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On the Mean-Square for the Approximate Functional Equation of the Riemann Zeta-Function

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§1. Introduction.

Let $d(n)$ be the number of positive divisors of n, and γ the Euler constant. The problem of estimating the quantity

$$\Delta(x)=\sum_{n\leqq x}{}' d(n)-x(\log x+2\gamma-1)-1/4$$

is called the Dirichlet divisor problem, where the symbol $\sum'$ indicates that the last term is to be halved if x is an integer. G. F. Voronoi [9] proved two remarkable formulas concerning $\Delta(x)$. Besides giving an explicit expression for $\Delta(x)$, he (see also (2.3) of [2]) proved

$$\int_2^T \Delta(x)dx=4^{-1}T+(2\sqrt{2}\pi^2)^{-1}T^{3/4}\sum_{n=1}^{\infty} d(n)n^{-5/4}\sin(4\pi\sqrt{nT}-4^{-1}\pi)+O(1)\,. \quad (1.1)$$

The sharp estimate of the error term is to be noted.

Let $\zeta(s)$ be the Riemann zeta-function, and for $T\geqq 2$, let

$$E(T)=\int_0^T |\zeta(1/2+it)|^2dt-T\log(T/2\pi)-(2\gamma-1)T\,, \quad (1.2)$$

the error term in the mean-square formula for $\zeta(s)$. J. L. Hafner and A. Ivić [2] established the analogue of (1.1) for $E(T)$:

$$\int_2^T E(t)dt=\pi T+2^{-1}(2T/\pi)^{3/4}\sum_{n=1}^{\infty}(-1)^n d(n)n^{-5/4}\sin(4\pi(nT/2\pi)^{1/2}-4^{-1}\pi)$$
$$+O(T^{2/3}\log T) \qquad \text{(see (2.5) of [2])}\,. \quad (1.3)$$

We note that apart from the factor $(-1)^n$ the series in (1.3) is the same as the one in

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(1.1) with $T/2\pi$ in place of T. We now define, for $T\geqq 2$,

$$G(T)=\int_2^T E(t)dt-\pi T.$$

Then J. L. Hafner and A. Ivić (see (2.8) of [2]) proved the following formula:

$$\int_2^T G^2(t)dt=(5\pi\sqrt{2\pi})^{-1}\left(\sum_{n=1}^{\infty} d^2(n)n^{-5/2}\right)T^{5/2}+O(T^2), \tag{1.4}$$

and moreover

$$G(T)=\Omega_{\pm}(T^{3/4}). \tag{1.5}$$

From (1.4) it follows immediately that $G(T)=\Omega(T^{3/4})$, but (1.5), which means that both $G(T)=\Omega_+(T^{3/4})$ and $G(T)=\Omega_-(T^{3/4})$ are true, is of course sharper. They said that, by using (1.1), one can obtain the analogue of the formula (1.4) for $\Delta(T)$, but did not state the result explicitly in [2]. Let, for $T\geqq 2$,

$$M(T)=\int_2^T \Delta(t)dt-4^{-1}T,$$

then we can show the asymptotic formula for $\Delta(T)$:

$$\int_2^T M^2(t)dt=(40\pi^2)^{-1}\left(\sum_{n=1}^{\infty} d^2(n)n^{-5/2}\right)T^{5/2}+O(T^2). \tag{1.6}$$

From (1.6) it follows immediately that $M(T)=\Omega(T^{3/4})$, and we can also prove the sharper result

$$M(T)=\Omega_{\pm}(T^{3/4}). \tag{1.7}$$

Let, for $t\geqq 1$,

$$R(s\,;t/2\pi)=\zeta^2(s)-\sum_{n\leqq t/2\pi}{}' d(n)n^{-s}-\chi^2(s)\sum_{n\leqq t/2\pi}{}' d(n)n^{s-1},$$

where

$$\chi(s)=2^s\pi^{s-1}\sin(2^{-1}\pi s)\Gamma(1-s).$$

It has been shown by Y. Motohashi [7] (see also (2.4.13) of [8]) that

$$\chi(1-s)R(s\,;t/2\pi)=-\sqrt{2}\,(t/2\pi)^{-1/2}\Delta(t/2\pi)+O(t^{-1/4}). \tag{1.8}$$

We note that M. Jutila [4] gives another proof of Motohashi's result (1.8). The asymptotic formula

$$\int_1^T |R(1/2+it\,;t/2\pi)|^2dt=\sqrt{2\pi}\left(\sum_{n=1}^{\infty} d^2(n)h^2(n)n^{-1/2}\right)T^{1/2}+O(T^{1/4}\log T) \tag{1.9}$$

was proved by I. Kiuchi and K. Matsumoto [5], and the error term has been improved to $O(\log^5 T)$ by I. Kiuchi [6], where

$$h(n)=(2/\pi)^{1/2}\int_0^\infty (y+n\pi)^{-1/2}\cos(y+4^{-1}\pi)dy=O(n^{-1/2}). \tag{1.10}$$

In view of the relation (1.8), to search an analogue of (1.1) for $\chi(1-s)R(s\,;\,t/2\pi)$ is an interesting problem in itself and the following result was proved by Y. Motohashi (see (3.4.7) of [8]):

$$\begin{aligned}&\int_0^T \chi(1/2-it)R(1/2+it\,;\,t/2\pi)dt\\ &\quad=(6\pi\sqrt{2})^{-1}\int_1^T (t/2\pi)^{-1/2}\{\log(t/2\pi)+2\gamma\}dt+2\sqrt{2}(T/2\pi)^{1/2}\\ &\quad\quad-(\pi\sqrt{2})^{-1}(T/2\pi)^{1/4}\sum_{n=1}^\infty d(n)n^{-3/4}\mathrm{Re}(g_0(n\pi))\cos(2\sqrt{2\pi nT}+4^{-1}\pi)+c_1\\ &\quad\quad+O(T^{-1/4}),\end{aligned} \tag{1.11}$$

where c_1 is a constant, and

$$\mathrm{Re}(g_0(n\pi))=-2\sqrt{2\pi}\,\mathrm{Im}\int_0^\infty \exp(-2^{-1}y^2+i(r\sqrt{2n\pi})y)dy \qquad (r=\exp(4^{-1}\pi i)). \tag{1.12}$$

In the next section (Lemma 3) we will give a proof of the fact

$$\mathrm{Re}(g_0(n\pi))=\pi\sqrt{2}\,h(n). \tag{1.13}$$

Therefore, Motohashi's formula (1.11) can be rewritten as

$$\begin{aligned}&\int_0^T \chi(1/2-it)R(1/2+it\,;\,t/2\pi)dt\\ &\quad=(3\sqrt{\pi})^{-1}T^{1/2}\{\log(T/2\pi)+2\gamma+4\}\\ &\quad\quad-(T/2\pi)^{1/4}\sum_{n=1}^\infty d(n)h(n)n^{-3/4}\cos(2\sqrt{2\pi nT}+4^{-1}\pi)+c+O(T^{-1/4}),\end{aligned} \tag{1.14}$$

where c is a constant. The formula (1.11) is used in the proof of Motohashi's "smoothed" version of Atkinson's formula for $E(T)$ (see Theorem 8 of [8]) as an application of the ζ^2-analogue of the Riemann-Siegel formula. Now, let

$$K(T)=\int_0^T \chi(1/2-it)R(1/2+it\,;\,t/2\pi)dt-(3\sqrt{\pi})^{-1}T^{1/2}\{\log(T/2\pi)+2\gamma+4\}, \tag{1.15}$$

then from (1.14) and (1.15) it follows that $K(T)=O(T^{1/4})$. The formulas (1.1) and (1.14) also support the analogy between $\chi(1-s)R(s\,;\,t/2\pi)$ and $(t/2\pi)^{-1/2}\Delta(t/2\pi)$. Analogously to Hafner and Ivić's (1.4) and (1.5), we shall prove the following results for $K(T)$.

THEOREM 1. *For $T\geqq 2$ we have*

$$\int_2^T K^2(t)dt=(3\sqrt{2\pi})^{-1}\left(\sum_{n=1}^{\infty} d^2(n)h^2(n)n^{-3/2}\right)T^{3/2}+O(T)\,. \tag{1.16}$$

THEOREM 2.

$$K(T)=\Omega_{\pm}(T^{1/4})\,. \tag{1.17}$$

REMARK. It follows immediately from (1.16) that $K(T)=\Omega(T^{1/4})$, so apart from the value of the numerical constants involved, the order of magnitude of $K(T)$ is precisely determined.

By using (1.1), we can obtain (1.6) and (1.7) as analogues of the above theorems, but the proofs of these results are quite similar and we omit it.

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§2. Some lemmas.

Firstly we show Lerch's formula; this is classically known, but here we give a proof for the convenience of readers.

LEMMA 1. *For $a>0$, we have*

$$\int_0^\infty e^{-u^2-(a/u)^2}du=2^{-1}\sqrt{\pi}\,e^{-2a}\,.$$

PROOF. Let

$$S=\int_0^\infty e^{-u^2-(a/u)^2}du=e^{-2a}\int_0^\infty e^{-\{u-(a/u)\}^2}du\,. \tag{2.1}$$

Now we put $t=a/u$, then

$$S=ae^{-2a}\int_0^\infty e^{-\{t-(a/t)\}^2}t^{-2}dt\,. \tag{2.2}$$

From (2.1) and (2.2), it follows that

$$S=2^{-1}e^{-2a}\int_0^\infty e^{-\{t-(a/t)\}^2}(1+at^{-2})dt\,.$$

Hence, putting $y=t-(a/t)$, we have

$$S=2^{-1}e^{-2a}\int_{-\infty}^{\infty}e^{-y^2}dy\,.$$

This completes the proof of Lemma 1.

LEMMA 2 (Lerch, see p. 279 in [10]). *For* $\mathrm{Re}(s)>0$ *and* $\mathrm{Re}(w)>0$, *we have*

$$\int_0^{\infty}\{1+4^{-1}(w/x)^2\}^{-2^{-1}s}e^{-x^2}dx=\sqrt{\pi}\,(\Gamma(2^{-1}s))^{-1}\int_0^{\infty}e^{-x^2-wx}x^{s-1}dx\,.$$

PROOF. We start from the obvious identity, valid for $z>0$ and $\mathrm{Re}(s)>0$,

$$\int_0^{\infty}e^{-zy}y^{s-1}dy=z^{-s}\Gamma(s)\,.$$

We put $z=1+4^{-1}(w/x)^2$ $(w>0)$ to get

$$\int_0^{\infty}e^{-\{1+4^{-1}(w/x)^2\}y}y^{2^{-1}s-1}dy=\Gamma(2^{-1}s)\{1+4^{-1}(w/x)^2\}^{-2^{-1}s}\,.$$

Multiplying this by e^{-x^2} and integrating over the interval $[0,\infty)$, we have

$$\int_0^{\infty}e^{-x^2}\{1+4^{-1}(w/x)^2\}^{-2^{-1}s}dx=(\Gamma(2^{-1}s))^{-1}\int_0^{\infty}e^{-y}y^{2^{-1}s-1}dy\int_0^{\infty}e^{-x^2-4^{-1}(w/x)^2y}dx\,.$$

By using Lemma 1 with $a=2^{-1}w\sqrt{y}$, we obtain that the right-hand side of the above is equal to

$$2^{-1}\sqrt{\pi}\,(\Gamma(2^{-1}s))^{-1}\int_0^{\infty}e^{-y-w\sqrt{y}}y^{2^{-1}s-1}dy\,.$$

Hence, for $\mathrm{Re}(s)>0$ and $w>0$, we have

$$\int_0^{\infty}\{1+4^{-1}(w/x)^2\}^{-2^{-1}s}e^{-x^2}dx=\sqrt{\pi}\,(\Gamma(2^{-1}s))^{-1}\int_0^{\infty}e^{-x^2-wx}x^{s-1}dx\,. \tag{2.3}$$

Next we shall prove that (2.3) is valid, by the analytic continuation, for $\mathrm{Re}(w)>0$. Let $w=u+iv$ $(u>0)$ and $s=\sigma+it$ $(\sigma>0)$. Since

$$\int_0^{\infty}|e^{-x^2-wx}x^{s-1}|dx\leqq\int_0^{\infty}e^{-x^2-ux}x^{\sigma-1}dx\,,$$

the integral on the right-hand side of (2.3) is absolutely convergent for $\sigma>0$ and any w. Since

$$2^{-1}t\arg(1+(w/2x)^2)\leqq c_1 \quad\text{and}\quad -2^{-1}\sigma\log|1+(w/2x)^2|\leqq c_2$$

for $u>0$ and $\sigma>0$, we have

$$\int_0^\infty |\{1+4^{-1}(w/x)^2\}^{-2^{-1}s}e^{-x^2}|dx$$

$$=\int_0^\infty e^{-2^{-1}\sigma\log|1+(w/2x)^2|+2^{-1}t\arg(1+(w/2x)^2)-x^2}dx$$

$$\leqq c_3\int_0^\infty e^{-x^2}dx\,,$$

where the constants c_j ($j=1,2,3$) depend on s and w. Therefore the left-hand side of (2.3) is absolutely convergent for $\sigma>0$ and $u>0$. This completes the proof of Lemma 2.

Now we can prove the identity (1.13):

LEMMA 3.

$$\mathrm{Re}(g_0(n\pi))=\pi\sqrt{2}\,h(n)\,.$$

PROOF. From (1.10) and (1.12), it suffices to show that

$$2^{-1/2}\,\mathrm{Im}\int_0^\infty \exp(-2^{-1}y^2+i(r\sqrt{2n\pi})y)dy=-2^{-1}\int_0^\infty (y+n\pi)^{-1/2}\cos(y+4^{-1}\pi)dy\,. \tag{2.4}$$

We put $s=1$, $\sqrt{2}\,x=y$ and $w=-2ir\sqrt{n\pi}$ in the right-hand side of Lemma 2, then it is equal to

$$2^{-1/2}\int_0^\infty \exp(-2^{-1}y^2+i(r\sqrt{2n\pi})y)dy\,. \tag{2.5}$$

The left-hand side in Lemma 2 with $s=1$ and $w=-2ir\sqrt{n\pi}$ is

$$\int_0^\infty x(x^2-r^2n\pi)^{-1/2}\exp(-x^2)dx=-2^{-1}i\int_0^{i\infty}(\xi+n\pi)^{-1/2}\exp(i(\xi+4^{-1}\pi))d\xi\,, \tag{2.6}$$

since

$$(x^2-r^2n\pi)^{1/2}=-ir(\xi+n\pi)^{1/2}\qquad(r=\exp(4^{-1}\pi i)\,,\quad \xi=-(x/r)^2)\,.$$

It can be easily seen that the integral

$$\int_{C_R}(\xi+n\pi)^{-1/2}\exp(i(\xi+4^{-1}\pi))d\xi$$

tends to 0 as R tends to infinity, where C_R denotes a quadrant of radius R from R to iR. Hence, we have

$$\int_0^{i\infty}(\xi+n\pi)^{-1/2}\exp(i(\xi+4^{-1}\pi))d\xi=\int_0^{\infty}(\xi+n\pi)^{-1/2}\exp(i(\xi+4^{-1}\pi))d\xi\,. \quad (2.7)$$

From (2.5)–(2.7), it follows that

$$2^{-1/2}\int_0^{\infty}\exp(-2^{-1}y^2+i(r\sqrt{2n\pi})y)dy=-2^{-1}i\int_0^{\infty}(\xi+n\pi)^{-1/2}\exp(i(\xi+4^{-1}\pi))d\xi\,.$$

Hence, taking the imaginary part, we obtain the identity (2.4).

§3. Proof of Theorem 1.

From (1.14), (1.15) and Schwarz's inequality, we have

$$\int_T^{2T}K^2(t)dt=I_1+I_2+I_3+O(T^{1/2}+T^{1/4}I_1^{1/2}+T^{1/4}I_3^{1/2})\,, \quad (3.1)$$

where

$$I_1=\int_T^{2T}\left|(t/2\pi)^{1/4}\sum_{n=1}^{\infty}d(n)h(n)n^{-3/4}\cos(2\sqrt{2\pi nt}+4^{-1}\pi)\right|^2dt\,,$$

$$I_2=2c\int_T^{2T}(t/2\pi)^{1/4}\sum_{n=1}^{\infty}d(n)h(n)n^{-3/4}\cos(2\sqrt{2\pi nt}+4^{-1}\pi)dt\,,$$

$$I_3=c^2\int_T^{2T}dt=O(T)\,. \quad (3.2)$$

Since

$$\int_T^{2T}t^{1/4}\exp(iu\sqrt{t})dt=O(u^{-1}T^{3/4}) \qquad (u\neq 0)\,,$$

we have, by (1.10),

$$I_2=O\left(T^{3/4}\sum_{n=1}^{\infty}d(n)h(n)n^{-5/4}\right)=O(T^{3/4})\,. \quad (3.3)$$

To evaluate I_1 we expand out the square, and get

$$I_1=2^{-1}\sum_{n=1}^{\infty}d^2(n)h^2(n)n^{-3/2}\int_T^{2T}(t/2\pi)^{1/2}dt$$

$$-2^{-1}\sum_{n=1}^{\infty}d^2(n)h^2(n)n^{-3/2}\int_T^{2T}(t/2\pi)^{1/2}\sin(4\sqrt{2\pi nt})dt$$

$$-2^{-1}\sum_{\substack{m=1\\ m\neq n}}^{\infty}\sum_{n=1}^{\infty} d(m)d(n)h(m)h(n)(mn)^{-3/4}\int_T^{2T}(t/2\pi)^{1/2}\sin(2\sqrt{2\pi t}(\sqrt{m}+\sqrt{n}))dt$$

$$+2^{-1}\sum_{\substack{m=1\\ m\neq n}}^{\infty}\sum_{n=1}^{\infty} d(m)d(n)h(m)h(n)(mn)^{-3/4}\int_T^{2T}(t/2\pi)^{1/2}\cos(2\sqrt{2\pi t}(\sqrt{m}-\sqrt{n}))dt$$

$$=I_{1,1}+I_{1,2}+I_{1,3}+I_{1,4}, \quad \text{say}.$$

Since

$$\int_T^{2T} t^{1/2}\exp(iu\sqrt{t})dt=O(u^{-1}T) \qquad (u\neq 0),$$

we have, by (1.10),

$$I_{1,2}=O\left(T\sum_{n=1}^{\infty} d^2(n)h^2(n)n^{-2}\right)=O(T), \tag{3.4}$$

and

$$I_{1,3}=O\left(T\sum_{\substack{m=1\\ m\neq n}}^{\infty}\sum_{n=1}^{\infty} d(m)d(n)h(m)h(n)(mn)^{-3/4}(\sqrt{m}+\sqrt{n})^{-1}\right)$$

$$=O\left(T\left(\sum_{n=1}^{\infty} d(n)h(n)n^{-1}\right)^2\right)=O(T). \tag{3.5}$$

It is seen that

$$\sum\sum_{\substack{m,n\leqq N\\ m\neq n}} d(m)d(n)h(m)h(n)(mn)^{-3/4}|\sqrt{m}-\sqrt{n}|^{-1}$$

$$=O\left(\sum\sum_{n<m\leqq N} d(m)d(n)h(m)h(n)m^{-1/4}n^{-3/4}(m-n)^{-1}\right)$$

$$=O\left(\sum\sum_{n,r\leqq N} d(n)d(n+r)n^{-13/8}r^{-11/8}\right)$$

$$=O\left(\sum\sum_{n,r\leqq N} n^{-(13/8)+\varepsilon}r^{-(11/8)+\varepsilon}\right)=O(1),$$

by using (1.10), the inequality $a+b\geqq 2\sqrt{ab}$ ($a\geqq 0$, $b\geqq 0$), and the fact $d(n)=O(n^{\varepsilon})$ for any $\varepsilon>0$. Hence, we have

$$I_{1,4}=O(T). \tag{3.6}$$

It is easy to see that

$$I_{1,1}=(3\sqrt{2\pi})^{-1}\left(\sum_{n=1}^{\infty} d^2(n)h^2(n)n^{-3/2}\right)\{(2T)^{3/2}-T^{3/2}\}. \tag{3.7}$$

From (3.4)–(3.7), we have

$$I_1=(3\sqrt{2\pi})^{-1}\left(\sum_{n=1}^{\infty} d^2(n)h^2(n)n^{-3/2}\right)\{(2T)^{3/2}-T^{3/2}\}+O(T). \tag{3.8}$$

From (3.2) and (3.8), we see that the error term in the right-hand side of (3.1) is $O(T)$. Therefore, we have

$$\int_T^{2T} K^2(t)dt=(3\sqrt{2\pi})^{-1}\left(\sum_{n=1}^{\infty} d^2(n)h^2(n)n^{-3/2}\right)\{(2T)^{3/2}-T^{3/2}\}+O(T).$$

Replacing T by $2^{-1}T$, $4^{-1}T$, and so on, and adding we obtain Theorem 1.

§4. Proof of Theorem 2.

From (1.10), integrating by parts, we have

$$h(n)=(2/\pi)^{1/2}\left(-(2\pi n)^{-1/2}+(2\pi n)^{-3/2}-(3/4)\int_0^{\infty}(y+n\pi)^{-5/2}\cos(y+4^{-1}\pi)dy\right),$$

which implies that

$$h(n)\leqq-(\pi\sqrt{n})^{-1}\{1-(1+\sqrt{2})(2\pi n)^{-1}\}, \tag{4.1}$$

and

$$h(n)\geqq-(\pi\sqrt{n})^{-1}-(\sqrt{2}-1)(2\pi^2)^{-1}n^{-3/2}. \tag{4.2}$$

From (1.14) and (1.15), it follows that

$$K(T)=-(T/2\pi)^{1/4}\sum_{n=1}^{\infty} d(n)h(n)n^{-3/4}\cos(2\sqrt{2n\pi T}+4^{-1}\pi)+c+O(T^{-1/4}).$$

To prove the omega-result (1.17), we proceed similarly as in the proof of J. L. Hafner and A. Ivić (see (6.4) of [2]). Set

$$J(u)=-\sum_{n=1}^{\infty} d(n)h(n)n^{-3/4}\cos(u\sqrt{n}+4^{-1}\pi).$$

Then, it suffices to show that

$$\limsup_{u\to\infty} J(u)>0, \tag{4.3}$$

and

$$\liminf_{u\to\infty} J(u) < 0\,. \tag{4.4}$$

We start from the proof of (4.3). Let M be a large positive integer, and let $\delta = M^{-1/2}$. For each $n \leqq M$, write $n = v^2 q$ where $q = q(n)$ is the square-free divisors of n. Let $Q_1 = \{q(n)\,;\, 1 \leqq n \leqq M\}$, and Q_2 be the set of all distinct elements of Q_1. Then $\{\sqrt{q}\,;\, q \in Q_2\}$ is the set of linearly independent numbers, as a special case of Besicovitch's theorem (see p. 204 of [1]). Then by Kronecker's approximation theorem (see Lemma 9.3 of [3]), there exists arbitrarily large u such that

$$|(2\pi)^{-1} u\sqrt{q} - m_q| < \delta_q$$

with some integer m_q and $0 < \delta_q < \delta$, where q ranges over the set Q_1. Hence we can deduce that for $n \leqq M$,

$$\cos(u\sqrt{n} + 4^{-1}\pi) = \cos(4^{-1}\pi) + O(\delta\sqrt{n})\,.$$

We have, by (4.1),

$$\begin{aligned}\limsup_{u\to\infty} J(u) &\geqq -\cos(4^{-1}\pi) \sum_{n\leqq M} d(n)h(n)n^{-3/4} + O(M^{-1/4}\log M)\\ &\geqq (\pi\sqrt{2})^{-1} \sum_{n\leqq M} d(n)n^{-5/4}(1 - (\sqrt{2}+1)(2\pi n)^{-1}) + O(M^{-1/4}\log M) > 0\,.\end{aligned}$$

Next, we prove (4.4). From Kronecker's approximation theorem, there exists arbitrarily large u such that

$$|(2\pi)^{-1} u\sqrt{q} - 1/2 - m_q| < \delta_q\,.$$

In this case

$$\cos(u\sqrt{n} + 4^{-1}\pi) = \lambda_n \cos(4^{-1}\pi) + O(\delta\sqrt{n})\,,$$

with $\lambda_n = 1$ if $n \equiv 0 \pmod 4$ and $\lambda_n = -1$ otherwise, so that we obtain, by (4.2),

$$\begin{aligned}\liminf_{u\to\infty} J(u) &\leqq -\cos(4^{-1}\pi) \sum_{n\leqq M} \lambda_n d(n)h(n)n^{-3/4} + O(M^{-1/4}\log M)\\ &\leqq (\pi\sqrt{2})^{-1} \sum_{n\leqq M} \lambda_n d(n)n^{-5/4} + (\sqrt{2}-1)(2\sqrt{2}\,\pi^2)^{-1} \sum_{n\leqq M} \lambda_n d(n)n^{-9/4}\\ &\quad + O(M^{-1/4}\log M)\,.\end{aligned}$$

Since

$$\sum_{n=1}^{\infty} \lambda_n d(n)n^{-s} = (3\cdot 2^{1-2s} - 1 - 2^{2-3s})\zeta^2(s) \qquad (s>1)\,,$$

which is negative at $s = 5/4$ and $9/4$ (see p. 186 of [2]), so that, letting M tend to infinity, we see that the resulting infinite series are negative. Hence we obtain $\liminf_{u\to\infty} J(u)<0$. This completes the proof of Theorem 2.

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