}(\Sigma_{\binom{a_n}{\varepsilon_n}} \cdots \Sigma_{\binom{a_{k+1}}{\varepsilon_{k+1}}}(\Sigma_{\binom{a_k}{\varepsilon_k}}\mathscr{U} - \mathscr{U})), \\ D'^{(n,k-1)} &= \pi_n \mathscr{K}(\Sigma_{\binom{a_n}{\varepsilon_n}} \cdots \Sigma_{\binom{a_{k+1}}{\varepsilon_{k+1}}}(\mathscr{U}')) + \pi_n \mathscr{K}(\Sigma_{\binom{a_n}{\varepsilon_n}} \cdots \Sigma_{\binom{a_{k+1}}{\varepsilon_{k+1}}}(\Sigma_{\binom{a_k}{\varepsilon_k}}\mathscr{U}' - \mathscr{U}')). \end{aligned}$$

By the definition of $\Sigma_{\binom{a_k}{\varepsilon_k}}$ we find out

$$\Sigma_{\binom{a_k}{\varepsilon_k}}\mathscr{U} - \mathscr{U} = \Sigma_{\binom{a_k}{\varepsilon_k}}\mathscr{U}' - \mathscr{U}'.$$

Therefore by the assumption we have $D^{(n,k)} = D'^{(n,k)}$. (q.e.d.)

Let us define the transformation $W^{(n,k)}$ on $D^{(n,k)}$ as follows:

$$W^{(n,k)}\boldsymbol{x} = \boldsymbol{x} - \pi_n \boldsymbol{f}_i^{(n,k)} \qquad \text{if} \quad \boldsymbol{x} \in D_i^{(n,k)}.$$

From Lemma 3.1 we see that

$$D_i^{(n,k)} - \pi_n \boldsymbol{f}_i^{(n,k)} = D_i'^{(n,k)}.$$

This means the transformations $W^{(n,k)}$ are well defined. We call these transformations *the domain exchange transformations* associated with modified Jacobi-Perron algorithm.

Let us denote the lattices $L^{(n,k)}$ on $\mathbb{P}(\gamma_n, \delta_n)$ as follows:

$$L^{(n,k)} = \{s(\pi_n \boldsymbol{f}_2^{(n,k)} - \pi_n \boldsymbol{f}_1^{(n,k)}) + t(\pi_n \boldsymbol{f}_3^{(n,k)} - \pi_n \boldsymbol{f}_1^{(n,k)}) \mid s, t \in \boldsymbol{Z}\}. \tag{3-3}$$

From the property that for all $n \in \boldsymbol{N}$

$$\bigcup_{\boldsymbol{l} \in L^{(n,n)}} (\pi_n \mathscr{K}(\mathscr{U}) + \boldsymbol{l}) = \mathbb{P}(\gamma_n, \delta_n) \quad \text{and}$$

$$\text{int.}(\pi_n \mathscr{K}(\mathscr{U}) + \boldsymbol{l}) \cap \text{int.}(\pi_n \mathscr{K}(\mathscr{U}) + \boldsymbol{l}') = \varnothing \qquad \text{if} \quad \boldsymbol{l} \neq \boldsymbol{l}', \ \boldsymbol{l}, \ \boldsymbol{l}' \in L^{(n,n)},$$

we know from Lemmas 2.2 and 2.3 that for all $n \in \boldsymbol{N}$

$$\bigcup_{\boldsymbol{l} \in L^{(n,k)}} (D^{(n,k)} + \boldsymbol{l}) = \mathbb{P}(\gamma_n, \delta_n) \quad \text{and} \tag{3-4}$$

$$\text{int.}(D^{(n,k)} + \boldsymbol{l}) \cap \text{int.}(D^{(n,k)} + \boldsymbol{l}') = \varnothing \qquad \text{if} \quad \boldsymbol{l} \neq \boldsymbol{l}', \ \boldsymbol{l}, \boldsymbol{l}' \in L^{(n,k)}.$$

Therefore we are able to identify $D^{(n,k)}$ and 2-dimensional torus $\boldsymbol{T}^2 = \mathbb{P}(\gamma_n, \delta_n)/L^{(n,k)}$. Moreover the domain exchange $W^{(n,k)}$ on $D^{(n,k)}$ can be identified to the quasi-periodic motion (Weyl automorphism) $Q^{(n,k)}$ on $D^{(n,k)}$ such that

$$Q^{(n,k)}x = x - \pi_n f_i^{(n,k)} \qquad (\mathrm{mod}\ L^{(n,k)}) .$$

From the fact that

$$\Sigma_{\binom{a_k}{\varepsilon_k}} \mathcal{U} \supset \mathcal{U}$$

we note that

$$D^{(n,k)} \subset D^{(n,k-1)} \quad \text{and} \quad D'^{(n,k)} \subset D'^{(n,k-1)} .$$

Now to show that the induced automorphism of $W^{(n,k-1)}$ to the set $D^{(n,k)}$ is equal to $W^{(n,k)}$, we introduce a second substitution $\sigma_{\binom{a}{\varepsilon}}$ on the set $S^* = \{1, 2, 3\}^* = \bigcup_{m=0}^{\infty} \prod_{k=1}^{m} \{1, 2, 3\}^k$ as follows:

$$\sigma_{\binom{a}{0}} : \begin{array}{l} 1 \to \overbrace{11 \cdots 1}^{a}2 \\ 2 \to 3 \\ 3 \to 1 \end{array}$$

and

$$\sigma_{\binom{a}{1}} : \begin{array}{l} 1 \to \overbrace{11 \cdots 1}^{a}3 \\ 2 \to 1 \\ 3 \to 2 \end{array}$$

and for $w = s_1 s_2 \cdots s_k \in S^*$

$$\sigma_{\binom{a}{\varepsilon}}(w) := \sigma_{\binom{a}{\varepsilon}}(s_1)\sigma_{\binom{a}{\varepsilon}}(s_2) \cdots \sigma_{\binom{a}{\varepsilon}}(s_k) .$$

Let f be the canonical homomorphism from S^* to $\boldsymbol{Z}^3$, that is,

(3-5)
$$f(i) := e_i ,$$
$$f(w) := \sum_{i=1}^{k} f(s_i) \qquad \text{for} \quad w = s_1 \cdots s_k \in S^* .$$

Then we have easily the following lemma.

LEMMA 3.1. *The following commutative relation holds: for each* $\varepsilon \in \{0, 1\}$

$$\begin{array}{ccc} S^* & \xrightarrow{\sigma_{\binom{a}{\varepsilon}}} & S^* \\ f \downarrow & & \downarrow f \\ \boldsymbol{Z}^3 & \xrightarrow{L_{\binom{a}{\varepsilon}}} & \boldsymbol{Z}^3 \end{array}$$

where $L_{\binom{a}{\varepsilon}}$ *is the linear map represented by* $A_{\binom{a}{\varepsilon}}$.

In general, under a recurrent transformation $T: X \to X$ and a transformation

$T_A: A \to A$ on a subset A of X, we say the transformation T_A is the induced transformation of T to the set A if the following relation holds:

$$T_A x = T^{n(x)} x \qquad \text{for any} \quad x \in A,$$

where $n(x) = \min\{n \mid T^n x \in A, n \geq 1\}$. We say the induced automorphism T_A of the transformation (X, T) to the set $A(\subset X)$ has a *substitution structure* σ:

$$\sigma: \begin{array}{l} 1 \to s_1^{(1)} s_2^{(1)} \cdots s_{k_1}^{(1)} \\ 2 \to s_1^{(2)} s_2^{(2)} \cdots s_{k_2}^{(2)} \\ 3 \to s_1^{(3)} s_2^{(3)} \cdots s_{k_3}^{(3)}, \end{array}$$

if there exist partitions $\{P_1, P_2, P_3\}$ of A and $\{X_1, X_2, X_3\}$ of X satisfying the following properties: there exist integers k_1, k_2, k_3 such that

$$T^{k-1} P_i \subset X_{s_k^{(i)}} \quad \text{for } 1 \leq k \leq k_i, \qquad T^{k-1} P_i \cap A = \emptyset \quad \text{for } 2 \leq k \leq k_i,$$
$$\text{and} \quad T^{k_i} P_i = T_A(P_i).$$

LEMMA 3.2. *For each n the induced automorphism of $W^{(n,n-1)}$ to the set $D^{(n,n)}$ is equal to $W^{(n,n)}$, and the induced automorphism has $\sigma_{\binom{a_n}{\varepsilon_n}}$-structure.*

PROOF. It is enough to see the assertion when $n=1$. Assume $\varepsilon_1 = 0$, then we see easily that

$$(W^{(1,0)})^k (D_1^{(1,1)}) = D_1^{(1,1)} - k\pi_1 f_1^{(1,1)} \subset D_1^{(1,0)} \qquad 0 \leq k \leq a_1 - 1,$$
$$(W^{(1,0)})^{a_1} (D_1^{(1,1)}) = D_2^{(1,0)},$$
$$(W^{(1,0)})^{a_1+1} (D_1^{(1,1)}) = W^{(1,0)}(D_2^{(1,0)}) = D_2^{(1,0)} - \pi_1 f_2^{(1,1)} = \pi_1 \mathscr{K}((0, E_1)) = D_1'^{(1,1)},$$
$$D_2^{(1,1)} \subset D_3^{(1,0)},$$
$$W^{(1,0)}(D_2^{(1,1)}) = D_2^{(1,1)} - \pi_1 f_3^{(1,1)} = \pi_1 \mathscr{K}((0, E_2)) = D_2'^{(1,1)},$$
$$D_3^{(1,1)} \subset D_1^{(1,0)},$$
$$W^{(1,0)}(D_3^{(1,1)}) = D_3^{(1,1)} - \pi_1 f_1^{(1,1)} = \pi_1 \mathscr{K}((0, E_3)) = D_3'^{(1,1)}.$$

This is nothing but the conclusion. (See Figure 4.)

LEMMA 3.3. *For each n and k, $1 \leq k \leq n$, the induced automorphism of $W^{(n,k-1)}$ to the set $D^{(n,k)}$ is equal to $W^{(n,k)}$, and the induced automorphism has $\sigma_{\binom{a_k}{\varepsilon_k}}$-structure.*

The proof is completely the same as the proof of Lemma 3.2. (See Figure 5.)

PROPOSITION 3.4. *Let $(\alpha, \beta, \gamma, \delta) \in \bar{X}$ and $(\alpha, \beta) \neq (0, 0)$. For each n the induced automorphism of $W^{(n,0)}$ to $D^{(n,n)}$ is equal to $W^{(n,n)}$, and the induced automorphism has $\sigma_{\binom{a_1}{\varepsilon_1}} \cdots \sigma_{\binom{a_n}{\varepsilon_n}}$-structure.*

The poof is obtained by repeated applications of Lemma 3.3.

COROLLARY 3.5. *Let* $(\alpha, \beta, \gamma, \delta) \in \bar{X}$ *and* $(\alpha_n, \beta_n) \neq (0, 0)$. *For each n, the cardinality of* $\{(x, E_j) \mid (x, E_j) \in \Sigma_{\binom{a_n}{\varepsilon_n}} \cdots \Sigma_{\binom{a_1}{\varepsilon_1}}(e_i, E_i)\}$ *is equal to* $n_{ij}^{(n)}$, *where* $A_{\binom{a_1}{\varepsilon_1}} \cdots A_{\binom{a_n}{\varepsilon_n}} = (n_{ij}^{(n)})$, $1 \leq i, j \leq 3$.

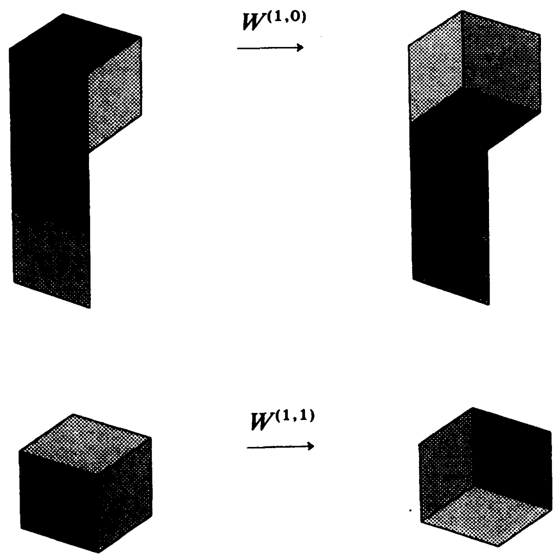

FIGURE 4. $W^{(1,1)}$ is induced from $W^{(1,0)}$

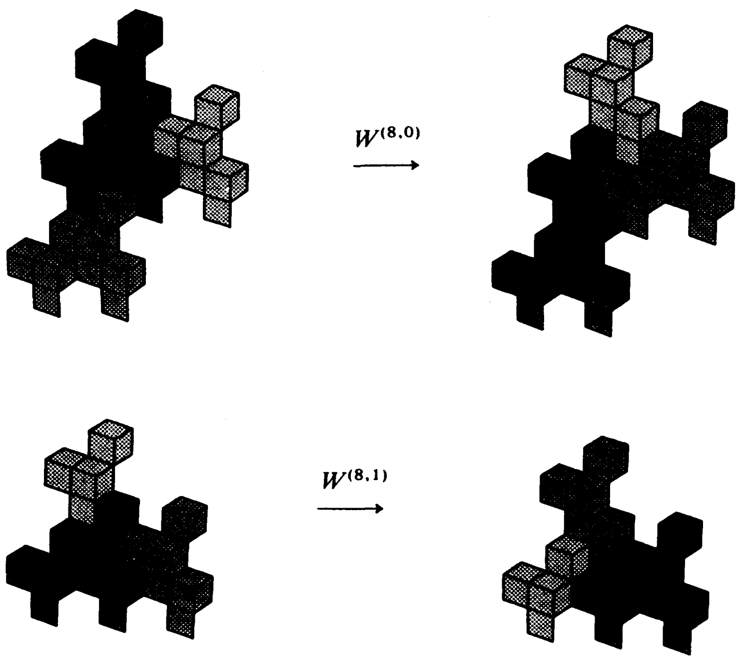

FIGURE 5. $W^{(8,1)}$ is induced from $W^{(8,0)}$ $(a_n = 1, \varepsilon_n = 0, n = 1, 2, \cdots, 8)$

4. Renormalization.

From now on, we make the following assumption:

ASSUMPTION P. For the point $(\alpha, \beta, \gamma, \delta)$ we assume that

(1) $(\alpha, \beta, \gamma, \delta)$ is a periodic point of $\bar{T}$ with period k,

(2) the eigenvalue $\lambda = \theta^{-1}\theta_1^{-1} \cdots \theta_{k-1}^{-1}$ of $A_{\binom{a_1}{\varepsilon_1}} \cdots A_{\binom{a_k}{\varepsilon_k}}$ is a Pisot number, that is,

$$\lambda > 1 \quad \text{and} \quad 0 < |\lambda_i| < 1 ,$$

where λ_i are the other eigenvalues of $A_{\binom{a_1}{\varepsilon_1}} \cdots A_{\binom{a_k}{\varepsilon_k}}$.

For the simplicity, we put as follows:

$$\Phi := \phi_{\binom{a_1}{\varepsilon_1}} \cdots \phi_{\binom{a_k}{\varepsilon_k}} = A_{\binom{a_1}{\varepsilon_1}} \cdots A_{\binom{a_k}{\varepsilon_k}} = {}^t(n_{ij})_{1 \le i,j \le 3} ,$$

$$(f_1^{(m)}, f_2^{(m)}, f_3^{(m)}) := (f_1^{(mk,0)}, f_2^{(mk,0)}, f_3^{(mk,0)}) = \Phi^{-m} ,$$

$$\mathbb{P} := \mathbb{P}(\alpha, \beta, \gamma, \delta) \qquad \text{(the orthogonal plane to } {}^t(1, \gamma, \delta)) ,$$

$$\pi := \pi_0 \qquad \text{(the projection to } \mathbb{P} \text{ along } {}^t(1, \gamma, \delta)) ,$$

$$\Sigma := \Sigma_{\binom{a_k}{\varepsilon_k}} \cdots \Sigma_{\binom{a_1}{\varepsilon_1}} \qquad \text{(the substitution on } \mathcal{G}(\alpha, \beta, \gamma, \delta)) ,$$

$$\sigma := \sigma_{\binom{a_1}{\varepsilon_1}} \cdots \sigma_{\binom{a_k}{\varepsilon_k}} ,$$

$$L := \{ m\pi(e_2 - e_1) + n\pi(e_3 - e_1) \mid m, n \in \mathbf{Z} \} ,$$

$$L^* := \{ \pi x \mid (x, S) \in \mathcal{S}(\gamma, \delta) \} ,$$

$$D^{(m)} := \pi\mathcal{K}(\Sigma^m \mathcal{U}) (= D^{(mk,0)}) ,$$

$$D_i^{(m)} := \pi\mathcal{K}(\Sigma^m(e_i, E_i)) (= D_i^{(mk,0)}) ,$$

$$W^{(m)} := W^{(mk,0)} \qquad \text{(domain exchange on } D^{(m)}) .$$

Then we have that

(4-1)
$$\Phi \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} = \frac{1}{\theta\theta_1 \cdots \theta_{k-1}} \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} \qquad \text{(by Prop. 1.1)} ,$$

$$ {}^t\Phi^{-1} \begin{pmatrix} 1 \\ \gamma \\ \delta \end{pmatrix} = \frac{1}{\eta\eta_1 \cdots \eta_{k-1}} \begin{pmatrix} 1 \\ \gamma \\ \delta \end{pmatrix} \qquad \text{(by Prop. 1.1)} ,$$

(4-2) $\mathbb{P}$ is Φ-invariant (by Cor. 1.3)

and for all $m \in \mathbf{N} \cup \{0\}$,

(4-3)
$$\bigcup_{z\in L}(D^{(m)}+\Phi^{-m}z)=\mathbb{P} \qquad \text{(by 3-4)},$$

$$\text{int.}(D^{(m)}+\Phi^{-m}z)\cap\text{int.}(D^{(m)}+\Phi^{-m}z')=\emptyset \qquad (z\neq z'\in L),$$

(4-4)
$$\bigcup_{\substack{(z,i):\\ z=\pi x,\\ (x,E_i)\in\mathscr{S}(\gamma,\delta)}}(D_i^{(m)}+\Phi^{-m}z)=\mathbb{P},$$

$$\text{int.}(D^{(m)}+\Phi^{-m}z)\cap\text{int.}(D^{(m)}+\Phi^{-m}z')=\emptyset \qquad (z\neq z'\in L^*).$$

(4-5) the induced automorphism of $W^{(m)}$ to the set $D^{(m-1)}$ is equal to $W^{(m-1)}$ and the induced automorphism $W^{(m)}\,|_{D^{(m-1)}}$ has a σ-structure.

Let us introduce subsets of L^* as follows:

$$L^{*(i,j)}=\{\pi x \mid (x, E_j)\in\Sigma_{\binom{a_k}{\varepsilon_k}}\cdots\Sigma_{\binom{a_1}{\varepsilon_1}}(e_i, E_i)\}.$$

Then we know from Cor. 3.5 that the cardinality of $L^{*(i,j)}$ is equal to n_{ij}. Therefore we denote the element of $L^{*(i,j)}$ by $z_k^{(i,j)}$, $1\leq k\leq n_{ij}$. Using this notation, we see that

$$D^{(1)}=\sum_{1\leq i,j\leq 3}\ \sum_{1\leq k\leq n_{ij}}(\pi\mathscr{K}(0, E_j)+z_k^{(i,j)}).$$

Moreover, generally we have

(4-6)
$$D^{(m)}=\sum\sum\pi\mathscr{K}(\Sigma^{m-1}(0, E_j)+\Phi^{-(m-1)}z_k^{(i,j)}).$$

Now let us define the sets on $\mathbb{P}$ as follows:

(4-7)
$$X=\lim_{m\to\infty}\Phi^m\pi\mathscr{K}(\Sigma^m(\mathscr{U})),$$

$$X_i=\lim_{m\to\infty}\Phi^m\pi\mathscr{K}(\Sigma^m(e_i, E_i)) \qquad \text{for} \quad i=1, 2, 3,$$

$$X_i'=\lim_{m\to\infty}\Phi^m\pi\mathscr{K}(\Sigma^m(0, E_i)) \qquad \text{for} \quad i=1, 2, 3,$$

where the limit is in the sense of Hausdorff metric on the family of compact subsets in $\mathbb{P}$. In the next section, we will see the existence of domains X, X_i, $i=1, 2, 3$ with fractal boundaries. On the assumption of the existence of domains X, X_i, X_i', $i=1, 2, 3$, we have the following theorem:

THEOREM 4.1. *For each points* $(\alpha, \beta, \gamma, \delta)\in\bar{X}$ *satisfying the Assumption* P, *the sets* X *and* X_i, $i=1, 2, 3$ *given by* (4-7) *have the following properties* (*see Figure* 6):

(1) (*Periodic space tiling*)

$$\bigcup_{x\in L}(X+x)=\mathbb{P},$$

$$\text{int.}(X+x)\cap\text{int.}(X+x')=\emptyset \qquad (x\neq x'\in L).$$

(2) (*Quasi-periodic space tiling*)

$$\bigcup_{(x,i):(x,E_i)\in\mathscr{S}(\gamma,\delta)} (X'_i+\pi x)=\mathbb{P}\,,$$

$$\text{int.}(X'_i+\pi x)\cap\text{int.}(X'_j+\pi x')=\varnothing \qquad \text{if} \quad (x,E_i)\neq(x',E_j)\in\mathscr{S}(\gamma,\delta)\,.$$

(3) *The following domain exchange transformation W on X is well defined*:

$$Wx=x-\pi e_i \qquad \text{if} \quad x\in X_i\,.$$

(4) (*Self-similarity of automorphism W*) *Let $X^{(1)}$ and $X_i^{(1)}$, $i=1, 2, 3$, be subsets of X given by*

$$X^{(1)}=\Phi X=\lim_{m\to\infty}\Phi^{m+1}\pi\mathscr{K}(\Sigma^m(\mathscr{U}))\,,$$

$$X_i^{(1)}=\Phi X_i=\lim_{m\to\infty}\Phi^{m+1}\pi\mathscr{K}(\Sigma^m(e_i,E_i))\,.$$

Let $W^{(1)}$ be the domain exchange transformation such that

$$W^{(1)}x=x-\pi\Phi e_i \qquad \text{if} \quad x\in X_i^{(1)}\,.$$

Then the induced transformation of W to the set $X^{(1)}$ is equal to $W^{(1)}$, that is, the following commutative relation holds:

$$\begin{array}{ccc} X & \xrightarrow{W} & X \\ \Phi\big\downarrow & & \big\downarrow\Phi \\ X^{(1)} & \xrightarrow{W|_{X^{(1)}}} & X^{(1)}\,. \end{array}$$

Moreover the induced automorphism has σ-structure, that is, denoting

$$\sigma_{\binom{a_1}{\varepsilon_1}}\cdots\sigma_{\binom{a_k}{\varepsilon_k}}:\ \begin{array}{l} 1\to s_1^{(1)}s_2^{(1)}\cdots s_{k_1}^{(1)} \\ 2\to s_1^{(2)}s_2^{(2)}\cdots s_{k_2}^{(2)} \\ 3\to s_1^{(3)}s_2^{(3)}\cdots s_{k_3}^{(3)}\,, \end{array}$$

then we have the relations: *for each $i=1, 2, 3$,*

$$W^{k-1}X_i^{(1)}\subset X_{s_k^{(i)}} \qquad 0\leq k\leq k_i-1\,,$$

$$W^{k_i}X_i^{(1)}=W|_{X^{(1)}}(X_i^{(1)})\ (=X_i^{(1)'})\,,$$

where k_i is given by

$$k_i=\sum_{j=1,2,3} n_{ij}\,.$$

(5) *The following Markov endomorphism T on X with structure matrix Φ is well defined*: *for $x\in X^{(1)}$*

$$Tx=\Phi^{-1}x-z_k^{(i,j)} \qquad \text{if} \quad \Phi^{-1}x\in z_k^{(i,j)}+X_j\,.$$

PROOF. For the relations of domains $D^{(m)}$ and $D_i^{(m)}$ in (4-3)–(4-6), we operate the renormalization operator Φ^m and take m to infinite, then we have the conclusions.

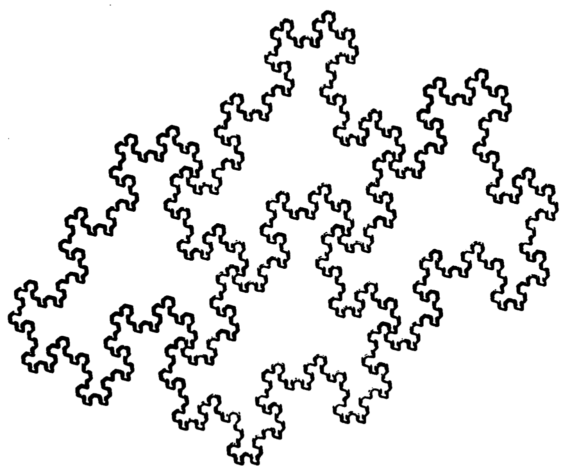

FIGURE 6-1. Tiling by X ($a_n = 1$, $\varepsilon_n = 0$ for all n)

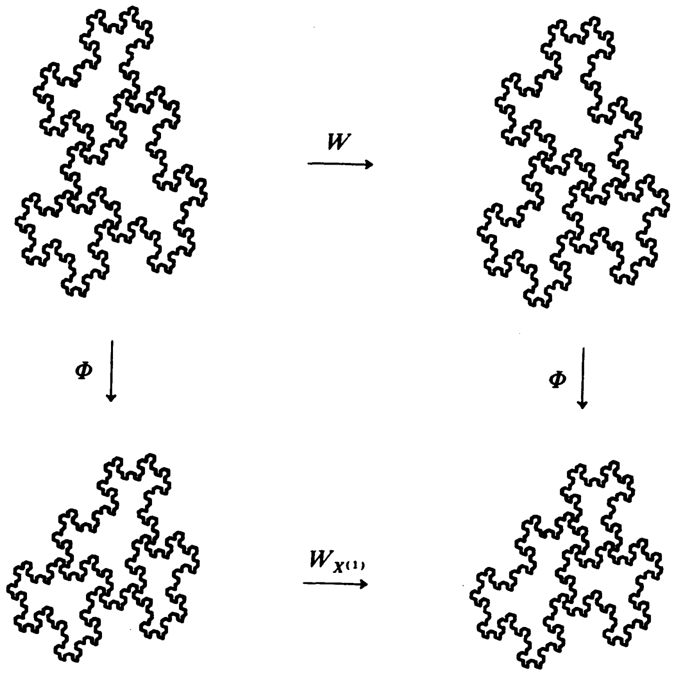

FIGURE 6-2. Self-similarity of the automorphism W

COROLLARY 4.2 (Recurrence rule of the origin point by the domain exchange W). *Under the Assumption* P, *let us consider the decreasing sequence* $X^{(m)}$ *of the neighborhoods of the origin point by* $X^{(m)} = \Phi^m X$, $m = 1, 2, \cdots$. *Put* $q_m = n_{11}^{(m)} + n_{21}^{(m)} + n_{31}^{(m)}$, *where* $\Phi^m = {}^t(n_{ij}^{(m)})$. *Then the following recurrence rule holds*:

$$W^k(\mathbf{0}) \notin X^{(m)} \qquad 1 \le k < q_m ,$$
$$W^{q_m}(\mathbf{0}) \in X^{(m)} .$$

PROOF. From ${}^t(1, \alpha, \beta) \in \mathscr{K}(e_1, E_1)$, we know that $\mathbf{0} \in \pi\mathscr{K}(e_1, E_1)$. From the definition of substitution σ, the head letter of $\sigma(1)$ is equal to 1, hence $\Sigma(e_1, E_1) \ni (e_1, E_1)$. This means that $\mathbf{0} \in D_1^{(m)}$ for all m, and so that $\mathbf{0} \in X_1$. By Theorem 4.1 (4), we already know the recurrence rule of the set $\Phi^m X_1$ to $\Phi^m X$ by using induced automorphism of W to the set $\Phi^m X$. Therefore we have the conclusion. (q.e.d.)

COROLLARY 4.3 (A construction of Markov partitions). *Under the Assumption* P, *Let S be the automorphism on* $\boldsymbol{T}^3$ *determined by* Φ. *Put*

$$I_i = \{\lambda(e_i - \pi e_i) \mid 0 \le \lambda < 1\} ,$$
$$P_i = \{x + y \mid x \in X_i, y \in I_i\} .$$

Then the set $\{P_i \mid i = 1, 2, 3\}$ *determines a partition of* $\boldsymbol{T}^3$ *which is a Markov partition with structure matrix* ${}^t\Phi$.

PROOF. Putting $P_i^0 = \{x + y \mid x \in \pi\mathscr{K}(e_i, E_i), y \in I_i\}$, then the union $D^0 (= \bigcup P_i^0)$ of parallelepipeds P_i^0, whose bottom is $\pi\mathscr{K}(e_i, E_i)$ and top is $e_i + \pi\mathscr{K}(0, E_i)$, satisfies the property

$$\bigcup_{z \in \boldsymbol{Z}^3} (D^0 + z) = \boldsymbol{R}^3 ,$$
$$\text{int.}(D^0 + z) \cap \text{int.}(D^0 + z') = \varnothing \qquad (z \ne z' \in \boldsymbol{Z}^3) .$$

Therefore putting $D = \bigcup_{i=1,2,3} P_i$, we see that

$$\bigcup_{z \in \boldsymbol{Z}^3} (D + z) = \boldsymbol{R}^3 ,$$
$$\text{int.}(D + z) \cap \text{int.}(D + z') = \varnothing .$$

The image by Φ of the contractive boundary $e_i + \{x - \pi e_i \mid x \in X_i\}$ of P_i is included in

$$\Phi e_i + \{\Phi x - \pi\Phi e_i \mid x \in X_i\} \subset \Phi e_i + X .$$

On the other hand the image by Φ^{-1} of the expanding boundary of P_i is included in the union of boundaries

$$\bigcup_{j=1,2,3} \bigcup_{z_k^{(i,j)} \in L^*(i,j)} (\partial^+ P_j + z_k^{(i,j)})$$

by Theorem 4.1 (5), where $\partial^+ P_j$ is the expanding boundary of P_j. This means the partition P is a Markov partition of T_Φ with structure matrix ${}^t\Phi$. (P is a Markov partition of $T_{\Phi^{-1}}$ with structure matrix Φ.) (q.e.d.)

5. Boundaries with fractal curves.

Instead of the proof for the existence of the limit set

$$X = \lim_{m\to\infty} \Phi^m \pi \mathscr{K}(\Sigma^m(\mathscr{U})),$$

we prove under Assumption P the existence of the limit set of boundaries

$$B = \lim_{n\to\infty} \Phi^m \pi \partial \mathscr{K}(\Sigma^m(\mathscr{U}))$$

as a 'generalized' simply closed Jordan curve on $\mathbb{P}$. For this purpose we introduce a free group G and an endomorphism η of G as follows. Let I_i, $i=1, 2, 3$, be oriented intervals generated by e_1, e_2, e_3, that is,

$$I_i := \{\lambda e_i \mid 0 \le \lambda \le 1\}, \qquad i=1, 2, 3.$$

Let us introduce a free abelian group G generated by $Z^3 \times \{I_1, I_2, I_3\}$, that is,

$$H = \left\{ \sum_{\lambda\in\Lambda} m_\lambda(z_\lambda, I_\lambda) \;\middle|\; \begin{matrix} z_\lambda \in Z^3, & I_\lambda \in \{I_1, I_2, I_3\}, \\ m_\lambda \in Z, & \#\Lambda < \infty \end{matrix} \right\}.$$

For each (a, ε), let us define an endomorphism $\eta_{\binom{a}{\varepsilon}}$ of G by

$$\eta_{\binom{a}{0}}: \begin{array}{lcl} (0, I_1) & \to & (0, I_3) \\ (0, I_2) & \to & (0, I_1) - \sum_{1\le k\le a} (e_1 - k e_3, I_3) \\ (0, I_3) & \to & (0, I_2), \end{array}$$

$$\eta_{\binom{a}{1}}: \begin{array}{lcl} (0, I_1) & \to & (0, I_2) \\ (0, I_2) & \to & (0, I_1) - \sum_{1\le k\le a} (e_1 - k e_2, I_2) \\ (0, I_3) & \to & (0, I_1), \end{array}$$

$$\eta_{\binom{a}{\varepsilon}}(z, I_i) = \varphi^{-1}_{\binom{a}{\varepsilon}}(z) + \eta_{\binom{a}{\varepsilon}}(0, I_i),$$

$$\eta_{\binom{a}{\varepsilon}}\left(\sum_{\lambda\in\Lambda} m_\lambda(z_\lambda, I_\lambda) \right) = \sum_{\lambda\in\Lambda} m_\lambda(\eta_{\binom{a}{\varepsilon}}(z_\lambda, I_\lambda)).$$

Let us define a map ∂ from $\mathscr{G}(\gamma, \delta)$ to G, called boundary map, as follows:

$$\partial(0, E_1) = (0, I_2) - (0, I_3) - (e_3, I_2) + (e_2, I_3),$$

$$\partial(0, E_2) = (0, I_3) - (0, I_1) - (e_1, I_3) + (e_3, I_1),$$

$$\partial(0, E_3) = (0, I_1) - (0, I_2) - (e_2, I_1) + (e_1, I_2),$$

$$\partial(x, E_i) = T_x \partial(0, E_i) \qquad \text{for} \quad (x, E_i) \in \mathscr{G}(\gamma, \delta),$$

$$\partial\left(\sum_{\lambda\in\Lambda}(\boldsymbol{x}_\lambda, E_\lambda)\right)=\sum_{\lambda\in\Lambda}\partial(\boldsymbol{x}_\lambda, E_\lambda) \qquad \text{for} \quad \sum_{\lambda\in\Lambda}(\boldsymbol{x}_\lambda, E_\lambda)\in\mathscr{G}(\gamma, \delta)\,.$$

Then we have the following proposition.

PROPOSITION 5.1. *For each* $(\alpha, \beta, \gamma, \delta)\in\bar{X}$, $(\alpha, \beta)\neq(0, 0)$, *the following diagram commutes*:

$$\begin{array}{ccc} \mathscr{G}(\gamma, \delta) & \xrightarrow{\Sigma_{\binom{a_1}{\varepsilon_1}}} & \mathscr{G}(\gamma_1, \delta_1) \\ \partial\downarrow & & \downarrow\partial \\ G & \xrightarrow{\eta_{\binom{a_1}{\varepsilon_1}}} & G\,, \end{array}$$

that is,

$$\eta_{\binom{a_1}{\varepsilon_1}}\circ\partial=\partial\circ\Sigma_{\binom{a_1}{\varepsilon_1}}\,.$$

The proof is easy from the definitions of $\Sigma_{\binom{a_1}{\varepsilon_1}}$, ∂ and $\eta_{\binom{a_1}{\varepsilon_1}}$.

Let $\mathscr{K}$ be the geometrical realization map on G, that is,

$$\mathscr{K}(\boldsymbol{x}, I_i):=\boldsymbol{x}+I_i\,,$$

$$\mathscr{K}\left(\sum_{\lambda\in\Lambda}m_\lambda(\boldsymbol{x}_\lambda, I_\lambda)\right)=\bigcup_{\lambda\in\Lambda,\, m_\lambda\neq 0}(\boldsymbol{x}_\lambda+I_\lambda)\,.$$

Then we have the following property:

$$\mathscr{K}(\partial(\zeta))=\partial(\mathscr{K}(\zeta)) \qquad \text{for} \quad \zeta\in\mathscr{G}(\gamma, \delta)\,,$$

where the second ∂ means the topological boundary of $\mathscr{K}(\zeta)$. Therefore we have the following proposition.

PROPOSITION 5.2. *For each* $(\alpha, \beta, \gamma, \delta)\in\bar{X}$, $(\alpha, \beta)\neq(0, 0)$ *the boundary of* $\mathscr{K}(\Sigma_{\binom{a_n}{\beta_n}}\cdots\Sigma_{\binom{a_1}{\varepsilon_1}}\mathscr{U})$ *is given by* $\mathscr{K}(\eta_{\binom{a_n}{\varepsilon_n}}\cdots\eta_{\binom{a_1}{\varepsilon_1}}\partial\mathscr{U})$.

We know a generating method of fractal curves by using endomorphisms of free group with finite generators, called Dekking method [3]. Let $G\langle 1, 2, 3\rangle$ be the free group generated by $\{1, 2, 3\}$. The free group $G\langle 1, 2, 3\rangle$ is mapped to G by a map F as follows:

$$F: \quad \begin{array}{ll} 1\to(0, I_1) & 1^{-1}\to-(-\boldsymbol{e}_1, I_1) \\ 2\to(0, I_2) & 2^{-1}\to-(-\boldsymbol{e}_2, I_2) \\ 3\to(0, I_3) & 3^{-1}\to-(-\boldsymbol{e}_3, I_3) \end{array}$$

and

$$w_1w_2\cdots w_s\to F(w_1)+(f(w_1), F(w_2))+\cdots+(f(w_1\cdots w_{s-1}), F(w_s))\,,$$

where $f: G\langle 1, 2, 3\rangle\to\boldsymbol{Z}^3$ is the canonical homomorphism given by (3-5). For an endomorphism η of G, if there exists an endomorphism θ of $G\langle 1, 2, 3\rangle$ such that

$$\eta(F(i)) = F(\theta(i))$$

then we call the endomorphism θ of $G\langle 1, 2, 3\rangle$ a *free representation* of η. For the endomorphisms $\eta_{\binom{a}{\varepsilon}}$ of G, free representations $\theta_{\binom{a}{\varepsilon}}$ are given by

$$\theta_{\binom{a}{0}}: \quad \begin{aligned} &1 \to 3 \\ &2 \to 1\,\overbrace{3^{-1}3^{-1}\cdots 3^{-1}}^{a} \\ &3 \to 2\,, \end{aligned}$$

$$\theta_{\binom{a}{1}}: \quad \begin{aligned} &1 \to 2 \\ &2 \to 3 \\ &3 \to 1\,\overbrace{2^{-1}\cdots 2^{-1}}^{a}\,. \end{aligned}$$

Let $\mathscr{K}$ be a geometrical realization map from $G\langle 1, 2, 3\rangle$ to the family of compact sets of $\boldsymbol{R}^3$ given by

$$\mathscr{K}(s) = \{\lambda f(s) \mid 0 \le \lambda \le 1\} \qquad \text{for} \quad s \in \{1^{\pm 1}, 2^{\pm 1}, 3^{\pm 1}\}\,,$$

$$\mathscr{K}(w_1 \cdots w_s) = \bigcup_{k=1}^{s} (f(w_1 \cdots w_{k-1}) + \mathscr{K}(w_k))\,.$$

Then we have that

$$\mathscr{K}(F(W)) = \mathscr{K}(W) \qquad \text{for} \quad W \in G\langle 1, 2, 3\rangle\,.$$

Using this endomorphism $\theta_{\binom{a}{\varepsilon}}$, the boundaries of the domains

$$D^{(n,0)} = \pi_n \mathscr{K}(\Sigma_{\binom{a_n}{\varepsilon_n}} \cdots \Sigma_{\binom{a_1}{\varepsilon_1}}(\mathscr{U}))$$

and

$$D_i^{(n,0)} = \pi_n \mathscr{K}(\Sigma_{\binom{a_n}{\varepsilon_n}} \cdots \Sigma_{\binom{a_1}{\varepsilon_1}}(e_i, E_i))$$

are given by

$$\partial D^{(n,0)} = \pi_n(f_1^{(n,0)} + \mathscr{K}(\theta_{\binom{a_n}{\varepsilon_n}} \cdots \theta_{\binom{a_1}{\varepsilon_1}}(21^{-1}32^{-1}13^{-1})))\,,$$

$$\partial D_i^{(n,0)} = \pi_n(f_i^{(n,0)} + \mathscr{K}(\theta_{\binom{a_n}{\varepsilon_n}} \cdots \theta_{\binom{a_1}{\varepsilon_1}}(jkj^{-1}k^{-1}))) \qquad (j \equiv i+1,\ k \equiv i+2 \pmod 3)\,.$$

If $(\alpha, \beta, \gamma, \delta) \in \bar{X}$ is a periodic point with period k, then using the notation

$$\theta := \theta_{\binom{a_k}{\varepsilon_k}} \cdots \theta_{\binom{a_1}{\varepsilon_1}}$$

the boundaries of $D^{(m)}$ and $D_i^{(m)}$ are given by

$$\partial D^{(m)} = \pi(f_1^{(m)} + \mathscr{K}(\theta^m(21^{-1}32^{-1}13^{-1})))\,,$$

$$\partial D_i^{(m)} = \pi(f_i^{(m)} + \mathscr{K}(\theta^m(jkj^{-1}k^{-1})))\,.$$

Therefore our goal is to see the existence of limit sets as a 'generalized' simply closed Jordan curves

$$\partial X = \lim_{m\to\infty} \Phi^m \pi(f_1^{(m)} + \mathscr{K}(\theta^m(21^{-1}32^{-1}13^{-1}))) ,$$

$$\partial X_i = \lim_{m\to\infty} \Phi^m \pi(f_i^{(m)} + \mathscr{K}(\theta^m(jkj^{-1}k^{-1}))) .$$

For each $\theta_{\binom{a}{\varepsilon}}$ we introduce an endomorphism $\Theta_{\binom{a}{\varepsilon}}$, called a *lifting* of $\theta_{\binom{a}{\varepsilon}}$, since our endomorphisms θ given by the composition of $\theta_{\binom{a}{\varepsilon}}$'s may have some cancellations.

Let us consider a free group $G\langle \mathrm{A}, \mathrm{B}, \cdots, \mathrm{K}\rangle$ of rank 11 and let us define a homomorphism $\psi: G\langle \mathrm{A}, \mathrm{B}, \cdots, \mathrm{K}\rangle \to G\langle 1, 2, 3\rangle$ as follows:

$$\begin{array}{ll} \psi(\mathrm{A}) = 21^{-1} & \psi(\mathrm{F}) = 2 \\ \psi(\mathrm{B}) = 32^{-1}1 & \psi(\mathrm{G}) = 3 \\ \psi(\mathrm{C}) = 13^{-1} & \psi(\mathrm{H}) = 213^{-1} \\ \psi(\mathrm{D}) = 32 & \psi(\mathrm{I}) = 231^{-1} \\ \psi(\mathrm{E}) = 23 & \psi(\mathrm{J}) = 321^{-1} \\ & \psi(\mathrm{K}) = 312^{-1} . \end{array}$$

Let us introduce an endomorphism $\Theta_{\binom{a}{\varepsilon}}$ of $G\langle \mathrm{A}, \mathrm{B}, \cdots, \mathrm{K}\rangle$ as follows:

$$\Theta_{\binom{a}{0}}(\mathrm{A}) = \mathrm{CG}^{-a}$$

$$\Theta_{\binom{a}{0}}(\mathrm{B}) = \begin{cases} \mathrm{EG}^{a-2}\mathrm{C}^{-1} & \text{if } a \neq 1 \\ \mathrm{I} & \text{if } a = 1 \end{cases}$$

$$\Theta_{\binom{a}{0}}(\mathrm{C}) = \mathrm{B}$$

$$\Theta_{\binom{a}{0}}(\mathrm{D}) = \mathrm{HG}^{-(a-1)}$$

$$\Theta_{\binom{a}{0}}(\mathrm{E}) = \mathrm{CG}^{-(a-1)}\mathrm{F}$$

$$\Theta_{\binom{a}{0}}(\mathrm{F}) = \mathrm{CG}^{-(a-1)}$$

$$\Theta_{\binom{a}{0}}(\mathrm{G}) = \mathrm{F}$$

$$\Theta_{\binom{a}{0}}(\mathrm{H}) = \begin{cases} \mathrm{CG}^{-(a-3)}\mathrm{E}^{-1} & \text{if } a \neq 1, 2 \\ \mathrm{I}^{-1} & \text{if } a = 2 \\ \mathrm{A}^{-1} & \text{if } a = 1 \end{cases}$$

$$\Theta_{\binom{a}{0}}(\mathrm{I}) = \mathrm{CG}^{-(a-1)}\mathrm{B}^{-1}$$

$$\Theta_{\binom{a}{0}}(\mathrm{J}) = \mathrm{HG}^{-a}$$

$$\Theta_{\binom{a}{0}}(\mathrm{K}) = \mathrm{EG}^{a-1}\mathrm{C}^{-1}$$

$$\Theta_{\binom{a}{1}}(\mathrm{A}) = \mathrm{B}$$

$$\Theta_{\binom{a}{1}}(\mathrm{B}) = \begin{cases} \mathrm{A}^{-1}\mathrm{F}^{-(a-2)}\mathrm{D}^{-1} & \text{if } a \neq 1 \\ \mathrm{J}^{-1} & \text{if } a = 1 \end{cases}$$

$$\Theta_{\binom{a}{1}}(\mathrm{C}) = \mathrm{F}^a\mathrm{A}$$

$$\Theta_{\binom{a}{1}}(\mathrm{D}) = \mathrm{A}^{-1}\mathrm{F}^{-(a-1)}\mathrm{G}$$

$$\Theta_{\binom{a}{1}}(\mathrm{E}) = \mathrm{KF}^{-(a-1)}$$

$$\Theta_{\binom{a}{1}}(\mathrm{F}) = \mathrm{G}$$

$$\Theta_{\binom{a}{1}}(\mathrm{G}) = \mathrm{A}^{-1}\mathrm{F}^{-(a-1)}$$

$$\Theta_{\binom{a}{1}}(\mathrm{H}) = \mathrm{DF}^{a-1}\mathrm{A}$$

$$\Theta_{\binom{a}{1}}(\mathrm{I}) = \mathrm{KF}^{-a}$$

$$\Theta_{\binom{a}{1}}(\mathrm{J}) = \mathrm{A}^{-1}\mathrm{F}^{-(a-1)}\mathrm{B}$$

$$\Theta_{\binom{a}{1}}(\mathrm{K}) = \begin{cases} \mathrm{A}^{-1}\mathrm{F}^{-(a-3)}\mathrm{D}^{-1} & \text{if } a \neq 1, 2 \\ \mathrm{J}^{-1} & \text{if } a = 2 \\ \mathrm{C} & \text{if } a = 1 \end{cases}$$

Then from the definitions of $\Theta_{\binom{a}{\varepsilon}}$ and ψ we have the following proposition.

PROPOSITION 5.3. (1) *The following commutative relation holds*:

$$\begin{array}{ccc} G\langle \mathrm{A}, \mathrm{B}, \cdots, \mathrm{K}\rangle & \xrightarrow{\Theta_{\binom{a}{\varepsilon}}} & G\langle \mathrm{A}, \mathrm{B}, \cdots, \mathrm{K}\rangle \\ \psi \downarrow & & \downarrow \psi \\ G\langle 1, 2, 3\rangle & \xrightarrow{\theta_{\binom{a}{\varepsilon}}} & G\langle 1, 2, 3\rangle . \end{array}$$

(2) *In particular, we have*

$$\psi \circ \Theta^n(\mathrm{ABC}) = \theta^n(21^{-1}32^{-1}13^{-1}),$$

where Θ is given by $\Theta_{\binom{a_k}{\varepsilon_k}} \cdots \Theta_{\binom{a_1}{\varepsilon_1}}$ for $\theta_{\binom{a_k}{\varepsilon_k}} \cdots \theta_{\binom{a_1}{\varepsilon_1}}$.

PROPOSITION 5.4. *For each n, Θ^n(ABC) does not have a cancellation.*

PROOF. Let us consider the following set $\mathscr{F}$ of $G\langle \mathrm{A}, \mathrm{B}, \cdots, \mathrm{K}\rangle$:

$\mathscr{F} = \{$AA, AB, AB^{-1}, $\mathrm{A}^{-1}\mathrm{B}$, AC^{-1}, $\mathrm{A}^{-1}\mathrm{C}^{-1}$, AD, $\mathrm{A}^{-1}\mathrm{D}^{-1}$, AF,
$\mathrm{A}^{-1}\mathrm{F}^{-1}$, AG, $\mathrm{A}^{-1}\mathrm{G}$, AJ, $\mathrm{A}^{-1}\mathrm{J}^{-1}$, AK, AK^{-1},
BB, BC, BC^{-1}, $\mathrm{B}^{-1}\mathrm{C}$, BD, $\mathrm{B}^{-1}\mathrm{E}$, BF^{-1}, $\mathrm{B}^{-1}\mathrm{F}$, BG, $\mathrm{B}^{-1}\mathrm{G}^{-1}$,
BI^{-1}, $\mathrm{B}^{-1}\mathrm{I}$, $\mathrm{B}^{-1}\mathrm{I}^{-1}$, BJ, BJ^{-1}, $\mathrm{B}^{-1}\mathrm{J}$,
CC, CE^{-1}, $\mathrm{C}^{-1}\mathrm{E}$, CF, $\mathrm{C}^{-1}\mathrm{F}$, CG^{-1}, $\mathrm{C}^{-1}\mathrm{G}$, $\mathrm{C}^{-1}\mathrm{H}$, $\mathrm{C}^{-1}\mathrm{H}^{-1}$,
CI^{-1}, $\mathrm{C}^{-1}\mathrm{I}$,
DF, $\mathrm{D}^{-1}\mathrm{F}$, $\mathrm{D}^{-1}\mathrm{G}^{-1}$,
$\mathrm{E}^{-1}\mathrm{F}^{-1}$, EG, $\mathrm{E}^{-1}\mathrm{G}$, $\mathrm{E}^{-1}\mathrm{I}^{-1}$,
FF, $\mathrm{F}^{-1}\mathrm{G}$, FH, FI, $\mathrm{F}^{-1}\mathrm{I}^{-1}$, $\mathrm{F}^{-1}\mathrm{J}$, $\mathrm{F}^{-1}\mathrm{J}^{-1}$, FK^{-1}, $\mathrm{F}^{-1}\mathrm{K}$,
GG, GH^{-1}, $\mathrm{G}^{-1}\mathrm{H}$, $\mathrm{G}^{-1}\mathrm{I}$, $\mathrm{G}^{-1}\mathrm{I}^{-1}$, GJ, $\mathrm{G}^{-1}\mathrm{J}^{-1}$, GK,
$\mathrm{H}^{-1}\mathrm{I}^{-1}$, II, JJ, JK$\}$.

For each element $UV \in \mathscr{F}$, it is easy to see that $\Theta_{\binom{a}{\varepsilon}}(U)\Theta_{\binom{a}{\varepsilon}}(V)$ does not have a cancellation and $\Theta_{\binom{a}{\varepsilon}}(UV)$ is constructed by $\mathscr{F}$, that is, if we put $\Theta_{\binom{a}{\varepsilon}}(UV) = S_1 S_2 \cdots S_l$ ($S_i \in \{\mathrm{A}, \cdots, \mathrm{K}\}$) then $S_i S_{i+1}$ ($1 \le i \le l-1$) belong to $\mathscr{F}$. Therefore, Θ^n(ABC) is constructed by $\mathscr{F}$ for all n, and so Θ^n(ABC) does not have a cancellation. (q.e.d.)

We see easily that $\psi(U)\psi(V)$ does not have a cancellation for all $UV \in \mathscr{F}$. Therefore, we have rearranged the endomorphism θ which may have a cancellation into the endomorphism Θ which does not have a cancellation. We call the endomorphism Θ a *lifting* of θ.

Therefore, by Theorem 1 of Dekking [3] or by Theorem 5.1 of Ito-Kimura [5], we have the following theorem.

THEOREM 5.1. *Under Assumption* P, *there exists a closed curve, which may have double points,*

$$B=\lim_{m\to\infty}\Phi^m\pi\mathscr{K}(f_1^{(m)}+\theta^m(21^{-1}32^{-1}13^{-1}))$$

as the boundary of X, *and its Hausdorff dimension is estimated by*

$$\dim_{\mathrm{H}}B\leq\frac{\log\lambda_N}{\log\lambda_B}$$

where $\lambda_B(=(\theta\theta_1\cdots\theta_{k-1})^{-1})$ *is a positive eigenvalue of* Φ *and* λ_N *is the maximum positive eigenvalue of* $N=(N_{ij})$, *where* N_{ij} *is the cardinality of* j *or* j^{-1} *in* $\Theta(i)$, $i, j\in\{\mathrm{A}, \mathrm{B}, \cdots, \mathrm{K}\}$. *The equality of the estimate holds in the case that* Φ *has complex eigenvalues.*

6. A generalization.

In this section, we leave the setting by modified Jacobi-Perron algorithm and try to discuss a generalization in an algebraic framework.

Let $\mathscr{H}=\{A_i \mid i=1, 2, 3, 4\}$ be the following generators of $SL(3, \boldsymbol{Z})$:

$$A_1=\begin{pmatrix}1&0&1\\1&0&0\\0&1&0\end{pmatrix},\quad A_2=\begin{pmatrix}1&1&0\\0&0&1\\1&0&0\end{pmatrix},\quad A_3=\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix},\quad A_4=\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}$$

and let us give the following definition.

DEFINITION. A matrix $B\in SL(3, \boldsymbol{Z})$ is said to be *special hyperbolic* if the following conditions hold:

(1) $B\geq 0$ and there exists m such that $B^m>0$.

(2) The maximum eigenvalue of B is a Pisot number.

(3) There exists a sequence $(i_1, i_2, \cdots, i_k)$, $i_j\in\{1, 2, 3, 4\}$ such that

$$B=A_{i_1}A_{i_2}\cdots A_{i_k}.$$

In this section we assume that the matrix B satisfies the special hyperbolicity. Let us denote the column and row eigenvectors of B by ${}^t(1, \alpha, \beta)$, $(1, \gamma, \delta)$ respectively. Let us denote

$$\Phi_B^{-1}(x):=B^{-1}x,$$

$$\phi_i^{-1}(x):=A_i^{-1}x,$$

$$\mathbb{P}_B:=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\,\middle|\,\left(\begin{pmatrix}1\\\gamma\\\delta\end{pmatrix},\begin{pmatrix}x\\y\\z\end{pmatrix}\right)=0\right\},$$

π be the projection to $\mathbb{P}_B$ along ${}^t(1, \alpha, \beta)$,

$\mathscr{G}_B$ be the family of sets of tips of stepped surface of $\mathbb{P}_B$.

Let us introduce the substitution Σ_i, $i=1, 2, 3, 4$ on $\mathscr{G}_B$ as follows (see Figure 7):

$$\Sigma_1 : \begin{cases} (0, E_1) \to (0, E_3) + (e_1 - e_3, E_1) \\ (0, E_2) \to (0, E_1) \\ (0, E_3) \to (0, E_2), \end{cases}$$

$$\Sigma_2 : \begin{cases} (0, E_1) \to (0, E_2) + (e_1 - e_2, E_1) \\ (0, E_2) \to (0, E_3) \\ (0, E_3) \to (0, E_1), \end{cases}$$

$$\Sigma_3 : \begin{cases} (0, E_1) \to (0, E_1) + (e_2 - e_1, E_2) \\ (0, E_2) \to (0, E_2) \\ (0, E_3) \to (0, E_3), \end{cases}$$

$$\Sigma_4 : \begin{cases} (0, E_1) \to (0, E_1) + (e_3 - e_1, E_3) \\ (0, E_2) \to (0, E_2) \\ (0, E_3) \to (0, E_3), \end{cases}$$

$$\Sigma_i(x, S) := \phi_i^{-1}(x) + \Sigma_i(0, S),$$

$$\Sigma_i\left(\sum_{\lambda \in \Lambda} (x_\lambda, S_\lambda)\right) := \sum_{\lambda \in \Lambda} (\Sigma_i(x_\lambda, S_\lambda)),$$

$$\Sigma_B := \Sigma_{i_k} \Sigma_{i_{k-1}} \cdots \Sigma_{i_1}.$$

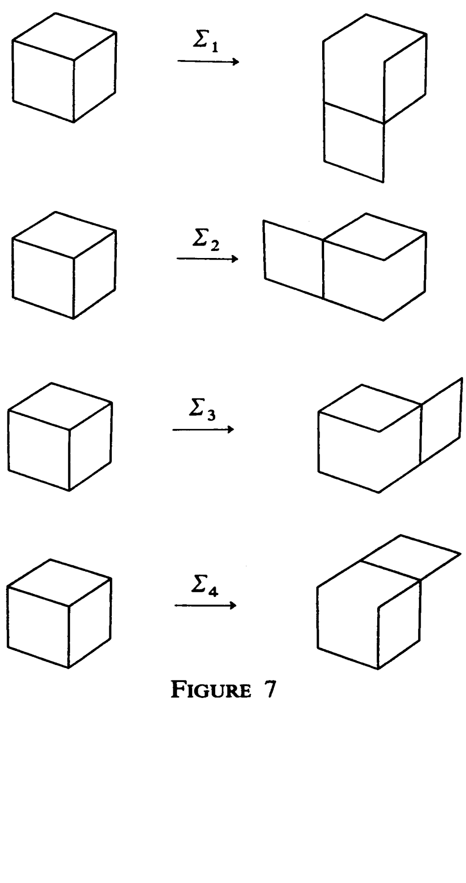

FIGURE 7

Let us introduce the substitution associated with Σ_i as follows:

$$\sigma_1 := \begin{cases} 1 \to 1\,2 \\ 2 \to 3 \\ 3 \to 1\,, \end{cases} \qquad \sigma_2 := \begin{cases} 1 \to 1\,3 \\ 2 \to 1 \\ 3 \to 2\,, \end{cases}$$

$$\sigma_3 := \begin{cases} 1 \to 1 \\ 2 \to 1\,2 \\ 3 \to 3\,, \end{cases} \qquad \sigma_4 := \begin{cases} 1 \to 1 \\ 2 \to 2 \\ 3 \to 1\,3 \end{cases}$$

and define the substitution associated with Σ_B by

$$\sigma_B = \sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_k}\,.$$

Under the renormalization by Φ_B, we have the following theorem.

THEOREM 6.1. *For the special hyperbolic matrix B the limit sets on $\mathbb{P}_B$ are well defined*:

$$X := \lim_{m\to\infty} \Phi_B^m \pi \mathscr{K}(\Sigma_B^m(\mathscr{U}))\,,$$

$$X_i := \lim_{m\to\infty} \Phi_B^m \pi \mathscr{K}(\Sigma_B^m(e_i, E_i))\,,$$

$$X_i' := \lim_{m\to\infty} \Phi_B^m \pi \mathscr{K}(\Sigma_B^m(0, E_i))$$

and satisfies the same statements as in Theorem 4.1 (1)–(5).

Remarking that Σ_i preserves $\mathscr{C}$-coveredness, the proof is obtained by complete analogy.

NOTE. The generator $\mathscr{K}$ is induced from the following decomposition properties:

$$A_{\binom{a}{0}} = A_3^{a-1} A_1\,,$$
$$A_{\binom{a}{1}} = A_4^{a-1} A_2\,.$$

References

[1] T. BEDFORD, Generating special Markov partitions for hyperbolic toral automorphisms using fractals, Ergodic Theory Dynamical Systems, **6** (1986), 325–333.

[2] R. BOWEN, Markov partitions are not smooth, Proc. Amer. Math. Soc., **71** (1978), 130–132.

[3] F. M. DEKKING, Recurrent sets, Adv. in Math., **44** (1982), 78–104.

[4] Sh. ITO and M. KIMURA, On Rauzy fractal, Japan J. Indust. Appl. Math., **8** (1991), 461–486.

[5] Sh. ITO and M. OHTSUKI, On the fractal curves induced from endomorphisms on the free group of rank 2, Tokyo J. Math., **14** (1992), 277–304.

[6] E. V. PODSYPANIN, A generalization of the continued fraction algorithm that is related to the Viggo Brun algorithm (Russian), Studies in Number Theory (LOMI), 4, Zap. Naucn. Sem. Leningrad.

Otdel. Mat. Inst. Steklov, **67** (1977), 184–194.
[7] G. Rauzy, Nombres algébriques et substitutions, Bull. Soc. Math. France, **110** (1982), 147–178.

Present Address:
Department of Mathematics, Tsuda College
Tsuda-machi, Kodaira-shi, Tokyo 187, Japan